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# PRIME IDEALS OF RINGS GRADED BY A COMMUTATIVE SEMIGROUP WHICH IS A NILPOTENT EXTENSION OF A GROUP

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**Abstract.** We study the prime ideal structure of the direct sum of directed system of rings indexed by a commutative semigroup which is nilpotent extension of a group.

Throughout this paper $S$ will be the commutative semigroup with 1 idempotent element, for each $s \in S, s^m = s^{n+m}$ for some $n, m \geq 1$.

Let $m(s)$ be the least positive integer such that $s^{m(s)} = s^{m(s)+n}$ for some $n$.

Let $t(s)$ be the least positive integer $n(s)$ such that $s^{m(s)} = s^{m(s)+n(s)}$.

Let $G_s = \{s^{m(s)}, s^{m(s)+1}, \cdots, s^{m(s)+t(s)-1}\}$ then each $G_s$ is a group with identity element $f = f^2$.

We denote $G = \cup_{s \in S} G_s$. At first we start with elementary properties related with $S$ and $G$.

**Lemma 1.** *$G$ is a subsemigroup of $S$, and moreover $G$ is a group.*

*Proof.* Let $\gamma, \delta \in G$, then $\gamma = \gamma^{1+t(\gamma)}$ and $\delta = \delta^{1+t(\delta)}$. Hence $(\gamma\delta)^{1+t(\gamma)t(\delta)} = \gamma^{1+t(\gamma)t(\delta)}\delta^{1+t(\gamma)t(\delta)} = \gamma\delta$, so that $\gamma\delta \in G_{\gamma\delta}$. Thus $G$ is a subsemigroup of $S$. Since $f$ is an identity element for each $G_s$, then $G = \cup_{s \in S} G_s$ has $f$ as an identity element. Also for each $\alpha \in G, \alpha \in G_s$ for some $s$. We have $\alpha^n = f$ for some $n$. Thus $\alpha^{-1} = \alpha^{n-1} \in G_s \subseteq G$. □

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We consider direct sum of the directed system of rings indexed by $S$. i.e., $R = \sum_{s\in S} R_s$ be a ring strongly graded by $S$ in the sense that there exist ring homomorphisms $\varphi_{s,t} : R_s \longrightarrow R_t$ such that (1) $\varphi_{s,s} = 1$ (2) $\varphi_{t,u}\varphi_{s,t} = \varphi_{s,u}$ for $s \leq t \leq u$ (3) $r_s \cdot r_t = \varphi_{s,st}(r_s)\varphi_{t,st}(r_t)$ where $\leq$ will denote the ideal order on $S$, defined by $s \leq t$ if $t$ is in the ideal $(s)$ of $S$ generated by $s$. A system of rings over $(S, \leq)$ is a collection $(R_s)_{s\in S}$ of rings, together with ring homomorphisms.

**Lemma 2.** *(1) If $g, h \in G$, then $\varphi_{g,h}$ is an isomorphism.*
*(2) $G$ is an ideal of $S$.*

*Proof.* (1) For each $g, h \in G$, since $G$ is a group $g \leq h$ and $h \leq g$. Thus $\varphi_{g,h}\varphi_{h,g} = 1_{R_g}$ and $\varphi_{h,g}\varphi_{g,h} = 1_{R_h}$, which implies that $\varphi_{g,h}$ is one to one, onto map.

(2) Let $\alpha \in S, \gamma \in G$. Then $\alpha^n = f$ for some $n \geq m(\alpha)$ and $\gamma^k = f$ for some $k$. Now $(\alpha\gamma)\gamma^{k-1} = \alpha\gamma^k = \alpha f = \alpha\alpha^n = \alpha^{n+1} \in G$. Therefore $\alpha \cdot \gamma = \alpha(\gamma f) = \alpha(\gamma\gamma^k) = (\alpha\gamma \cdot \gamma^{k-1})\gamma \in G$, since $G$ is closed under multiplication. Similarly, $\gamma\alpha \in G$. □

Using Lemma 2(2), we have an ideal in $R$.

**Lemma 3.** *$I = \sum_{g\in G} R_g$ is an ideal of $\sum_{s\in S} R_s = R$.*

*Proof.* $R_s R_g \subseteq R_{sg}$ and $R_g R_s \subseteq R_{gs}$ and apply Lemma 2(2), we have the result. □

The following Lemma shows that for a given ideal of $R = \sum_{s\in S} R_s$ or $R_g$ for $g \in G$, we correspond new ideal in $R$.

**Lemma 4.** *(1) If $A$ is an ideal of $R = \sum_{s\in S} R_s$, then $\sum_{g\in G}(A \cap R_g)$ is an ideal of $R$.*

*(2) If $A$ is an ideal of $R_g$ for some $g \in G$, then $\sum_{h\in G} \varphi_{g,h}(A)$ is an ideal of $R$.*

*Proof.* (1) Let $r \in R_s$ and $a \in A \cap R_h$ for some $h \in G$. Then $r \cdot a = \varphi_{s,sh}(r)\varphi_{g,sh}(a) \in A \cap R_{sh} \subseteq \sum(A \cap R_g)$.

(2) Let $s \in S, r_s \in R_s$ and $a \in A$, then

$$\begin{aligned} r_s \cdot \varphi_{g,h}(a) &= \varphi_{s,sh}(r_s)\varphi_{h,sh}(\varphi_{g,h}(a)) \\ &= \varphi_{s,sh}(r_s)\varphi_{g,sh}(a). \end{aligned}$$

But $\varphi_{g,sh}$ is an isomorphism (by Lemma 2). Hence there exists $r \in R_g$ such that $\varphi_{g,sh}(r) = \varphi_{s,sh}(r_s)$. Thus

$$r_s \cdot \varphi_{g,h}(a) = \varphi_{g,sh}(r)\varphi_{g,sh}(a) = \varphi_{g,sh}(ra) \in \varphi_{g,sh}(A).$$

Similarly, $\varphi_{g,h}(a)r_s \in \varphi_{g,hs}(A)$. ($sh, hs \in G$ by Lemma 2(2)) □

In order to describe prime ideal of this ring $R$, we try the following strategy; at first for the given prime ideal $P$ of $R$ we try to explain some properties of the set of ideals $\{P \cap R_\alpha \mid \alpha \in S\}$ in $R$.

**Lemma 5.** *(1) Let $P$ be an ideal of $R = \sum_{s\in S} R$, then $P \cap R_g$ is an ideal of $R_g$ for each $g \in G$.*

*(2) Let $P$ be an ideal of $R = \sum_{s\in S} R$, then $\sum_{\gamma\in G} \varphi_{\gamma,g}(P \cap R_g)$ is an ideal of $R$.*

*Proof.* (1) Let $r_g \in R_g, p \in P \cap R - g$. by Lemma 2(1), $\varphi_{f,g}$ is an isomorphism for each $f, g \in G$, there exists $r_f \in R_f$ such that $\varphi_{f,g}(r_f) = r_g$. Hence

$$\begin{aligned} r_g \cdot p \text{ (multiplication in } R_g) &= \varphi_{f,g}(r_f)\varphi_{g,g}(p) \\ &= \varphi_{f,fg}(r_f)\varphi_{g,fg}(p) \\ &= r_f \cdot p \text{ (multiplication in } R) \\ &\in P \text{ (since } P \text{ is an ideal of } R). \end{aligned}$$

Similarly $p \cdot r_g \in P$.

(2) Apply the result of (1) to Lemma 4(2). □

The following Lemmas show us how each prime ideal of $R = \sum_{s\in S} R_s$ impact in $R_g$ for each $g \in G$.

**Lemma 6.** *Let $P$ be a prime ideal of $R = \sum_{s\in S} R_s$. Then $\sum_{g\in G} \varphi_{\gamma,g}(P\cap R_\gamma) \subseteq P$ for any $\gamma \in G$.*

*Proof.* Let $I = \sum_{g\in G} R_g$, and consider terms of the product :

$I \cdot (\sum_{g\in G} \varphi_{\gamma,g}(P\cap R_\gamma))$, for $h \in G, R_h \cdot \varphi_{\gamma,g}(P\cap R_\gamma)$ is typical, at first we want to show that $R_h \cdot \varphi_{\gamma,g}(P\cap R_\gamma) \subseteq P$.

$$\begin{aligned} R_h \cdot \varphi_{\gamma,g}(P\cap R_\gamma) &= \varphi_{h,hg}(r_h)\varphi_{g,hg}(\varphi_{\gamma,g}(P\cap R_\gamma)) \\ &= \varphi_{h,hg}(R_h)\varphi_{\gamma,hg}(P\cap R_\gamma) \\ &= R_{hg}\cdot \varphi_{\varphi,hg}(P\cap R_\gamma) \\ &= \varphi_{hg\gamma^{-1},hg\gamma^{-1}\gamma}(R_{hg\gamma^{-1}\gamma})\cdot \varphi_{\gamma,hg\gamma^{-1}\gamma}(P\cap R_\gamma) \\ &= R_{hg\gamma^{-1}}(P\cap R_\gamma) \subseteq P. \end{aligned}$$

Consequently, $I \cdot \sum_{g\in G} \varphi_{\gamma,g}(P\cap R_\gamma) \subseteq P$.

Since $I$ is an ideal of $R$ (by Lemma 3) and $\sum_{g\in G} \varphi_{\gamma,g}(P\cap R_\gamma)$ is an ideal of $R$ (by Lemma 5(2)). Since $P$ is a prime ideal of $R$, we have either $I \subseteq P$ or $\sum_{g\in G} \varphi_{\gamma,g}(P\cap R_\gamma) \subseteq P$ by the primeness of $P$. So in either case we have $\sum_{g\in G} \varphi_{\gamma,g}(P\cap R_\gamma) \subseteq P$ for any $\gamma\in G$. For $I = \sum_{g\in G} R_g \subseteq P$, $R_g = \varphi_{\gamma,g}(R_\gamma) \supseteq \varphi_{\gamma,g}(R_\gamma\cap P)$. Thus $I = \sum_{g\in G} R_g \supseteq \sum_{g\in G} \varphi_{\gamma,g}(R_\gamma \cap P)$. □

**Lemma 7.** *If $P$ is a prime ideal of $R$ then $P\cap R_g$ is a prime ideal of $R_g$ for each $g\in G$.*

*Proof.* By Lemma 5(1), $P\cap R_g$ is an ideal of $R_g$, for each $g \in G$. Let $A$ and $B$ be ideals of $R_g$ such that $AB \subseteq P\cap R_g$. We want to show that $A \subseteq P\cap R_g$ or $B\subseteq P\cap R_g$. □

Consider $(RA+A)(RB+B) = RARB+RAB+ARB+AB$. Since $P$ is prime, it is sufficient to show that this product is in $P$, for $RA+A\subseteq P$ implies $A\subseteq P\cap R_g$. Also note that $RA+A$ and $RB+B$ are left ideals of $R$. We check that each term of the product is in $P$ :

(i) $RARB \subseteq P$ :

Let $s, t \in S$,

$$\begin{aligned}
R_sAR_tB &= \varphi_{s,sgtg}(R_s)\varphi g, sgtg(A)\varphi_{t,sgtg}(R_t)\varphi_{g,sgtg}(B) \\
&\subseteq \varphi_{sgtg}\varphi_{g,sgtg}(A)R_{sgtg}\varphi_{g,sgtg}(B) \\
&= \varphi_{g,sgtg}(R_g)\varphi_{g,sgtg}(A)\varphi_{g,sgtg}(R_g)\varphi_{g,sgtg}(B) \quad \text{(by Lemma 2)} \\
&= \varphi_{g,sgtg}(R_gAR_gB) \subseteq \varphi_{g,sgtg}(AB) \subseteq \varphi_{g,sgtg}(P \cap R_g) \\
&\subseteq P \quad \text{(by Lemma 6)}
\end{aligned}$$

(ii)

$$\begin{aligned}
R_sAB &= \varphi_{s,sgg}(R_s)\varphi_{g,sgg}(A)\varphi_{g,sgg}(B) \\
&\subseteq \varphi_{sg^2}\varphi_{g,sg^2}(AB) \\
&= \varphi_{g,sg^2}(R_g)\varphi_{g,sg^2}(AB) \\
&= \varphi_{g,sg^2}(R_gAB) \quad \text{(by Lemma 2)} \\
&\subseteq \varphi_{g,sg^2}(AB) \\
&\subseteq \varphi_{g,sg^2}(P \cap R_g) \\
&\subseteq P. \quad \text{(by Lemma 6)}
\end{aligned}$$

(iii)

$$\begin{aligned}
AR_sB &= \varphi_{g,gsg}(A)\varphi_{s,gsg}(R_s)\varphi_{g,gsg}(B) \\
&\subseteq \varphi_{g,gsg}(A)R_{gsg}\varphi_{g,gsg}(B) \\
&= \varphi_{g,gsg}(R_g)\varphi_{g,sg^2}(AB) \\
&= \varphi_{g,sg^2}(R_gAB) \quad \text{(by Lemma 2)} \\
&\subseteq \varphi_{g,sg^2}(AB) \\
&\subseteq \varphi_{g,sg^2}(P \cap R_g) \\
&\subseteq P. \quad \text{(by Lemma 6)}
\end{aligned}$$

(iv) $AB \subseteq P \cap R_g \subseteq P$.

**Lemma 8.** *Let $P$ be a prime ideal of $R$. Then for each $\gamma, g \in G, \varphi_{\gamma,g}(P \cap R_\gamma) = P \cap R_g$.*

*Proof.* By Lemma 6, $\varphi_{g,\gamma}(P \cap R_g) \subseteq P \cap R_\gamma$. Hence

$$\begin{aligned} P \cap R_g &= \varphi_{g,g}(P \cap R_g) \\ &= \varphi_{\gamma,g}\varphi_{g,\gamma}(P \cap R_g) \\ &\subseteq \varphi_{\gamma,g}(P \cap R_\gamma) \\ &\subseteq P \cap R_g. \quad \text{(by Lemma 5)} \end{aligned}$$

This forces $\varphi_{\gamma,g}(P \cap R_\gamma) = P \cap R_g$. □

Next lemma establishes that we have one more different type ideal in $R = \sum_{s \in S} R_s$. (compare to Lemma 3)

**Lemma 9.** $J = \sum_{s \leq t} \varphi_{s,t}(R_s)$ *is an ideal of $R$.*

*Proof.* $J$ is closed under $+$ and $-$ by the definition. We have to show that for $r_u \in R_u$, $r_u \cdot \varphi_{s,t}(r_s) \in J$ and $\varphi_{s,t}(r_s) \cdot r_u \in J$.

$$\begin{aligned} r_u \cdot \varphi_{s,t}(r_s) &= \varphi_{u,ut}(r_u)\varphi_{t,ut}(\varphi_{s,t}(r_s)) \\ &= \varphi_{u,ut}(r_u)\varphi_{s,ut}(r_s) \\ &= (\varphi_{us,ut}\varphi_{u,us}(r_u))(\varphi_{us,ut}\varphi_{s,us}(r_s)) \\ &= \varphi_{us,ut}(\varphi_{u,us}(r_u)\varphi_{us,ut}(r_s)) \\ &\in \varphi_{us,ut}(R_{us}). \end{aligned}$$

By symmetry $\varphi_{s,t}(r_s) \cdot r_u \in \varphi_{su,tu}(R_{su})$. □

For the given prime ideal $P$ of $R = \sum_{s \in S} R_s$, the following Lemmas show as how the ideal $P$ acts in $P \cap R_\alpha, \varphi_{s,t}(P \cap R)$ or $\varphi^{-1}_{\alpha,\beta}(P \cap R)$.

**Lemma 10.** *Let $P$ be a prime ideal of $R$. Then $\sum_{s \leq t} \varphi_{s,t}(P \cap R_s) \subseteq P$, where $s, t \in S$.*

*Proof.* For the convenience we denote $K = \sum_{s \le t} \varphi_{s,T}(P \cap R_s)$. Consider the left ideal $L = RK + K$. Then $L^2 = RKRK + RKK + KRK + KK$ multiplying these terms, every term contains a factor of the form : either $R_u \cdot \varphi_{s,t}(P \cap R_s)$ or $\varphi_{r,u}(P \cap R_r) \cdot \varphi_{s,t}(P \cap R_s)$.

Since $\varphi_{s,t}(P \cap R_r) \cdot \varphi_{s,t}(P \cap R_s) \subseteq R_u \cdot \varphi_{s,t}(P \cap R_s)$, it is sufficient to show that $R_u \varphi_{s,t}(P \cap R_s) \subseteq P$ for all $u, s, t$, i.e., in order to get $L^2 \subseteq P$. Then $P$ is prime implies that $L \subseteq P$. But

$$\begin{aligned}
R_u \cdot \varphi_{s,t}(P \cap R_s) &= \varphi_{u,ut}(R_u)(\varphi_{t,ut}\varphi_{s,t}(P \cap R_s)) \\
&= \varphi_{u,ut}(R_u)\varphi_{s,ut}(P \cap R_s) \\
&= \varphi_{uw,ut}\varphi_{u,uw}(R_u)\varphi_{s,ut}(P \cap R_s)) \\
&= \varphi_{uw}(R_u) \cdot (P \cap R_s) \\
&= \varphi_{uw}(R_u) \cdot (P \cap R_s) \quad \text{(multiplication in } R). \\
&\subseteq P.
\end{aligned}$$

□

**Lemma 11.** *If $P$ is a prime ideal of $R$. $\alpha \le \beta \le \gamma$, then*

$$\varphi^{-1}_{\alpha,\gamma}(P \cap R_\gamma) \supseteq \varphi^{-1}_{\alpha,\beta}(P \cap R_\beta).$$

*Proof.* Let $x \in \varphi^{-1}_{\alpha,\beta}(P \cap R_\beta)$. Then $\varphi_{\alpha,\beta}(x) \in P \cap R_\gamma$. So by Lemma 10, we have $\varphi_{\alpha,\gamma}(x) = \varphi_{\beta,\gamma}\varphi_{\alpha,\beta}(x) \subseteq \varphi_{\beta,\gamma}(P \cap R_\beta) \subseteq P \cap R_\gamma$. Thus we have $x \in \varphi^{-1}_{\alpha,\gamma}(P \cap R)$. □

**Lemma 12.** *Let $P$ be a prime ideal of $R$. For any $\alpha \le \beta, \varphi^{-1}_{\alpha,\beta}(R_\beta \cap P) \subseteq P \cap R_\alpha$.*

*Proof.* Write $\beta^m = f, \alpha^n = f$ for some $m, n$. Then $\alpha \le \beta \le f$ ; so Lemma 10 says that it is sufficient to show that $\varphi^{-1}_{\alpha,f}(R_f \cap P) \subseteq P$. To avoid trivialities, we may assume that $\alpha \ne f$ and hence $m > 1$. Let $\sigma \in (\alpha^{m-1})$. Consider the product of left ideals : $(RA + A)(RR_\sigma + R_\sigma)$, where $A = \varphi^{-1}_{\alpha,f}(R_f \cap P)$. After multiplying out the product, we see that it is sufficient to show $AR_tR_\sigma \subseteq P$ and $AR_\sigma \subseteq P$ for $(RA + A)(RR_\sigma +$

$R_\sigma) \subseteq P$.

(1)

$$\begin{aligned} A \cdot R_\sigma &= \varphi_{\alpha,\alpha\sigma}(A)\varphi_{\sigma,\alpha\sigma}(R_\sigma) \\ &= \varphi_{\alpha,f\rho}(A)\varphi_{\sigma,f\rho}(R_\sigma) \quad \text{for some} \rho \\ &= \varphi_{f,f\rho}(A)\varphi_{\alpha,f}(A)\varphi_{\sigma,f\rho}R_\sigma) \\ &\subseteq \varphi_{f,f\rho}(R_f \cap P)R_{f\rho}. \quad \text{(by Lemma 7)} \end{aligned}$$

Hence $A \cdot R_\sigma \subseteq (R_f \cap P) \cdot R_{f\rho} \subseteq P$ (by $P$ is an ideal of $R$).

(2) $A \cdot R_t \cdot R_\sigma = \varphi_{\alpha,\alpha t\sigma}(A)\varphi_{t,\alpha t\sigma}(R_t)\varphi_{\sigma,\alpha t\sigma}(R_\sigma)$. Note that

(a) $\alpha^m = f \Longrightarrow \alpha \leq f$

(b) $t^k = f \Longrightarrow t \leq f$

(c) $\sigma^j = f \Longrightarrow \sigma \leq f$ ;

(d) $\sigma = \alpha^{m-1}\rho \Longrightarrow \alpha^{m-1}\sigma$ ; also $\alpha \leq \alpha t$ ; $\Longrightarrow f = \alpha \cdot \alpha^{m-1} \leq \alpha t\sigma$.

Thus

$$\begin{aligned} A \cdot R_t \cdot R_\sigma &= [\varphi_{f,\alpha t\sigma}\varphi_{\alpha,t}(A)][\varphi_{f,\alpha t\sigma}\varphi_{t,f}(R_t)][\varphi_{f,\alpha t\sigma}\varphi_{\sigma,f}(R_\sigma)] \\ &= \varphi_{f,\alpha t\sigma}[\varphi_{\alpha,f}(A)\varphi_{t,f}(R_t)\varphi_{\sigma,f}(R_\sigma)]. \end{aligned}$$

But $\varphi_{\alpha,f}(A) \subseteq R_f \cap P$, so that $\varphi_{\alpha,f}(A)\varphi_{t,f}(R_t)\varphi_{\sigma,f}(R_\sigma) \subseteq P$ by Lemma 6. Hence

$$\begin{aligned} A \cdot R_t \cdot R_\sigma &= \varphi_{f,\alpha t\sigma}[\varphi_{\alpha,f}(A)\varphi_{t,f}(R_f)\varphi_{\alpha,f}(R_\sigma)] \\ &\subseteq P \quad \text{(by Lemma 5).} \end{aligned}$$

We now have $(RA + A)(RR_\sigma + R_\sigma) \subseteq P$. Since $P$ is prime, either $A \subseteq RA + A \subseteq P$ or $R_\sigma \subseteq RR_\sigma + R_\sigma \subseteq P$ for all $\sigma \in (\alpha^{m-1})$. If $A \subseteq P$, we are done. Thus we may suppose that $R_\sigma \subseteq P$, where $\sigma \in (\alpha^{m-1})$. Let $k$ be the least positive integer such that $R_\sigma \subseteq P$ for each $\sigma \in (\alpha^k)$. We may assume that such a $k$ exists by our supposition about $\alpha^{m-1}$. If $k = 1$, then $A \subseteq R_\sigma \subseteq P$ and we are done. So we assume $k > 1$. Consider the product of left ideals of $R$ for $\nu \in (\alpha^{k-1})$ : $(RA + A)(RR_\nu + R_\nu)$. Writing out, the product, we have

$$RARR_\nu + RAR_\nu + ARR_\nu + AR_\nu.$$

To get each of these terms in $P$, it is sufficient to show that $AR_\nu \subseteq P$ and $AR_tR_\nu \subseteq P$ for each $t \in S$.

(1) Since $\alpha^{k-1} \leq \nu$, then $\alpha\nu \in (\alpha^k)$ ; so $AR_\nu \subseteq R_{\alpha\nu} \subseteq P$ by our choice of $k$.

(2) Since $\alpha \leq \alpha t$ and $\alpha^{k-1} \leq \nu$, then $\alpha^k = \alpha \cdot \alpha^{k-1} \leq \alpha t\nu$ ; so $\alpha t\nu \in (\alpha^k)$. Hence $AR_tR_\nu \subseteq R_{\alpha t\nu} \subseteq P$ by our choice of $k$.

Therefore, $(RA + A)(RR_\nu + R_\nu) \subseteq P$. Since $P$ is prime either $A \subseteq RA + A \subseteq P$ or else $R_\nu \subseteq RR_\nu + R_\nu \subseteq P$ for each $\nu \in (\alpha^{k-1})$. By our choice of $k$, the latter choice is not a possibility, hence $A \subseteq P$, as desired. □

Following [1], we use the following terminology; a set $\mathcal{P} = \{P_\alpha \mid \alpha \in S\}$ of $R$ is a prime set if

(a) $P_\gamma$ is a prime ideal of $R_\gamma$ for each $\gamma \in G$ ;

(b) $\varphi^{-1}_{\beta,\delta}(P_\delta) = P_\beta$ for each $\beta, \delta \in S$ with $\beta \leq \delta$.

**Proposition 13.** *If $P$ is a prime ideal of $R$, then $\mathcal{P} = \{P \cap R_\alpha \mid \alpha \in S\}$ is a prime set of $R$.*

*Proof.* Condition (a) holds by Lemma 7, and condition (b) holds by Lemma 12. □

The following Lemma is essential to the proof of main Theorem of this paper.

**Lemma 14.** *Let $P$ be a prime ideal of $R$ and let $\alpha \notin G$. Write $\alpha^{m(\alpha)} = \alpha^{m(\alpha)+t(\alpha)}$ for minimal $m(\alpha), t(\alpha) > 0$. Then $r - \varphi_{\alpha,\alpha^{t(\alpha)+1}}(r) \in P$ for each $r \in R_\alpha$.*

*Proof.* $\alpha \notin G$ implies $m(\alpha) > 1$. Consider the product of ideals :

$$< R_{\alpha^{m(\alpha)-1}} >< r - \varphi_{\alpha,\alpha^{t(\alpha)+1}}(r) \mid R_\alpha > .$$

Multiplying out the product, we obtain a sum of terms, each of which has a factor of one of the following forms :

$$a \cdot (r - \varphi_{\alpha,\alpha^{t(\alpha)+1}}(r)) \qquad \text{for } a \in R_{\alpha^{m(\alpha)-1}}, r \in R$$

or

$$a \cdot b \cdot (r - \varphi_{\alpha,\alpha^{t(\alpha)+1}}(r)) \qquad \text{for } b \in R_\sigma, \sigma \in S.$$

Let $\mu = \alpha^{m(\alpha)-1}\sigma\sigma^{t(\alpha)+1} = \sigma\alpha^{m(\alpha)-t(\alpha)} = \sigma\alpha^{m(\alpha)}$. Also $\mu = \alpha^{m(\alpha)-1}\sigma\alpha = \sigma\alpha^{m(\alpha)}$. Then

$$\begin{aligned} & a \cdot (r - \varphi_{\alpha,\alpha^{t(\alpha)+1}}(r)) \\ = {} & \varphi_{\alpha^{m(\alpha)-1},\alpha^{m(\alpha)}}(a)\varphi_{\alpha,\alpha^{m(\alpha)}}(r) - \varphi_{\alpha^{m(\alpha)-1},\alpha^{m(\alpha)}}(a)\varphi_{\alpha^{t(\alpha)+1},\alpha^{m(\alpha)}}\varphi_{\alpha,\alpha^{t(\alpha)+1}}(r) \\ = {} & \varphi_{\alpha^{m(\alpha)-1},\alpha^{m(\alpha)}}(a)\varphi_{\alpha,\alpha^{m(\alpha)}}(r) - \varphi_{\alpha^{m(\alpha)-1},\alpha^{m(\alpha)}}\varphi_{\alpha,\alpha^{m(\alpha)}}(r) \\ = {} & 0 \end{aligned}$$

and

$$\begin{aligned} & a \cdot b \cdot (r - \varphi_{\alpha,\alpha^{t(\alpha)+1}}(r)) \\ = {} & \varphi_{\alpha^{m(\alpha)-1},\mu}(a)\varphi_{\sigma,\mu}(b)\varphi_{\alpha,\mu}(r) - \varphi_{\alpha^{m(\alpha)-1},\mu}(a)\varphi_{\sigma,\mu}(b)\varphi_{\alpha^{t(\alpha)+1},\mu}\varphi_{\alpha,\alpha^{t(\alpha)+1}}(r) \\ = {} & \varphi_{\alpha^{m(\alpha)-1},\mu}(a)\varphi_{\sigma,\mu}(b)\varphi_{\alpha,\mu}(r) - \varphi_{\alpha^{m(\alpha)-1},\mu}(a)\varphi_{\sigma,\mu}(b)\varphi_{\alpha,\mu}(r) \\ = {} & 0 \end{aligned}$$

Hence $< R_{\alpha^{m(\alpha)-1}} >< r - \varphi_{\alpha,\alpha^{t(\alpha)+1}}(r) \,|\, r \in R_\alpha >= 0 \subseteq P$. Since $P$ is prime, either $R_{\alpha^{m(\alpha)-1}} \subseteq P$ or else $r - \varphi_{\alpha,\alpha^{t(\alpha)+1}}(r) \in P$ for all $r \in R_\alpha$. In the latter case, we are done ; so suppose that $R_{\alpha^{m(\alpha)-1}} \subseteq P$. Then Lemma 12 gives

$$\begin{aligned} R_\alpha &= \varphi^{-1}_{\alpha,\alpha^{m(\alpha)-1}}(R_{\alpha^{m(\alpha)-1}}) \\ &= \varphi^{-1}_{\alpha,\alpha^{m(\alpha)-1}}(P \cap R_{\alpha^{m(\alpha)-1}}) \\ &\subseteq P. \end{aligned}$$

Thus Lemma 10 gives $\varphi_{\alpha,\alpha^{t(\alpha)-1}} \in P$ for each$r \in R_\alpha$. Hence $r - \varphi_{\alpha,\alpha^{t(\alpha)+1}}(r) \in P$ for each $r \in R_\alpha$, as desired. □

Let $I' =< r - \varphi_{\alpha,\alpha^{t(\alpha)+1}}(r) >$ if $S - G \neq 0$ ; $I' = 0$ if $S - G = 0$. By Lemma 14, $I' \subseteq P$ for every prime ideal $P$ of $R$. Thus, by the Correspondence Theorem for Homomorphisms, to find the prime ideals of $R$, we only need to find the prime ideals of $R/I'$. And the following proposition describe $R/I'$ related with $I = \sum_{g\in G} R_g$.

**Proposition 15.** $R/I' \cong (\sum_{g\in G} R_g)/(I' \cap \sum_{g\in G} R_g)$.

*Proof.* Let $g \notin G$ and $r \in R_\gamma$. We have $r + I' = \varphi_{g,g^{t(g)+1}}(r) + I'$. Since $t(g) \geq 1$, then $\varphi_{g,g^{t(g)+1}}(r) \in R_{g^{t(g)+1}} \neq R_g$. If $g^{t(g)+1} \in G$, done. If $g^{t(g)+1} \notin G$, let $\mu = t(g^{t(g)+1}) = g^k$ for some $k > t(g) + 1$. Then

$$\begin{aligned} r + I' &= \varphi_{g,g^{t(g)+1}}(r) + I' \\ &= \varphi_{g^{t(g)+1},(g^{t(g)+1})^{\mu+1}}\varphi_{g,g^{t(g)+1}}(r) \\ &= \varphi_{g,(g^{t(g)+1})^{\mu+1}}(r) + I' \end{aligned}$$

If $(g^{t(g)+1})^{\mu+1} \in G$, done. If not, repeat inductively, since some power of $g$ is in $G$, eventually we have $r + I' = a + I'$ for some $a \in R_g$, $g \in G$. Hence

$$\begin{aligned} R/I' &= (\sum_{s\in S} R_s + I')/I' \\ &= (\sum_{g\in G} R_g + I')/I' \\ &\cong (\sum_{g\in G})/(I' \cap \sum_{g\in G} R_g). \end{aligned}$$

□

**Lemma 16.** *(1)* $I' \cap \sum_{g\in G} R_g$ *is contained in prime radical* $\mathcal{P}(\sum_{g\in G} R_g)$ *of* $I = \sum_{g\in G} R_g$.

*(2) If* $P$ *is a prime ideal of* $R$, *then* $P \cap \sum_{g\in G} R_g$ *is a prime ideal of* $I = \sum_{g\in G} R_g$.

*Proof.* (1) $I'$ is an ideal $R$ that is contained in $\mathcal{P}(R)$ ; so $I'$ is a radical ideal of $R$. Also $I = \sum g \in GR_g$ is an ideal of $R$ by Lemma 3. Hence

$I \cap I'$ is an ideal of $I$. Since the prime radical is hereditary, $I \cap I'$ is a radical ideal of $I$. Thus every prime ideal of $I = \sum_{g \in G} R_g$ contains $I' \cap \sum_{g \in G} R_g$.

(2) Let $A, B$ be ideals of $I$, and let $AB \subseteq P$.

Consider $[(RA + A)(RB + B)]^3$. Writing out the product, we get asum terms of the form :

$$\underbrace{\sim A \sim B \sim}_{\subseteq I \text{ (by Lemma 3)}} \; A \; \underbrace{\sim B \sim A \sim B}_{\subseteq I \text{ (by Lemma 3)}} \subseteq AB \subseteq P.$$

Since $P$ is a prime ideal of $R$, then either $A \subseteq RA + A \subseteq P$ or else $B \subseteq RB + B \subseteq P$. Thus $A \subseteq P \cap I$ or $B \subseteq P \cap I$. □

**Theorem 17.** *$P$ is a prime ideal of $R$ if and only if $P = I' + P'$, where $P'$ is a prime ideal of $I = \sum_{g \in G} R_g$.*

*Proof.* ($\Rightarrow$) Lemma 14 implies that $I' \subseteq P$. Hence $P \supseteq I' + (P \cap I)$. Lemma 16(2) yields $P' = I \cap P$ is a prime ideal of $I$. Conversely, if $p \in P$, then $p + I' = p' + I'$ for some $p' \in I \cap P$ by our definition of $I'$ ; so $p = p' + x$ for some $x \in I'$. Thus $P \subseteq I' + P'$.

($\Leftarrow$) Claim 1 : If $P'$ is a prime ideal of $I$, then $P'$ is an ideal of $R$. Let $J = RP' + P'R + RP'R + P'$, which is a sum of ideals of $I$ by Lemma 4. Consider the ideal $J^3$ of $R$.

Writing out the product for $J^3$, we have a sum of terms of the form :

$$\underbrace{\sim P' \sim}_{\subseteq I \text{ (by Lemma 3)}} \quad P' \quad \underbrace{\sim P' \sim}_{\subseteq I \text{ (by Lemma 3)}} \quad \subseteq \; IP'I \; \subseteq \; P'.$$

Thus $J^3 \subseteq P'$. Since $J$ is an ideal of $I$ and $P'$ is a prime ideal of $I$, then we must have

$$RP' + P'R + RP'R + P' = J \subseteq P'.$$

Therefore, $P'$ is an ideal of $R$.

Claim 2 : $I' + P'$ is an ideal of $R$. This is clear by Lemma 3 and Claim 1, as a sum of two ideals is an ideal.

Claim 3 : $I'+P'$ is a prime ideal of $R$. By Lemma 15, $P' \supseteq I \cap I'$. By the Correspondence Theorem for Homomorphisms, $[P'+(I \cap I')]/(I \cap I')$ is a prime ideal of $I/(I \cap I')$. So under the natural isomorphism $I/(I \cap I') \cong (I+I')/I'$, we have $(P'+I')/I'$ is a prime ideal of $(I+I')/I'$. But $(I+I')/I' = R/I'$ by the definition of $I'$ ; hence $(P'+I')/I'$ is a prime ideal of $R/I'$. By the Correspondence Theorem for Homomorphisms, $P'+I'$ must be a prime ideal of $R$. □

## References


[1] A. D. Bell, S. Stalder and M. L. Teply, *Prime ideals and radicals in semigroup-graded rings*, Proc. Edinburgh Math. Soc. **39**(1996), 1-25.

[2] J. M. Howie, *An introduction to semigroup theory*, London Math. Soc. Monographs, Academic Press, London (1976).

[3] T. W. Hungerford, *Algebra, Graduate Texts in Math*, Springer-Verlag, New York-Heidelbert-Berlin **73**(1980).

[4] S. Stalder, *The properties of strong band graded rings*, Ph.D. Thesis, University of Wisconsin-Milwaukee (1993).

[5] M. L. Teply, *On semigroup algebras and semisimple semilattice sums of rings*, Edinburgh Math. Soc. **24**(1981), 87-90.

[6] M. L. Teply, *Prime ideals and radicals of semigroup rings and semigroup graded rings*, TGRC-Workshop Proceeding (1994).



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