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ON THE UNIQUENESS FOR ENTIRE FUNCTIONS

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Abstract. We establish two uniqueness theorems for non-constant entire functions which extends results of J. S. Hwang in J. Math. Anal. and its Appl. Vol. 37 (1972) 452-456.

1. Introduction

If $f(z)$ is an entire function in the complex plane we use $M(r, f)$ to denote the modulus function of f over the closed disk $\{z : |z| \le r\}$. R. P. Boas essentially obtained the following result in his book "Entire Function".

Theorem A. (see [1, p.75, Theorem 5.4.2]) If $f_1(z)$ and $f_2(z)$ are entire functions of exponential type and, for some θ, $|f_1(re^{i\theta}) - f_2(re^{i\theta})| \le e^{-rA(r)}$, where $A(r)$ is a positive increasing function with $A(r) \to +\infty$ as $r \to +\infty$, then $f_1 \equiv f_2$.

Theorem B. (see [2, Theorem 1]) Let f_1, f_2 be entire functions of exponential type τ. If on a curve Γ having a limit point at infinity we have

$$|f_1(z) - f_2(z)| \le e^{-A(|z|)|z|} \tag{1.1}$$

for $z \in \Gamma$, where $A(r)$ is a positive increasing function satisfying the condition $\lim_{|z| \to +\infty} A(|z|) = +\infty$, then $f_1 \equiv f_2$.

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Theorem C. (see [2, Theorem 2]) Let f_1, f_2 be entire functions. If $\log M(r, f_1)$ and $\log M(r, f_2)$ are of finite positive order λ and of finite type τ and $A(r)$ is a positive increasing function with

$$\liminf_{r\to+\infty} \frac{A(r)}{r^\lambda} > 0. \tag{1.2}$$

Suppose that on a curve Γ having a limit point at infinity we have

$$|f_1(z) - f_2(z)| \le e^{-A(|z|)|z|} \tag{1.3}$$

for $z \in \Gamma$, then $f_1 \equiv f_2$.

Theorem D. (see [2, Theorem 3]) Let f_1, f_2 be entire functions with $\log M(r, f_1)$ and $\log M(r, f_2)$ being of finite positive order λ and of finite type τ. If $A(r)$ is an unbounded increasing function defined in $r > 0$, such that on a curve Γ having a limit point at infinity we have

$$|f_1(z) - f_2(z)| \le e^{-A(|z|)\ |z|^\lambda} \tag{1.4}$$

for $z \in \Gamma$, then $f_1 \equiv f_2$.

The purpose of this article is to try to extend above known results for entire functions of finite positive order to the case of transcendental entire functions of arbitrary growth. To do this we need to use some terminologies as follows:

Definition. Let $S(r)$ and $\varphi(r)$ be two unbounded non-negative increasing functions defined for $r > 0$, and $\varphi'(r)$ exists and is positive continuous except on a set of isolated points. *The φ-order of $S(r)$* is a number λ $(0 \le \lambda \le +\infty)$ for which

$$\limsup_{r\to+\infty} \log S(r)/\log\varphi(r) = \lambda. \tag{1.5}$$

If $S(r)$ is of finite φ-order λ, the number

$$\tau = \limsup_{r\to+\infty} \frac{S(r)}{(\varphi(r))^\lambda} \tag{1.6}$$

is called *the φ-type* of $S(r)$.

For any nonconstant entire functions $f(z)$, if we put $\varphi(r) = \log M(r, f)$, then $\log M(r, f)$ is of φ-order one and of φ-type one.

Theorem 1. Let f_1, f_2 be entire functions. If $\log M(r, f_1)$ and $\log M(r, f_2)$ are of finite positive φ-order λ and of finite φ-type τ and $A(r)$ is a positive increasing function with

$$\liminf_{r \to +\infty} \frac{A(r)}{(\varphi(2r))^{\lambda}} > 0, \tag{1.7}$$

and if on a curve Γ having a limit point at infinity we have

$$|f_1(z) - f_2(z)| \leq e^{-A(|z|)\ \varphi(|z|)} \tag{1.8}$$

for $z \in \Gamma$, then $f_1 \equiv f_2$.

Remark. Above Theorem 1 extends a result of J. S. Hwang [2, Theorem 2] from entire functions of finite order to entire functions of arbitrary growth.

Theorem 2. Let f_1, f_2 be entire functions with $\log M(r, f_1)$ and $\log M(r, f_2)$ being of finite positive φ-order λ. If $\varphi(r)$ satisfies the following restriction

$$\liminf_{r \to +\infty} \frac{\varphi(r)}{(\varphi(2r))} > K > 0 \tag{1.9}$$

and if $A(r)$ is an unbounded increasing function defined in $r > 0$, such that on a curve Γ having a limit point at infinity we have

$$|f_1(z) - f_2(z)| \leq e^{-A(|z|)\ (\varphi(|z|))^{\lambda}} \tag{1.10}$$

for $z \in \Gamma$, then $f_1 \equiv f_2$.

Remarks. 1. The number 2 which is appeared in (1.7) can be replaced by any number greater than 1 and so it is not essential. **2.** If $\varphi(r) = r$ then Theorem 2 reduces a result of J. S. Hwang [2, p. 455, Theorem 3], furthermore if $\lambda = 1$ and if $\Gamma(r) = re^{i\theta}$ Theorem 2 reduces a result of R. P. Boas in [1, p. 75, Theorem 5.4.2].

2. The Proofs

To prove Theorem 1 we need a result of D. C. Rung in the following.

Lemma 2.1. (see [3]) Let $D = \{z|\ |z| < 1\}$ and let Γ be a Jordan arc in D tending to a point on the boundary of D. If $\min_{z\in\Gamma} |z| = a$, then for each $z \in D - \Gamma$, we have

$$w(z;\Gamma) \geq \frac{1}{8\pi}(1 - |z|^2)(1 - a^2), \tag{2.11}$$

where $w(z;\Gamma)$ is the harmonic measure at the point z of the arc Γ relative to the domain $D - \Gamma$.

Proof of Theorem 1. Put

$$F(z) = f_1(z) - f_2(z). \tag{2.12}$$

It suffices to show that $F(z) \equiv 0$. Without loosing the generality we assume that the curve Γ is Jordan and begins with the origin. For each positive integer n, let $\Gamma_n = \{z|z \in \Gamma, 2^{n-1} \leq |z| \leq 2^n\}$. If we set

$$A_n = \inf_{2^{n-1}\leq|z|\leq 2^n} A(|z|),$$

and $G_n = \{z|\ |z| < 2^n\}$, then we have

$$A_n > M(\varphi(2^n))^\lambda \text{ and } \lim_{n\to+\infty} A_n = +\infty; \tag{2.13}$$

$$|F(z)| \leq \exp(-A_n\varphi(2^{n-1})) \text{ for } z \in \Gamma_n; \tag{2.14}$$

$$|F(z)| \leq \exp((\tau + \epsilon)(\varphi(2^n))^\lambda) \text{for } z \in G_n; \tag{2.15}$$

where n is sufficiently large, M and ϵ are some positive numbers.

Applying the result of Nevanlinna on harmonic measure [4, p.42], we have

$$\log|F(z)| \leq -A_n\varphi(2^{n-1})w(z;\Gamma_n) + (\tau + \epsilon)(\varphi(2^n))^\lambda \tag{2.16}$$

where $w(z;\Gamma_n)$ is the harmonic measure at z of Γ_n relative to $G_n - \Gamma_n$.

Based on the above Lemma of Rung, Hwang [2, p. 454] obtained

$$w(z;\Gamma_n) \geq (\frac{3}{32\pi})(1-\frac{|z|^2}{2^{2n}}). \tag{2.17}$$

(2.6) and (2.7) give

$$\log|F(z)| \leq (-A_n)\varphi(2^{n-1})\{\frac{3}{32\pi}(1-\frac{|z|^2}{2^{2n}}) - \frac{(\tau+\epsilon)(\varphi(2^n))^\lambda}{A_n(\varphi(2^{n-1}))}\} \tag{2.18}$$

It follows from (2.3) that we have

$$\lim_{n\to+\infty} \frac{(\tau+\epsilon)(\varphi(2^n))^\lambda}{A_n(\varphi(2^{n-1}))} \leq \lim_{n\to+\infty} \frac{\tau+\epsilon}{M\varphi(2^{n-1})} = 0; \tag{2.19}$$

so for any $|z| \leq 1$, there is another positive M' and a natural number N such that for all n with $n \geq N$ we have

$$\frac{3}{32\pi}(1-\frac{|z|^2}{2^{2n}}) - \frac{(\tau+\epsilon)(\varphi(2^n))^\lambda}{A_n(\varphi(2^{n-1}))} > M'. \tag{2.20}$$

(2.8) yields

$$\log|F(z)| \leq (-A_n)\varphi(2^{n-1})M' \quad \text{for } |z| \leq 1 \text{ and } n \geq N. \tag{2.21}$$

(2.3) gives $F(z) \equiv 0$ for any z with $|z| \leq 1$, since F is entire, hence $F(z) \equiv 0$ for any $z \in \mathbb{C}$. This completes the proof of Theorem 1.

Proof of Theorem 2. We observe that the condition (1.10) gives

$$|F(z)| \leq e^{-A_n(\varphi(2^{n-1}))^\lambda}, \qquad z \in \Gamma_n \tag{2.22}$$

where $F(z)$, A_n, Γ_n are the same as in the proof of Theorem 1. Replacing $A_n(\varphi(2^{n-1}))$ by $A_n(\varphi(2^{n-1}))^\lambda$ in (2.6), we obtain

$$\log|F(z)| \quad \leq \quad w(z;\Gamma_n)(-A_n(\varphi(2^{n-1}))^\lambda) + (\tau+\epsilon)(\varphi(2^n))^\lambda, \tag{2.23}$$

where $w(z;\Gamma_n)$ is the harmonic measure at z of Γ_n relative to $G_n - \Gamma_n$; and

$$\log|F(z)| \leq -A_n(\varphi(2^{n-1}))^\lambda\{\frac{3}{32\pi}(1-\frac{|z|^2}{2^{2n}}) - \frac{(\tau+\epsilon)}{A_n}K^\lambda\}. \tag{2.24}$$

By the same argument as in Theorem 1, the conclusion of Theorem 2 follows from expression (2.14).

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