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A GENERALIZATION OF THE SCHUR-COHN TEST

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ABSTRACT. In this note we provide a generalization of the Schur-Cohn test. If p_n be a polynomial of degree n then p_n is a reduced-stable polynomial whose reduced degree is equal to r if and only if $D_n := P_n Q_n^{-1}$ is a contraction such that $\text{rank}\,(I - D_n D_n^*) = r$.

Consider first the following interpolation problem, called the *Carathéodory-Schur interpolation problem* (CSIP). Given $c_0, c_1, \cdots, c_{N-1}$ in $\mathbb{C}$, find an analytic function k on the open unit disc $\mathbb{D}$ such that

(i) $\widehat{k}(j) = c_j, \quad j = 0, \cdots, N-1 \quad$ ($\widehat{k}(j)$ denotes the j-th Fourier coefficient of k)

and

(ii) $\sup_{z\in\mathbb{D}} |k(z)| \le 1$.

CSIP can be analyzed by a matricial argument (cf. [4]): CSIP is solvable if and only if the Toeplitz matrix

$$C := \begin{pmatrix} c_0 & 0 & 0 & \dots & 0 \\ c_1 & c_0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_{N-2} & & \ddots & \ddots & 0 \\ c_{N-1} & c_{N-2} & \dots & c_1 & c_0 \end{pmatrix}$$

is a contraction, that is, $||C|| \le 1$, or equivalently, $I - CC^* \ge 0$. Today this result is also called the Carathéodory–Fejér Theorem. A proof of this result can be obtained by means of the Commutant Lifting Theorem (cf. [3, Proposition XXVII.7.2]). In particular, by Pick's Theorem (cf. [2, Corollary I.2.3]) CSIP has a unique solution k if and only if $\det\,(I - CC^*) = 0$. Moreover, in the cases

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where $\det(I - CC^*) = 0$, the unique solution is a finite Blaschke product k of the form

$$k(z) = e^{i\theta} \prod_{j=1}^{n} \frac{z - \zeta_j}{1 - \overline{\zeta_j} z} \quad (|\zeta_j| < 1 \text{ for } j = 1, \cdots, n;\ \theta \in [0, 2\pi)),$$

such that $\deg(k) = \operatorname{rank}(I - CC^*)$, where $\deg(k)$ denotes the *degree* of k, that is, the number of zeros of k in the open unit disc $\mathbb{D}$ (see also [5, Theorem]).

As an application of the preceding result, we will derive a generalization of the Schur-Cohn test.

A polynomial p is called *stable* if all the zeros of p are in the open unit disk $\mathbb{D}$, and is called *reduced-stable* if for every zero ζ of p such that $|\zeta| > 1$, the number $1/\overline{\zeta}$ is a zero of p in the open unit disk $\mathbb{D}$ of multiplicity greater than or equal to the multiplicity of ζ. If p is reduced-stable, we define the *reduced degree* of p as the integer $Z_{\mathbb{D}} - Z_{\mathbb{C}\setminus\overline{\mathbb{D}}}$, where $Z_{\mathbb{D}}$ and $Z_{\mathbb{C}\setminus\overline{\mathbb{D}}}$ are the number of zeros of p in $\mathbb{D}$ and in $\mathbb{C} \setminus \overline{\mathbb{D}}$ counting multiplicity. Thus if p is a stable polynomial of degree n then evidently the reduced degree of p is n. Let p_n denote the polynomial $p_n(z) = c_0 + c_1 z + \cdots c_n z^n$. The $n \times n$ Toeplitz matrices P_n and Q_n are defined by

$$P_n = \begin{pmatrix} c_0 & 0 & 0 & \dots & 0 \\ c_1 & c_0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & 0 \\ c_{n-1} & c_{n-2} & \dots & c_1 & c_0 \end{pmatrix} \quad \text{and} \quad Q_n = \begin{pmatrix} \overline{c_n} & 0 & 0 & \dots & 0 \\ \overline{c_{n-1}} & \overline{c_n} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \overline{c_2} & & \ddots & \ddots & 0 \\ \overline{c_1} & \overline{c_2} & \dots & \overline{c_{n-1}} & \overline{c_n} \end{pmatrix}.$$

Then the *Schur-Cohn test* for the stability of a polynomial states that if p_n is a polynomial of degree n then the following statements are equivalent ([1]).

1. p_n is a stable polynomial.
2. $P_n Q_n^{-1}$ is a strict contraction.

The following theorem is a generalization of the Schur-Cohn test.

THEOREM 1. *Let p_n be a polynomial of degree n . Then the following statements are equivalent.*

1. *p_n is a reduced-stable polynomial whose reduced degree is equal to r.*
2. *$\frac{p_n}{z^n \overline{p_n}}$ is a finite Blaschke product of degree r.*
3. *$D_n := P_n Q_n^{-1}$ is a contraction such that* $\operatorname{rank}(I - D_n D_n^*) = r$.

Proof. (1) $\Leftrightarrow$ (2): If $p_n = \alpha\prod_{j=1}^n (z-\zeta_j)$ then $z^n\overline{p_n} = \overline{\alpha}\prod_{j=1}^n (1-\overline{\zeta_j}z)$. Then $\frac{p_n}{z^n\overline{p_n}}$ is a finite Blaschke product of degree r if and only if for every zero ζ of p_n such that $|\zeta| > 1$, the number $1/\overline{\zeta}$ is a zero of p_n in the open unit disk $\mathbb{D}$ of multiplicity greater than or equal to the multiplicity of ζ and $r = Z_{\mathbb{D}} - Z_{\mathbb{C}\setminus\overline{\mathbb{D}}}$ if and only if p_n is a reduced-stable polynomial of reduced degree r.

(2) $\Leftrightarrow$ (3): Let $p_n = c_0 + c_1 z + \cdots c_n z^n$ $(c_n \neq 0)$. Then $z^n\overline{p_n} = \overline{c_n} + \overline{c_{n-1}}z + \cdots \overline{c_0}z^n$ and so $\frac{p_n}{z^n\overline{p_n}}$ is analytic in a neighborhood of 0. Let A be the $n\times n$ matrix given by

$$\begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ 1 & 0 & & & \vdots \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

Using the usual functional calculus, we have that $p_n(A) = P_n$ and $(z^n\overline{p_n})(A) = Q_n$. In particular, since A is nilpotent, the expression $\left(\frac{p_n}{z^n\overline{p_n}}\right)(A)$ makes sense. Note that if $\frac{p_n}{z^n\overline{p_n}} = \sum_{j=0}^\infty d_j z^j$ then

$$\left(\frac{p_n}{z^n\overline{p_n}}\right)(A) = \begin{pmatrix} d_0 & 0 & 0 & \dots & 0 \\ d_1 & d_0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ d_{n-2} & & \ddots & \ddots & 0 \\ d_{n-1} & d_{n-2} & \dots & d_1 & d_0 \end{pmatrix}.$$

Thus by CSIP, $D_n := \left(\frac{p_n}{z^n\overline{p_n}}\right)(A)$ is a contraction if and only if $\frac{p_n}{z^n\overline{p_n}}$ is analytic on $\mathbb{D}$ if and only if $\frac{p_n}{z^n\overline{p_n}}$ is a finite Blaschke product. By the functional calculus we see that $\left(\frac{p_n}{z^n\overline{p_n}}\right)(A) = P_nQ_n^{-1}$. The argument for degree comes from an argument of S. Takahashi [5, Theorem], which states that if $I - CC^* \geq 0$ then there exists a finite Blaschke product whose degree is equal to the rank of $I - CC^*$. □

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