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# VARIATIONAL METHODS ON WARPED PRODUCT MANIFOLDS

YOON-TAE JUNG, EUNHEE CHOI AND SANG CHUL LEE

ABSTRACT. In this paper, we consider the problem of achieving constant scalar curvature on warped product manifolds according to fiber manifolds with constant scalar curvature.

## 1. Introduction

In [2], F. Dobarro and E. L. Dozo have studied from the viewpoint of partial differential equations and variational methods, the problem of showing when a Riemannian metric of constant scalar curvature can be produced on a product manifold $B \times F$ by a warped product construction applied to the two Riemannian manifolds $(B, g_B)$ and $(F, g_F)$, especially in the case when the fibre $(F, g_F)$ is of constant curvature. Particularly, in Theorem 3.6 of [2], the uniqueness of the warping function is considered. In [2], the eigenvalue problem of the elliptic operator $Lu = -\frac{4n}{n+1}\Delta u + Ru$ for a warped product $B \times_f F$ of Riemannian manifolds $B$ and $F$, where $\Delta$ is a Laplacian on $B$ and $R$ is the scalar curvature on $B$, is studied. Basically, the fact that the operator $L - \lambda I : C^{2,\alpha}(M) \to C^{\alpha}(M)$ is an isomorphism for some $\lambda$, is employed.

In this paper, we consider the problem of achieving constant scalar curvature according to fiber manifolds with constant scalar curvature on warped product manifolds. The class studied consists of taking $(B, g_B)$ to be a compact Riemannian manifold and $(F, g_F)$ to be a pseudo-Riemannian manifold. In our study of this case, we apply different variational considerations in comparison to the previous various results(cf. [2],[3]).

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Although we will assume throughout this paper that all data ($M$, metric $g$, and curvature, etc.) are smooth, this is merely for convenience. Our arguments go through with little or no change if one makes minimal smoothness hypotheses, such as assuming that the given data is Holder continuous.

## 2. Main Results

On a (semi)Riemannian product manifold $B \times F$, let $\pi$ and $\sigma$ be the projections of $B \times F$ onto $B$ and $F$, respectively, and let $f > 0$ be a smooth function on $B$.

DEFINITION. The warped product manifold $M = B \times_f F$ is the product manifold $M = B \times F$ furnished with metric tensor

$$g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F),$$

where $g_B$ and $g_F$ are metric tensors of $B$ and $F$, respectively. In other words, if $v$ is tangent to $M$ at $(p,q)$, then

$$g(v,v) = g_B(d\pi(v), d\pi(v)) + f^2(p) g_F(d\sigma(v), d\sigma(v)).$$

On the given warped product manifold $M = B \times_f F$, we also write $S^B$ for the pullback by $\pi$ of the scalar curvature $S_B$ of $B$ and similarly for $S^F$ and $S_F$. From now on, we denote the volume element of $g_B$ by $dV$, the gradient by $\nabla$, and the associated Laplacian by $\Delta$.

LEMMA 2.1. *If $S$ is the scalar curvature of $M = B \times_f F$ with $n = dimF > 1$, then*

$$S = S^B + \frac{S^F}{f^2} - 2n\frac{\Delta f}{f} - n(n-1)\frac{< \nabla f, \nabla f >}{f^2}, \tag{2.1}$$

*where $\Delta$ is the Laplacian on $B$.*

*Proof.* See Corollary 2.5 in [3] or p.214 in [4] □

Now we may pose the following question:

Question A: If $S_F(q) \equiv c$ (constant) on $F$, can we find a warping function $f > 0$ on $B$ such that the warped metric $g$ has constant scalar curvature $S(p,q) = k$ on $M = B \times_f F$?

If $S(p,q) \equiv k$ for all $(p,q) \in M$, then equation (2.1) is the pullback by $\pi$ of the following equation:

$$k = S_B(p) + \frac{c}{f^2} - 2n\frac{\Delta f}{f} - n(n-1)\frac{< \nabla f, \nabla f >}{f^2},$$

or equivalently,

$$\Delta f + \frac{1}{2n}(k - S_B(p))f - \frac{c}{2nf} + \frac{n-1}{2}\frac{< \nabla f, \nabla f >}{f} = 0. \tag{2.2}$$

By the change of variables $f = v^{\frac{2}{n+1}}$,

$$\nabla f = \frac{2}{n+1} \quad v^{\frac{2}{n+1}-1} \quad \nabla v$$

and

$$\Delta f = \frac{2}{n+1} \quad (\frac{2}{n+1} - 1) \quad v^{\frac{2}{n+1}-2} \quad |\nabla v|^2 + \frac{2}{n+1} \quad v^{\frac{2}{n+1}-1} \quad \Delta v.$$

Hence equation (2.2) is changed into

$$\Delta v + \frac{n+1}{4n}(k - S_B(p))v - \frac{n+1}{4n}cv^{1-\frac{4}{n+1}} = 0. \tag{2.3}$$

We treat the case of warped product manifolds with (semi-)Riemannian fiber such that for $M = B \times_f F$, $B$ is a compact Riemannian manifold and $F$ is a (semi)Riemannian manifold with constant scalar curvature $c$.

We let $H_{s.p}(B)$ denote the Sobolev space of functions on $B$ whose derivatives through order $s$ are in $L_p(B)$. The norm on $H_{s.p}(B)$ will be denoted by $\| \quad \|_{s,p}$. The usual norm $L_2(B)$ inner product will be written $\| \quad \|$.

Assuming that we denote $S_B(p)$ by $S(p)$ , by equation(2.3), on $B$, we consider the functional

$$\begin{aligned} J(v) &= \frac{\int_B |\nabla v|^2 dV + \frac{n+1}{4n}\int_B S(p)v^2 dV + \frac{(n+1)^2}{4n(n-1)}c\int_B v^{2-\frac{4}{n+1}}dV}{\int_B v^2 dV} \\ &= \int_B |\nabla v|^2 dV + \frac{n+1}{4n}\int_B S(p)v^2 dV + \frac{(n+1)^2}{4n(n-1)}c\int_B v^{2-\frac{4}{n+1}}dV \end{aligned}$$

on the set $D = \{v \in H_{1.2}(B)| \quad v \geq 0, \int_B v^2 dV = 1\}$.

THEOREM 2.2. *For each c, there exists* $inf_{v\in D}J(v)$.

*Proof.* Since $B$ is compact, there exists $m > 0$ such that $-m < S(p) < m$.

$$J(v) \geq -\frac{n+1}{4n}\int_B mv^2 dV - \frac{(n+1)^2}{4n(n-1)}|c|\int_B v^{2-\frac{4}{n+1}}dV$$

But since $B$ is compact, $(\int_B v^{2-\frac{4}{n+1}}dV)^{\frac{1}{2-\frac{4}{n+1}}} \leq C_1(\int_B v^2 dV)^{\frac{1}{2}}$. Hence for $v \in D$, $J(v) \geq -\frac{n+1}{4n} - \frac{(n+1)^2}{4n(n-1)}|c|(C_1)^{2-\frac{4}{n+1}}dV$. □

THEOREM 2.3. *If* $c < 0$, *then* $inf J(v) < 0$.

*Proof.* If $v \equiv d$( constant), then

$$\begin{aligned} J(v) &= \frac{\int_B |\nabla v|^2 dV + \frac{n+1}{4n}\int_B S(p)v^2 dV + \frac{(n+1)^2}{4n(n-1)}c\int_B v^{2-\frac{4}{n+1}}dV}{\int_B v^2 dV} \\ &= \frac{\frac{n+1}{4n}\int_B S(p)dV + \frac{(n+1)^2}{4n(n-1)}c\int_B d^{-\frac{4}{n+1}}dV}{Vol(B)} \leq 0 \end{aligned}$$

for small $d > 0$.

□

THEOREM 2.4. *If* $\int_B S(p)dV < 0$ , *then* $inf J(v) < 0$ *for all c.*

*Proof.* If $v \equiv d$( constant), then

$$\begin{aligned} J(v) &= \frac{\int_B |\nabla v|^2 dV + \frac{n+1}{4n}\int_B S(p)v^2 dV + \frac{(n+1)^2}{4n(n-1)}c\int_B v^{2-\frac{4}{n+1}}dV}{\int_B v^2 dV} \\ &= \frac{\frac{n+1}{4n}\int_B S(p)dV + \frac{(n+1)^2}{4n(n-1)}c\int_B d^{-\frac{4}{n+1}}dV}{Vol(B)} \leq 0 \end{aligned}$$

for large $d > 0$.

□

THEOREM 2.5. *If* $S(p) \geq 0(\not\equiv 0)$ *and* $c \geq 0$, *then* $inf J(v) > 0$.

*Proof.* Trivially, $inf J(v) \geq 0$. If $inf J(v) = 0$, then $|\nabla v| \equiv 0$. Since $\int_B v^2 dV = 1$, $v \equiv$ positive constant, which means $\int_B S(p)dV = 0$. But it is impossible.

□

THEOREM 2.6. *For each $c \leq 0$, if $\inf J(v) = \bar{c}$, then there exists a solution of equation (2.3) for such $k = \frac{4n}{n+1}\bar{c}$. That is, we can find a warping function $v > 0$ on $B$ such that the warped metric has a constant scalar curvature $k = \frac{4n}{n+1}\bar{c}$ on $M$.*

*Proof.* By Theorem 2.2, there exists $\inf J(v) = \bar{c}$. Let $\{v_i\} \in D = \{v \in H_{1,2}(B) | \quad v \geq 0, \int_B v^2 dV = 1\}$ be a minimizing sequence such that $J(v_i) \to \bar{c}$. We follow several steps.

Step1. $\{v_i\}$ is bounded in $H_{1,2}(B)$.

Since $J(v_i) \to \bar{c}$, we may assume that for all $i$,

$$J(v_i) = \int_B |\nabla v_i|^2 dV + \frac{n+1}{4n}\int_B S(p) v_i^2 dV + \frac{(n+1)^2}{4n(n-1)} c \int_B v_i^{2-\frac{4}{n+1}} \leq C_2$$

for some positive $C_2$. Just like as the proof of Theorem 2.2,

$$J(v_i) \geq \int_B |\nabla v_i|^2 dV - \frac{n+1}{4n} m - \frac{(n+1)^2}{4n(n-1)} |c| (C_1)^{2-\frac{4}{n+1}},$$

which means that $\int_B |\nabla v_i|^2 dV \leq C_3$ for some positive $C_3$. Hence $\{\nabla v_i\}$ is bounded and $||v_i|| = 1$, which implies that $\{v_i\}$ is bounded in $H_{1,2}(B)$

Step2. By Kondrokov's theorem for compact manifolds (cf. [1]), the imbedding $H_{1,2}(B) \hookrightarrow L_2(B)$ is compact. A bounded closed set in $H_{1,2}(B)$ is weakly compact, so there exist a subsequence $\{v_j\}$ of $\{v_i\}$ and a function $v_0$ such that

i) $v_j \to v_0$ strongly in $L_2(B)$,

ii) $v_j \to v_0$ weakly in $H_{1,2}(B)$,

iii) $v_j \to v_0$ almost everywhere pointwise.

By i) and iii), $\int_B v_0^2 dV = 1$ and $v_0 \geq 0$. And by ii), $\|v_0\|_{1,2} \leq \liminf_{j\to\infty} \|v_j\|_{1,2}$. Since $v_j \to v_0$ strongly in $L_2(B)$, we can see that $J(v_0) \leq \bar{c}$. The minimum property implies that $J(v_0) = \bar{c}$.

Step 3. Set $\varphi = v_0 + t\psi$, where $t$ is a small real number and $\psi \in C^\infty(B)$.

An asymptotic expansion gives

$$\begin{aligned}
J(\varphi) &= [\int_B |\nabla(v_0+t\psi)|^2 dV + \frac{n+1}{4n}\int_B S(p)(v_0+t\psi)^2 dV \\
&+ \frac{(n+1)^2}{4n(n-1)}c\int_B (v_0+t\psi)^{2-\frac{4}{n+1}} dV][\int_B (v_0+t\psi)^2 dV]^{-1} \\
&= [\int_B (|\nabla v_0|^2 + 2t\nabla v_0 \nabla\psi)dV + \frac{n+1}{4n}\int_B S(p)(v_0^2+2tv_0\psi)dV \\
&+ \frac{(n+1)^2}{4n(n-1)}c\int_B (v_0^{2-\frac{4}{n+1}} + \frac{2(n-1)}{n+1}tv_0^{1-\frac{4}{n+1}}\psi)dV + O(t^2)] \\
&\times [\int_B (v_0^2 + 2tv_0\psi + O(t^2))dV]^{-1} \\
&= \bar{c} + t[2\int_B \nabla v_0\nabla\psi)dV + \frac{2(n+1)}{4n}\int_B S(p)v_0\psi dV + \frac{2c(n+1)}{4n}\int_B v_0^{1-\frac{4}{n+1}}\psi dV \\
&- 2\bar{c}\int_B v_0\psi dV] + O(t^2)
\end{aligned}$$

From that $\frac{dJ(\varphi)}{dt}|_{t=0} = 0$, $v_0$ satisfies that for all $\psi \in C^\infty(B)$,

$$\int_B \nabla v_0 \nabla\psi + \frac{n+1}{4n}\int_B S(p)v_0\psi dV + \frac{n+1}{4n}\int_B cv_0^{1-\frac{4}{n+1}}\psi dV - \bar{c}\int_B v_0\psi = 0.$$

Putting $\bar{c} = \frac{n+1}{4n}k$, we have

$$\Delta v_0 + \frac{n+1}{4n}(k - S_B(p))v_0 - \frac{n+1}{4n}cv_0^{1-\frac{4}{n+1}} = 0.$$

Step 4. By the elliptic regularity theory, $v_0$ is a $C^2$ solution. For each $c \le 0$,

$$\Delta v_0 + \frac{n+1}{4n}(k - S_B(p))v_0 = \frac{n+1}{4n}cv_0^{1-\frac{4}{n+1}} \le 0.$$

Hence , by the maximum principle, $v_0$ cannot attain zero, which is our desired solution. (cf. [1. Prop.3.75]and the sign of Laplacian )

□

REMARK 2.7. *In Theorem 2.6, we can prove that for the case $c > 0$ there exists a solution $v_0 \ge 0$ satisfying our equation. But we cannot prove that such $v_0$ is positive.*

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Yoon-Tae Jung
Department of mathematics,
Chosun University,
Kwangju, 501-759, Korea
*E-mail*: ytajung@chosun.ac.kr

EunHee Choi
Department of mathematics,
Chosun University,
Kwangju, 501-759, Korea

Sang Chul Lee
Department of mathematics,
Chosun University,
Kwangju, 501-759, Korea
*E-mail*: l2372@paran.com