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LINDELÖFICATION OF FRAMES

Mee Kyung Khang

Abstract. We introduce a concept of countably strong inclusions $\lhd$ and that of $\lhd$-σ-ideals and prove that the subframe $S(\lhd)$ of the frame σIdL of σ-ideals is a Lindelöfication of a frame L. We also deal with conditions for which the converse holds. We show that any countably approximating regular $D(\aleph_1)$ frame has the smallest countably strong inclusion and any frame which has the smallest $D(\aleph_1)$ Lindelöfication is countably approximating regular $D(\aleph_1)$.

1. Introduction and preliminaries

This section is a collection of basic definitions and results on frames. For general notions and facts concerning frames, we refer to Johnstone[10].

1.1. Frames.

Definition 1.1. ([3]) A *frame* is a complete lattice L in which binary meet distributes over arbitrary join, that is, $x \wedge \bigvee S = \bigvee\{x \wedge s \in S\}$ for any x in L and any subset S of L.

We will denote the bottom element of a frame L by 0 or 0_L and the top element by e or e_L.

Example 1.2. (1) Every complete chain is a frame.
(2) Every complete Boolean algebra is a frame.
(3) For a topological space X, the open set lattice Ω(X) of X under the inclusion is a frame.
(4) For a topological space X, the regular open set lattice O_{reg}(X) is a complete Boolean algebra and hence it is a frame.

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DEFINITION 1.3. A *frame homomorphism* is a map $h : \mathrm{L} \to \mathrm{M}$ between frames L and M Preserving all finitary meets and binary joins.

For any element a of a frame L, the map $a \wedge \_ : \mathrm{L} \to \mathrm{L}$ preserves arbitraty joins; hence it has a right adjont, which will be denoted by $a \to \_ : \mathrm{L} \to \mathrm{L}$. In particular, $a \to 0$ exists for any a in L and we write $a \to 0 = a^*$, called the *pseudocomplement* of a.

PROPOSITION 1.4. *Let L be a frame and a,b in L. Then we have :*
(1) $0^* = e$ *and* $e^* = 0$.
(2) $a \wedge a^* = a^{**} \wedge a^* = 0$.
(3) $a \leq a^{**}$.
(4) *If* $a \leq b$, *then* $a^* \geq b^*$.
(5) $(\bigvee_{\iota \in I} a_\iota)^* = \bigwedge_{\iota \in I} {a_\iota}^*$ *for any family* $(a_\iota)_{\iota \in I}$ *in L.*

A frame homomorphism $h : \mathrm{L} \to \mathrm{M}$ has the right adjoint which will be denoted by h^* and given by

$$h^*(y) = \bigvee \{x \in \mathrm{L} \mid h(x) \leq (y)\}(y \in \mathrm{M}).$$

Then h^* has the following properties :
(1) h is 1-1 if and only $h^* \circ h = id_{\mathrm{L}}$.
(2) h is onto if and only if $h \circ h^* = id_{\mathrm{M}}$.

DEFINITION 1.5. A frame homomorphism $h : \mathrm{L} \to \mathrm{M}$ is said to be :
(1) *dense* if $h(a) = 0$ implies $a = 0$.
(2) *codense* if $h(a) = e$ implies $a = e$.

1.2. Some Special Frames.

DEFINITION 1.6. ([8])Let L be a frame and a, b in L. We say that a is *rather below* b, if there exists c in L such that $a \wedge c = 0$ and $b \vee c = e$, equivalently, $a^* \vee b = e$. In this case, we write $a \prec b$.

We note that $u \prec v$ in $\Omega(\mathrm{X})$ means $\overline{u} \subseteq v$, for a topological space $(\mathrm{X}, \Omega(\mathrm{X}))$.

PROPOSITION 1.7. *Let L be a frame and a, b, x, y in L. Then*
(1) $0 \prec x$ *and* $x \prec e$.
(2) $a \prec a$ *if and only if* a *is complemented.*
(3) $a \prec b$ *implies* $a \leq b$.
(4) *If* $x \leq a \prec b \leq y$, *then* $x \prec y$.

(5) *If* $a \prec b$ *and* $x \prec y$, *then* $a \wedge x \prec b \wedge y$ *and* $a \vee x \prec b \vee y$.
(6) $a \prec b$ *implies* $b^* \prec a^*$.

Proposition 1.8. *Let L and M be bounded distributive lattices and* $f : L \to M$ *a bounded lattice homomorphism. Then* f *preserves* $\prec$.

Definition 1.9. A frame L is said to be *regular* if for any a in L, $a = \bigvee\{b \in L | b \prec a\}$.

It is clear that a topological space $(\mathrm{X}, \Omega(\mathrm{X}))$ is regular if and only if $\Omega(\mathrm{X})$ is a regular frame.

Definition 1.10. Let L be a complete lattice and a, b in L. We say that a is *way below* b and write $a \ll b$, if for any subset S of L, $b \leq \bigvee \mathrm{S}$ implies $a \leq \vee\mathrm{E}$ for some finite subset E of S.

Example 1.11. Let $(\mathrm{X}, \Omega(\mathrm{X}))$ be a locally compact space. Then $u \ll v$ in $\Omega(\mathrm{X})$ if and only if there is a compact subset w of X such that $u \subseteq w \subseteq v$.

In [8], we have the following :

Definition 1.12. A complete lattice L is said to be *continuous*, if for any a in L, $a = \bigvee\{x \in \mathrm{L} \mid x \ll a\}$.

Proposition 1.13. *If L is a continuous frame, then the relation* $\ll$ *interpolates, i.e., for* $x \ll y$, *there is* z *in L with* $x \ll z \ll y$.

Definition 1.14. Let L be a complete lattice and a, b in L. We say that a is *countably way below* b and write $a \ll_c b$, if for any subset S of L, $b \leq \bigvee \mathrm{S}$ implies $a \leq \bigvee \mathrm{C}$ for some countable subset C of S.

Example 1.15. (1) Let A and B be subsets of a set X. Then $A \ll_c B$ in the frame $\wp(\mathrm{X})$ of the power set of X if and only if there is a countable subset C of X with $A \subseteq C \subseteq B$.
(2) In $\Omega(\mathrm{X})$ of a topological space $(\mathrm{X}, \Omega(\mathrm{X}))$, $u \ll_c v$ if there is a Lindelöf subset w of X with $u \subseteq w \subseteq v$. If X is locally Lindelöf, then the converse also holds.

Proposition 1.16. *Let L be a frame and* a, b, x, y *in L. Then*

(1) $0 \ll_c a$.
(2) $a \ll_c b$ *implies* $a \leq b$.
(3) *If* $x \leq a \ll_c b \leq y$, *then* $x \ll_c y$.
(4) *If* $a_n \ll_c b$ *for all* $n \in N$, *then* $\bigvee_{n \in N} a_n \ll_c b$.

(5) *If* $a \ll b$*, then* $a \ll_c b$*.*

Definition 1.17. ([11])A complete lattice L is said to be *countably approximating*, if for any x in L, $x = \bigvee\{a \in L \mid a \ll_c x\}$.

Example 1.18. (1) A continuous frame L is countably approximating by (5) of Proposition 1.16.
(2) If a frame is countable, then it is countably approximating.
(3) The frame $\Omega(X)$ of a locally Lindelöf space $(X, \Omega(X))$ is countably approximating.

2. Lindelöfication of a Frame

In this section, we deal with Lindelöfications of frames. We define countably strong inclusions on a frame and study relationship between Lindelöfications and countably strong inclusions.

2.1. Lindelöf Frames. The following definition is a natural generalization of compact frames.

Definition 2.1. A frame L is said to be a *Lindelöf frame*, if for any subset S of L with $\bigvee S = e$, there is a countable subset C of S such that $\bigvee C = e$.

Remark. (1) A topological space $(X, \Omega(X))$ is a Lindelöf space if and only if $\Omega(X)$ is a Lindelöf frame.
(2) A frame L is a Lindelöf frame if there is a countable subset B of L such that for any x in L, $x = \bigvee B'$ for some $B' \subseteq B$. Thus the regular open set lattice $O_{reg}(R)$ of the real line R is a non-spatial Lindelöf frame.

Using the definition of a closed sublocale and codense homomorphism, we obtain :

Proposition 2.2. *A closed sublocale of a Lindelöf frame is a Lindelöf frame.*

Proposition 2.3. *If* $h : L \to M$ *is a codence frame homomorphism and* M *is a Lindelöf frame, then* L *is a Lindelöf frame.*

A 1-1 frame homomorphism is clearly codence and therefore the following is immedate;

Corollary 2.4. *If $h : L \to M$ is a 1-1 frame homomorphism and M is a Lindelöf frame, then L is a Lindelöf frame.*

Definition 2.5. A frame L is said to be a $D(\aleph_1)$ *frame*, if for any a in L and any sequence $(b_n)_{n\in N}$ in L, $a \vee (\bigwedge_{n\in N} b_n) = \bigwedge_{n\in N} (a \vee b_n)$.

Example 2.6. (1) Suppose Ω(X) of a topological space X is closed under countable intersections, then Ω(X) is a $D(\aleph_1)$ frame. In particular, for any topological space X, the topology generated by the set of all G_δ-sets in the space is a $D(\aleph_1)$ frame.
(2) Every completely distributive frame is $D(\aleph_1)$.
(3) Every complete Boolean algebra is $D(\aleph_1)$ and hence atomless complete Boolean algebra is non-spatial $D(\aleph_1)$.
(4) The regular open set lattice $O_{reg}(R)$ of the real line R is a non-spatial Lindelöf $D(\aleph_1)$ frame. But the open set frame $\Omega(R)$ of the real line is not $D(\aleph_1)$.

Proposition 2.7. *If $x_n \prec y$ for all n in N in a $D(\aleph_1)$ frame L, then $\bigvee_{n\in N} x_n \prec y$ in L.*

Proof. By the assumption, $x_n^* \vee y = e$ for all n in N. Since L is $D(\aleph_1)$, $(\bigvee_{n\in N} x_n)^* \vee y = (\bigwedge_{n\in N} {x_n}^*) \vee y = \bigwedge_{n\in N} ({x_n}^* \vee y) = e$. Thus $\bigvee_{n\in N} x_n \prec y$. $\square$

2.2. Lindelöfication of a Frame. In this section, we introduce a concept of countably strong inclusions and using these, we construct Lindelöfication of frames.

Definition 2.8. A binary relation $\lhd$ on a frame L is said to be a *countably strong inclusion*, if it satisfies :
(1) if $x \le a \lhd b \le y$, then $x \lhd y$.
(2) $\lhd$ is closed under finite meets and countable joins.
(3) $a \lhd b$ implies $a \prec b$.
(4) $\lhd$ interpolates.
(5) $a \lhd b$ implies $b^* \lhd a^*$.
(6) $a = \bigvee\{x \in \mathrm{L} \mid x \lhd a\}$ for any a in L.

Remark. (1) A binary relation $\lhd$ on a frame with the condition 1) in the above definition satisfies 2) if and only if $\lhd$ satisfies :
(i) $e \lhd e$; if $x \lhd y_1$ and $x \lhd y_2$, then $x \lhd y_1 \wedge y_2$ and
(ii) if $x_n \lhd y$ for all n in N, then $\bigvee_{n\in N} x_n \lhd y$.

(2) Let X be a set. Then $\subseteq$ is a countably strong inclusion on $\wp(X)$.

By proposition 1.7, we have the following :

Proposition 2.9. *If L is a Lindelöf regular $D(\aleph_1)$ frame, then $\prec$ is a countably strong inclusion.*

Example 2.10. (1) $\wp(N)$ is a Lindelöf regular $D(\aleph_1)$ frame.

(2) Let X be an uncountable set and p a particular point of X. Define a topology on X by declaring open any set whose complement either is countable or includes p. Then $\Omega(X)$ is a Lindelöf regular $D(\aleph_1)$ frame.

We will investigate the relationship between countably strong inclusions and Lindelöfication.

Definition 2.11. A *Lindelöfication* of a frame L is a dense, onto frame homomorphism $h : L \to M$ such that M is a Lindelöf regular frame.

Consider a dense onto frame homomorphism $h : L \to M$ with the right adjoint $h^* : L \to M$. We define a relation $\lhd$ on L as follows : $x \lhd y$ if and only if $h^*(x) \prec h^*(y)$ for any x, y in L. Then since h is onto, $h \circ h^* = id_L$ and hence $\lhd \subseteq \overset{2}{h}(\prec)$, where $\overset{2}{h}$ denotes the map $h \times h$.

Using these notions, we have the following:

Proposition 2.12. $\lhd = \overset{2}{h}(\prec)$

Proof. Suppose $u \prec v$ in M and let $h(u) = x$ and $h(v) = y$. Then $v \leq h^*(y)$; hence $u \prec h^*(y)$. Thus there exists t in M such that $u \wedge t = 0$ and $h^*(y) \vee t = e$. Since h is onto, $h(h^*(x) \wedge t) = h(h^*(x)) \wedge h(t) = x \wedge h(t) = h(u) \wedge h(t) = h(u \wedge t) = 0$. Since h is dense, $h^*(x) \wedge t = 0$. Thus $h^*(x) \prec h^*(y)$; hence $x \lhd y$. □

The properties of pseudocomplements and dense homomorphisms lead us the following :

Lemma 2.13. *For a dense onto homomorphism $h : M \to L$, $(h^*(a))^* = h^*(a)^*$ for all a in L.*

Proposition 2.14. *Suppose that $h : M \to L$ is a Lindelöfication of a frame L and M is a $D(\aleph_1)$ frame. Then the relation $\lhd$ defined as above is a countably strong inclusion on L.*

In the following, we construct a Lindelöfication from a countably strong inclusion. For this, we study some properties of σ-ideals.

We recall that a subset D of a poset L is said to be *countably directed*, if every countable subset of D has an upper bound in L.

DEFINITION 2.15. A subset I of a frame L is said to be a *σ-ideal* if it is a countably directed lower set, equivalently, it is a lower set and closed under countable joins.

Let σIdL denote the set of all σ-ideals in L. Then σIdL is clearly closed under arbitrary intersections in the power set lattice $\wp$(L) of L and therefore it is a complete lattice.

EXAMPLE 2.16.
(1) Let X be an infinite set. Count(X) = {S $\subseteq$ X | S is a countable set } is a σ-ideal of $\wp$(X). Fin(X) = {F $\subseteq$ X | F is a finite set} is an ideal but not a σ-ideal of $\wp$X.
(2) If L is a $D(\aleph_1)$ frame, then for any a in L, $\{x \in \mathrm{L} \mid x \prec a\}$ is a σ-ideal of L by Proposition (2.7).
(3) For a frame L and a in L, $\downarrow_c a = \{x \in \mathrm{L} \mid x \ll_c a\}$ is a σ-ideal of L by (1), (3) and (4) in Proposition (1.16).

LEMMA 2.17. *For $(I_\lambda)_{\lambda\in\Lambda} \subseteq \sigma IdL$, $\bigvee_{\lambda\in\Lambda} I_\lambda = \{\bigvee_{k\in N} x_k \mid (x_k)_{k\in N}$ is a sequence in $\bigcup_{\lambda\in\Lambda} I_\lambda\}$ in σIdL.*

PROPOSITION 2.18. *σIdL is a Lindelöf frame.*

We now consider the set of all $\triangleleft$ *-σ-ideal* in a frame L as a candidate of a Lindelöfication of L.

DEFINITION 2.19. Let I be a σ-ideal and $\triangleleft$ a countably strong inclusion on L. Then I is said to be a $\triangleleft$ -σ-ideal if for any a in I, there is b in I such that $a \triangleleft b$.

PROPOSITION 2.20. *Let $\triangleleft$ be a countably strong inclusion on a frame L. Then for any a in L, $\{x \in L \mid x \triangleleft a\}$ is a $\triangleleft$ -σ-ideal.*

We note that for a Lindelöf regular $D(\aleph_1)$ frame L, $\prec$ is a countably strong inclusion on L and hence for any a in L, $\{x \in \mathrm{L} \mid x \prec a\}$ is a $\prec$ -σ-ideal.

In the following, let $\triangleleft$ denote a countably strong inclusion on a frame L and let $S(\triangleleft)$ denote the set of all $\triangleleft$ -σ-ideals in L.

PROPOSITION 2.21. *$S(\triangleleft)$ is a Lindelöf subframe of σIdL.*

Consider $j : \sigma\mathrm{IdL} \to \mathrm{L}$ defined by $j(\mathrm{J}) = \bigvee \mathrm{J}$ and the restriction $j_0 : S(\triangleleft) \to \mathrm{L}$ of j. Since j is a dense homomorphism, so is j_0.

Consider $k : \mathrm{L} \to S(\triangleleft)$ defined by $k(a) = \{x \in \mathrm{L} \mid x \triangleleft a\}$. Then k is well defined by Proposition (2.20), and $\bigvee k(a) = a$, for $\triangleleft$ is a countably strong inclusion. Thus $j_0(k(a)) = a$ for all $a \in \mathrm{L}$. So j_0 in onto.

The following is immediate from the definition of $\triangleleft$ -σ-ideals.

LEMMA 2.22. *For any J in $S(\triangleleft)$, $J = \bigcup_{a \in J} k(a)$.*

LEMMA 2.23. *If $x \triangleleft a$, then $k(x) \prec k(a)$.*

PROPOSITION 2.24. *$S(\triangleleft)$ is regular.*

Proof. For any J in $S(\triangleleft)$, $\mathrm{J} = \bigcup_{a \in \mathrm{J}} k(a)$ and for any a in J, there is x in J such that $a \triangleleft x$. By the above lemma, $k(a) \prec k(x) \leq \mathrm{J}$. Thus $\mathrm{J} = \bigcup_{a \in \mathrm{J}} k(a) \leq \bigvee \{\mathrm{I} \in S(\triangleleft) \mid \mathrm{I} \prec \mathrm{J}\} \leq \mathrm{J}$; hence $\mathrm{J} = \bigvee \{\mathrm{I} \in S(\triangleleft) \mid \mathrm{I} \prec \mathrm{J}\}$. Therefore $S(\triangleleft)$ is regular. □

Collecting the above results, we have the following :

THEOREM 2.25. *Suppose $\triangleleft$ is a countably strong inclusion on a frame L. Then $j_0 : S(\triangleleft) \to L$ is a Lindelöfication of L.*

PROPOSITION 2.26. *If L is $D(\aleph_1)$, then so is σIdL.*

By Proposition 2.21, 2.24 and 2.26, we have the following :

COROLLARY 2.27. *Suppose L is $D(\aleph_1)$ and $\triangleleft$ is closed under countable meets. Then $S(\triangleleft)$ is $D(\aleph_1)$ and hence $j_0 : S(\triangleleft) \to L$ is a $D(\aleph_1)$ Lindelöfication of L.*

We note that if L is $D(\aleph_1)$, then $\prec$ is closed under countable meets. Indeed, $x \prec y_n$ for all $n \in N$ in L, then $x^* \vee (\bigwedge_{n \in N} y_n) = \bigwedge_{n \in N} (x^* \vee y_n) = e$ and therefore, $x \prec \bigwedge_{n \in N} y_n$.

NOTATION 2.28. In the following, $j_0 : S(\triangleleft) \to \mathrm{L}$ will be denoted by $\mathcal{I}_{\mathrm{L}} : S(\triangleleft) \to \mathrm{L}$.

Let CS(L) be the set of all countably strong inclusions on a frame L. Then $(\mathrm{CS(L)}, \subseteq)$ is a poset.

DEFINITION 2.29. Let $f : \mathrm{M} \to \mathrm{L}$ and $g : \mathrm{N} \to \mathrm{L}$ be Lindelöfications of a frame L. If there is a frame homomorphism $h : \mathrm{M} \to \mathrm{N}$ with $g \circ h = f$, then we say that f is *smaller than* g and write $f \leq g$.

Note that if h exists in the above definition, it is unique since g is dense and hence a monomorphism for N is a regular frame. Moreover, if the reverse also holds, then h is an isomorphism, which shows that the equivalence relation associated with this preorder is just an isomorphism of Lindelöfications.

Let Lind(L) be the set of all equivalent classes of Lindelöfications of L. Then $(\mathrm{Lind(L)}, \leq)$ is a poset. Consider maps $\varphi : \mathrm{Lind}^*(\mathrm{L}) \to \mathrm{CS(L)}$ defined by $\varphi(h : \mathrm{M} \to \mathrm{L}) = \overset{2}{h}(\prec_{\mathrm{M}})$ and $\psi : \mathrm{CS(L)} \to \mathrm{Lind(L)}$ defined by $\psi(\lhd) = (\mathcal{I}_{\mathrm{L}} : S(\lhd) \to \mathrm{L})$, where $\mathrm{Lind}^*(\mathrm{L})$ denotes the set of all $D(\aleph_1)$ Lindelöfications of L.

PROPOSITION 2.30. *The map φ and ψ are isotones.*

THEOREM 2.31. *Suppose $\lhd$ is a countably strong inclusion on a frame L such that $S(\lhd)$ is $D(\aleph_1)$. Then $\varphi(\psi(\lhd)) = \lhd$.*

Proof. Take any countably strong inclusion $\lhd$ on L and let $\lhd_0 = \varphi(\psi(\lhd))$ the countably strong inclusion determined by $\psi(\lhd)$. Note that $\bigvee \mathrm{J} \leq a$ if and only if $\mathrm{J} \subseteq k(a) = \{x \in \mathrm{L} \mid x \lhd a\}$ for all J in $S(\lhd)$ and a in L because J is a $\lhd$ -σ-ideal. Thus k is the right adjoint of $\mathcal{I}_{\mathrm{L}}$, the join map. Thus $x \lhd_0 y$ if and only if $k(x) = \mathcal{I}^*_{\mathrm{L}}(x) \prec \mathcal{I}^*_{\mathrm{L}}(y) = k(y)$ by Proposition 2.12. Suppose $x \lhd_0 y$, then there is J in $S(\lhd)$ such that $k(x) \wedge \mathrm{J} = \{0\}$ and $k(x) \vee \mathrm{J} = \mathrm{L}$. Thus we have $x \wedge \bigvee \mathrm{J} = 0$, and $z \vee t = e$ for some $z \in k(y)$ and $t \in \mathrm{J}$; hence $x \wedge t \leq x \wedge \bigvee \mathrm{J} = 0$. Since z is in $k(y)$, $z \lhd y$. Thus $x \prec z \lhd y$; hence $x \lhd y$. Therefore $\lhd_0 \subseteq \lhd$. If $x \lhd y$, then $k(x) \prec k(y)$ i.e., $x \lhd_0 y$. Hence $\lhd \subseteq \lhd_0$. In all, $\lhd_0 = \lhd$. □

LEMMA 2.32. *Let L be a regular frame. Then any codense homomorphism $h : L \to M$ is 1-1.*

LEMMA 2.33. *Let L be a regular $D(\aleph_1)$ frame and M a Lindelöf frame. Then any dense homomorphism $h : L \to M$ is codense.*

For a frame homomorphism $h : L \to M$, let $\sigma\mathrm{Idh} : \sigma\mathrm{IdL} \to \sigma\mathrm{IdM}$ be the frame homomorphism assigning each σ-ideal J in L, to the σ-ideal

$\downarrow h(\mathrm{J}) = \bigcup_{a\in \mathrm{J}}\{\downarrow h(a) \mid a \in \mathrm{J}\}$. Then $\sigma\mathrm{Id}h$ is a frame homomorphism. Thus σId is a functor from Frm to Frm.

LEMMA 2.34. *Let M be a regular $D(\aleph_1)$ frame. Then $s : M \to \sigma IdM$ defined by $s(a) = \{x \in M \mid x \prec a\}$ is a frame homomorphism.*

THEOREM 2.35. *For a $D(\aleph_1)$ Lindelöfication $h : M \to L$ of a frame L, $\psi \circ \varphi(h) \cong M$.*

Proof. Let $\varphi(h) = \lhd$, then $x \lhd y$ if and only if $h^*(x) \prec h^*(y)$. It is enough to show that there is an isomorphism $f : \mathrm{M} \to S(\lhd)$ such that $\mathcal{I}_{\mathrm{L}} \circ f = h$.

Define $f : \mathrm{M} \to S(\lhd)$ by $f(a) =\downarrow h\{x \in M \mid x \prec a\}$, which is clearly a σ-ideal in L. Since M is a Lindelöf regular $D(\aleph_1)$ frame, $x \prec a$ implies that $x \prec y \prec a$, for some y in M and hence $h(x) \lhd h(y)$ and $h(y)$ is in $f(a)$. Thus for any z in $f(a)$, $z \le h(x)$ for some $x \prec a$; hence $z \lhd h(y)$ for some $h(y)$ in $f(a)$. Therefore $f(a)$ is in $S(\lhd)$. Thus f is well-defined. For any a in M, $\bigvee f(a) = \bigvee\{h(x) \mid x \prec a\} = h(a)$. Hence we have $\mathcal{I}_{\mathrm{L}} \circ f = h$.

As noted above $f = (\sigma\mathrm{Id}h) \circ s$, where $s(a) = \{x \in M \mid x \prec a\}$. Thus by lemma 2.34, f is a frame homomorphism.

Since h is dense, so is f ; hence f is 1-1 by the above lemma.

For any J in $S(\lhd)$, let $a = \bigvee h^{-1}(\mathrm{J})$. Assume that $x \prec a$ then $e = x^* \vee a = x^* \vee \bigvee h^{-1}(\mathrm{J})$ and therefore $e = x^* \vee \bigvee E$, for some countable subset E of $h^{-1}(\mathrm{J})$. Hence $x = x \wedge e = x \wedge (x^* \vee \bigvee E) = x \wedge \bigvee E$, so that $x \le \bigvee E \in h^{-1}(\mathrm{J})$. Therefore $h(x)$ is in J. And if $h(x)$ is in J, then there is $h(y)$ in J such that $h(x) \lhd h(y)$, because h is onto. Thus $x \le h^*h(x) \prec h^*h(y)$ and $h^*h(y) = \bigvee\{z \in M \mid h(z) \le h(y)\} \le a$. Hence $x \prec a$. This show that $h(x) \in \mathrm{J}$ if and only if $x \prec a$. Since h is onto, $f(a) =\downarrow (h\{x \in M \mid x \prec a\}) = h(\{x \in M \mid x \prec a\})$, so that $f(a) = \mathrm{J}$. In all, f is an isomorphism. This completes the proof. □

2.3. The Smallest Lindelöfication. In this section, we deal with Lindelöfications of countably approximating frames.

PROPOSITION 2.36. *Let L be a countably approximating regular $D(\aleph_1)$ frames, then $x \ll_c y$ if and only if $x \prec y$ and $\uparrow x^*$ is a Lindelöf frame.*

Proof. ($\Rightarrow$) Suppose $x \ll_c y$. Since L is regular, $y = \bigvee\{z \in \mathrm{L} \mid z \prec y\}$. Since L is $D(\aleph_1)$, $\{z \in y \mid z \prec y\}$ is countably directed and hence there is z in L such that $x \prec z$ and $x \le z$, which implies $x \prec y$.

Since L is countably approximating, there exists z in L such that $x \ll_c z \ll_c y$. Take any $S \subseteq \uparrow x^*$ with $e = \bigvee S$. Then $y \leq e = \bigvee S$. Since $z \ll_c y$, there is $(a_n)_{n\in N}$ in S such that $z \leq \bigvee_{n\in N} a_n$. And $x \ll_c z$ implies $x \prec z$; hence $x \prec \bigvee_{n\in N} a_n$. Thus $e = x^* \vee (\bigvee_{n\in N} a_n) = \bigvee_{n\in N} (x^* \vee a_n) = \bigvee_{n\in N} a_n$. Hence $\uparrow x^*$ is a Lindelöf frame.

($\Leftarrow$) Suppose that $y \leq \bigvee$S for a subest S of L. Since $x \prec y$, $e = x^* \vee y = x^* \vee (\bigvee \mathrm{S}) = \bigvee_{s\in \mathrm{S}} (x^* \vee s)$. Since $\uparrow x^*$ is a Lindelöf frame, there is $(a_n)_{n\in N}$ in S such that $e = \bigvee_{n\in N} (x^* \vee s_n) = x^* \vee (\bigvee_{n\in N} s_n)$. Hence $x = x \wedge e = x \wedge (x^* \vee (\bigvee_{n\in N} s_n)) = x \wedge (\bigvee_{n\in N} s_n)$, which implies that $x \leq \bigvee_{n\in N} s_n$. Therefore $x \ll_c y$. □

Remark. In any frame L, if $x \prec y$ and $\uparrow x^*$ is a Lindelöf frame, then $x \ll_c y$. Furthermore, the relation $\prec$ on any Lindelöf frame L implies $\ll_c$ for $\uparrow x^*$ for any $x \in$ X. Thus every Lindelöf regular frame is countably approximating, because $x = \bigvee\{y \in \mathrm{L} \mid y \prec x\} \leq \bigvee\{y \in \mathrm{L} \mid y \ll_c x\} \leq x$ for any x in L.

But the converse does not hold since the frame of the discrete topology on the real line is countably approximating but not a Lindelöf frame.

Using the above remark, we have the following corollary.

Corollary 2.37. *Let L be a Lindelöf regular $D(\aleph_1)$ frame and x, y in L. Then $x \ll_c y$ if and only if $x \prec y$.*

We define $a \lhd\!\lhd b$ in a frame L if and only if $a \prec b$, and $\uparrow a^*$ or $\uparrow b$ is a Lindelöf frame.

Proposition 2.38. *Let L be a frame and a, b, x, y in L. Then*

(1) *If $x \leq a \lhd\!\lhd b \leq y$, then $x \lhd\!\lhd y$.*
(2) *If $x \lhd\!\lhd a$ and $x \lhd\!\lhd b$, then $x \lhd\!\lhd a \wedge b$.*
(3) *If $x \lhd\!\lhd a$, then $a^* \lhd\!\lhd x^*$.*
(4) *Suppose L is a $D(\aleph_1)$ frame, and $x_n \lhd\!\lhd a$ for all $n \in N$, then $\bigvee_{n\in N} x_n \lhd\!\lhd a$.*

Proposition 2.39. *Suppose L is a countably approximating regular $D(\aleph_1)$ frame. Then $\lhd\!\lhd$ interpolates.*

Proof. Suppose that $x \lhd\!\lhd a$. Then $x \prec a$.

Assume that $\uparrow x^*$ is a Lindelöf frame. Then $x \ll_c a$ by Proposition 2.36, and since L is countably approximating, there is b in L such that $x \ll_c b \ll_c a$ and hence $x \prec b \prec a$. Since $x \prec b$ and $\uparrow x^*$ is a Lindelöf frame, $x \lhd\!\lhd b$. Since $b \ll_c a$ and $\uparrow b^*$ is a Lindelöf frame, $b \lhd\!\lhd a$. In all, $x \lhd\!\lhd b \lhd\!\lhd a$.

Assume that $\uparrow a$ is a Lindelöf frame. Since $x \prec a$, there is z in L such that $x \wedge z = 0$ and $a \vee z = e$. Thus $e = a \vee z = a \vee \bigvee\{u \in \mathrm{L} \mid u \ll_c z\}$. Since $\uparrow a$ is a Lindelöf frame, there is a sequence $(u_n)_{n\in N}$ in L such that $u_n \ll_c z$ and $a \vee \bigvee_{n\in N} u_n = e$. Let $u = \bigvee_{n\in N} u_n$. Then $u \ll_c z$ and hence $u \prec z$ and $z^* \prec u^*$. Since $x \leq z^*$, $x \prec u^*$. Since $u \ll_c z$, $\uparrow u^*$ is a Lindelöf frame and hence $x \lhd\!\lhd u^*$. Since $a \vee z = e$, $u^* \prec a$ and hence $u^* \lhd\!\lhd a$. In all, $x \lhd\!\lhd u^* \lhd\!\lhd a$. □

Collecting the above propositions, we can conclude the following:

PROPOSITION 2.40. *If L is a countably approximating regular $D(\aleph_1)$ frame, then the relation $\lhd\!\lhd$ is a countably strong inclusion.*

THEOREM 2.41. *A countably approximating regular $D(\aleph_1)$ frame L has a smallest countably strong inclusion.*

Proof. By the above proposition, it is enough to show that a countably strong inclusion $\lhd$ on L contains $\lhd\!\lhd$.

Suppose that $x \lhd\!\lhd y$. Then $x \prec y$.

Assume that $\uparrow x^*$ is a Lindelöf frame, then by Proposition 2.36, $x \ll_c y = \bigvee\{z \in \mathrm{L} \mid z \lhd y\}$ and $\{z \in \mathrm{L} \mid z \lhd y\}$ is countably directed. Thus there is z in L such that $x \leq z$ and $z \lhd y$ and hence $x \lhd y$.

Suppose that $\uparrow y$ is a Lindelöf frame. Since $x \prec y$, there is u in L such that $x \wedge u = 0$ and $y \vee u = e$. Thus $e = y \vee u = y \vee \bigvee\{v \in \mathrm{L} \mid v \lhd u\}$. Since $\uparrow y$ is a Lindelöf frame, there is a sequence $(v_n)_{n\in N}$ in L such that $y \vee (\bigvee_{n\in N} v_n) = e$ and $v_n \lhd u$ for all $n \in N$. Let $v = \bigvee_{n\in N} v_n$. Then $v \lhd u$ and $y \vee v = e$. Thus $x \leq u^* \lhd v^* \leq y$; hence $x \lhd y$. This completes the proof. □

We will show that a frame which has the smallest $D(\aleph_1)$ Lindelöfication is countably approximating regular.

LEMMA 2.42. *Let M be a Lindelöf regular $D(\aleph_1)$ frame and a an element of M. Then $M_a = \{x \in M \mid x \leq a$ or $x \vee a = e\}$ is a regular subframe of M.*

LEMMA 2.43. *Let M be a countably approximating frame and a in M. Then $\downarrow a$ is countably approximating.*

THEOREM 2.44. *Suppose that a frame L has the smallest Lindelöfication $h : M \to L$ such that M is $D(\aleph_1)$. Then L is a countably approximating regular frame.*

Proof. Case 1. L is a Lindelöf frame.

By Lemma 2.32 and 2.33, h is 1-1 and hence an isomorphism. Thus L is countably approximating regular because M is by above remark.

Case 2. L is not a Lindelöf frame.

Then h is not an isomorphism and hence not codense by Lemma 2.32. Thus there is a in M with $a < e$ and $h(a) = e$. Then M_a is a regular subframe of M where $\mathrm{M}_a = \{x \in \mathrm{M} \mid x \leq a \text{ or } x \vee a = e\}$, by Lemma 2.42.

Consider $\overline{h} : \mathrm{M}_a \to \mathrm{L}$ defined by $\overline{h}(x) = h(x)$, which is a frame homomorphism since M_a is a subframe of M. Since h is onto and $h(a) = e$, $\overline{h}$ is also onto. Since M is a Lindelöf frame, so is M_a by Corollary 2.4. Moreover $\overline{h}$ is dense for h is dense. Thus $\overline{h}$ is a Lindelöfication of L. Since M_a is also closed under countable meets in M, M_a is also $D(\aleph_1)$. Since M is the smallest $D(\aleph_1)$ Lindelöfication, $\mathrm{M}_a = \mathrm{M}$ and hence for any x in M, $x \leq a$ or $x \vee a = e$.

Consider $\widetilde{h} :\downarrow a \to \mathrm{L}$ defined by $\widetilde{h}(x) = h(x)$ which is an onto frame homomorphism by the same argument for $\overline{h}$.

If $b \leq a$ and $h(b) = e$, then $\mathrm{M}_b = \mathrm{M} = \mathrm{M}_a$; hence $a \leq b$ or $b \vee a = e$. Since $b \leq a$ implies $a \vee b = a$, and $a < e$, $a \vee b \neq e$. Thus $a = b$. Therefore $\widetilde{h}$ is codense and 1-1 by Lemma 2.32 so that $\widetilde{h}$ is an isomorphism.

Since M_a is a Lindelöf regular frame and a is in M_a, $\downarrow a$ is countably approximating. Hence L is countably approximating for $\widetilde{h}$ is an isomorphism. This completes the proof. □

REMARK. In the above proof, a is, in fact, a maximal element in M. Indeed, suppose that $a \leq b < e$. Then since $b \in \mathrm{M} = \mathrm{M}_a$, $b \leq a$ or $b \vee a = e$. But $b \vee a \neq e$, because $a \vee b = b < e$. Thus $a = b$.

References

[1] B. Banaschewski, *Another look at the localic Tychonoff theorem*, Comment. Math. Univ. Carolinae, **26**, 1988, pp. 1-14.

[2] B. Banaschewski, *Compactification of frames*, Math. Nach.,**149**, 1990, pp. 105-116.

[3] B. Banaschewski, *Frames and compactifications, Extension theory of topological structures and its applications*, Deutscher Verlag der Wissenschaften, Berlin, 1969, pp. 29-33.
[4] B. Banaschewski, and G. C. L. Brümmer, *Stably continuous frames*, Math. Proc. Camb. Phil. Soc., **104**, 1988, no. 1, pp. 7-19.
[5] B. Banaschewski, and G. C. L. Brümmer, K. A. Hardie, *Biframes and bispaces*, Quaestiones Math., **6**, 1983, pp. 13-25.
[6] B. Banaschewski, and C. J. Mulvey, *Stone-Čech compactification of locales I*, Houston J. Math., **6**, 1980, pp. 301-312.
[7] X. Chen, *Closed homomorphism*, Doctoral Dissertation, McMaster University, Hamilton, 1991.
[8] G. Gierz, K. H. Hoffman, K. Keimel, J. D. Lawson, M. Mislove and D. S. Scott, *A compendium of continuous lattices*, Springer-Verlag, Berlin, 1980.
[9] J. R. Isbell, *Atomless parts of locales*, Math. Scand.,**31**, 1972, pp. 5-32.
[10] P. T. Johnstone, *Stone spaces*, Cambridge Univ. Press, Cambridge, 1982.
[11] S. O. Lee, *On countably approximating lattices*, J. Korean Math, Soc., **25**, no. **1**, 1988, pp. 11-23.

Department of Applied Mathematics
Paichai University
Daejeon 302-735, Korea
E-mail: mkkhang@pcu.ac.kr