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# KRULL RING WITH UNIQUE REGULAR MAXIMAL IDEAL

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Abstract. Let $R$ be a Krull ring with the unique regular maximal ideal $M$. We show that $R$ has a regular prime element and reg-dim$R = 1 \Leftrightarrow R$ is a factorial ring and reg-dim$(R) = 1 \Rightarrow M$ is invertible $\Leftrightarrow R \subsetneq [R : M] \Leftrightarrow M$ is divisorial $\Leftrightarrow$ reg-ht$M = 1 \Rightarrow R$ is a rank one discrete valuation ring. We also show that if $M$ is generated by regular elements, then $R$ is a rank one discrete valuation ring $\Rightarrow$ $R$ is a factorial ring and reg-dim$(R) = 1$.

## 1. Introduction

Let $R$ be a commutative ring with identity, and let $T(R)$ be the total quotient ring of $R$. An element of $R$ is said to be *regular* if it is not a zero divisor. An ideal of $R$ is *regular* if it contains a regular element of $R$. The regular height of a regular prime ideal $P$ of $R$, denoted by reg-ht$P$, is defined to be the supremum of the lengths of chains consisting of regular prime ideals contained in $P$ plus 1. The regular dimension of $R$, reg-dim$(R)$, is sup{reg-ht$P | P$ is a regular prime ideal of $R$}. Thus reg-dim$(R) = 1$ if and only if each regular prime ideal of $R$ is a maximal ideal. Also, if $R$ is an integral domain, then dim$(R)$ = reg-dim$(R)$ and ht$P$ = reg-ht$P$ for all nonzero prime ideals $P$ of $R$.

It is easy to see that if $R$ is a quasi-local Krull domain with maximal ideal $M$, then $R$ is a factorial domain of dim$(R) = 1 \Leftrightarrow M$ is invertible $\Leftrightarrow M$ is divisorial $\Leftrightarrow$ ht$M = 1 \Leftrightarrow R$ is a rank one discrete valuation ring. In [1, Proposition 2.13], Alajbegovic and Osmanagic proved that if $R$ is a Krull ring with the unique regular maximal ideal $M$, then $M$ is invertible

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$\Rightarrow M$ is divisorial $\Rightarrow$ reg-ht$M = 1 \Rightarrow R$ is a rank one discrete valuation ring. They also showed that if each regular ideal of $R$ is generated by regular elements, then $R$ is a rank one discrete valuation ring $\Rightarrow M$ is invertible. In this short note, we show that $R$ has a regular prime element and reg-dim$R = 1 \Leftrightarrow R$ is a factorial ring and reg-dim$(R) = 1$ $\Rightarrow M$ is invertible $\Leftrightarrow R \subsetneq [R : M] \Leftrightarrow M$ is divisorial $\Leftrightarrow$ reg-ht$M = 1$ $\Rightarrow R$ is a rank one discrete valuation ring. We also show that if $M$ is generated by regular elements, then $R$ is a rank one discrete valuation ring $\Rightarrow R$ is a factorial ring and reg-dim$(R) = 1$.

Let $G$ be a totally ordered abelian group and let $T$ be a commutative ring with 1. Extend $G$ by the symbol $\infty$ by defining $g < \infty$ and $g + \infty = \infty + g = \infty$ for all $g \in G$. A valuation $v$ on $T$ with value group $G$ is a map from $T$ onto $G \cup \{\infty\}$ such that for all $x, y \in T$,

1. $v(xy) = v(x) + v(y)$.
2. $v(x + y) \geq \min\{v(x), v(y)\}$.
3. $v(1) = 0$ and $v(0) = \infty$.

Note that $0 = v(1) = v((-1)^2) = v(-1) + v(-1)$ by (1), and since $G$ is totally ordered, $v(-1) = 0$; hence $v(-x) = v(x)$ for all $x \in T$. Also, if $x$ is a unit in $T$, then $v(\frac{1}{x}) = -v(x)$. Let $x, y \in T$ such that $v(x) > v(y)$. Then $v(x + y) \geq \min\{v(x), v(y)\} = v(y)$ by (2), and hence $v(y) = v(y + x - x) \geq \min\{v(x + y), v(-x) = v(x)\} \geq v(y)$. Thus $v(x + y) = v(y) = \min\{v(x), v(y)\}$.

Let $R$ be a subring of the ring $T$. It is known that there exists a valuation $v$ on $T$ such that $R = \{x \in T | v(x) \geq 0\}$ and $P = \{x \in T | v(x) > 0\}$ if and only if, for each $x \in T \setminus R$, there exists $x' \in P$ such that $xx' \in R \setminus P$ [4, Theorem 5.1]. In this case, $(R, P)$ is called the *valuation pair of* $T$ associated with valuation $v$. We call $R$ a *valuation ring* if $T = T(R)$. In particular, if $G$ is an additive group of integers $\mathbb{Z}$, then $R$ is called a rank one *discrete valuation ring* (DVR).

Let $\mathcal{F}(R)$ be the set of all regular fractional ideals of $R$. If $A, B \in \mathcal{F}(R)$, then $[A : B] = \{x \in T(R) | xB \subseteq A\}$ is also in $\mathcal{F}(R)$. For any $A \in \mathcal{F}(R)$, let $A_v = [R : [R : A]]$. If $A = A_v$, then $A$ is said to be *divisorial*. It is well known that $A \subseteq A_v$, $(A_v)_v = A_v$, and $(AB)_v = (AB_v)_v$ (see, for example, [2]). For any $A, B \in \mathcal{F}(R)$, define $A * B = (AB)_v$; then $D(R)$, the set of all divisorial ideals of $R$, is a commutative semigroup under the operation $*$. Moreover, $D(R)$ is a group if and only if $R$ is completely integrally closed (cf. [1, Proposition 1.8]).

## 2. Main results

A ring $R$ is called a *Krull ring* if either $R = T(R)$ or there exists a family $\{(V_\alpha, P_\alpha)\}$ of rank one discrete valuation rings such that

1. $R = \cap_\alpha V_\alpha$ and
2. for each regular element $x$ of $R$, $xV_\alpha = V_\alpha$ for all but a finite number of the $V_\alpha$'s.

It is well known that $R$ is a Krull ring if and only if $R$ is completely integrally closed and the ascending chain condition on regular divisorial ideals holds (cf. [1, Theorem 2.2]). A ring $R$ is called a *factorial ring* if each regular element of $R$ can be expressed by a finite product of (regular) prime elements. Next, we give the main result of this paper.

THEOREM 1. *Let $R$ be a Krull ring with the unique regular maximal ideal $M$. Consider the following statements.*

1. *$R$ has a regular prime element and reg-dim$(R) = 1$.*
2. *$R$ is a factorial ring and reg-dim$(R) = 1$.*
3. *$M$ is invertible.*
4. *$R \subsetneq [R : M]$.*
5. *$M$ is divisorial.*
6. *reg-ht$M = 1$.*
7. *$R$ is a rank one DVR.*

*Then the implications* (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Rightarrow$ (7) *hold, and if $M$ is generated by a set of regular elements, then the implication (7) $\Rightarrow$ (1) holds.*

*Proof.* (1) $\Rightarrow$ (2) and (3): Since reg-dim$(R) = 1$, $M$ is the unique regular prime ideal of $R$. So $M = pR$ for a regular prime element $p$ of $R$ by (1). Hence $M$ is invertible and $R$ is a factorial ring.

(2) $\Rightarrow$ (1): This is clear.

(3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7): These appear in [1, Proposition 2.13].

(5) $\Rightarrow$ (3): Note that $R \subsetneq [R : M]$ because $R = [R : M]$ implies $M = M_v = [R : [R : M]] = R$. Suppose that $M[R : M] = M$, and let $x \in [R : M] \setminus R$. Then $xM \subseteq M$, and hence $x^n M \subseteq M \subseteq R$ for all positive integers $n$. Thus if $r \in M$ is a regular element of $R$, then $rx^n \in R$, and so $x$ is almost integral over $R$. But, since $R$ is completely integrally closed [1, Theorem 2.2], $x \in R$, a contradiction. Hence $M \subsetneq M[R : M]$, and since $M$ is a maximal ideal, $M[R : M] = R$.

(6) ⇒ (5): It follows from [1, Corollary 2.9] because $R$ is a Krull ring and $M$ is a regular minimal prime ideal.

(7) ⇒ (1): Here we assume that $M$ is generated by a set of regular elements. If reg-ht$M \geq 2$, there exists a regular prime ideal $P$ of $R$ such that $P \subsetneq M$. Since $R$ is a rank one DVR, $R = R_{[P]} := \{x \in T(R) | sx \in R$ for some $s \in R \setminus P\}$ [1, Proposition 2.11]. Also, since $M$ is generated by regular elements, there exists a regular element $a \in M \setminus P$. So $\frac{1}{a} \in R_{[P]} = R$ and $1 = \frac{1}{a}a \in M$, a contradiction. Thus reg-ht$M = 1$.

Next, let $v$ be the valuation on $T(R)$ such that $R = \{x \in T(R) | v(x) \geq 0\}$ and $M = \{x \in T(R) | v(x) > 0\}$. Let $\{a_\alpha\}$ be the set of regular elements of $R$ in $M$; then $M = (\{a_\alpha\})$ by (7). Note that $\{v(a_\alpha)\}$ is a set of natural numbers; so by the well-ordering property of $\mathbb{N}$, $\{v(a_\alpha)\}$ has a minimal element, say $v(x)$. Then, for all $a_\alpha \in \{a_\alpha\}$, we have $0 < v(x) \leq v(a_\alpha)$, and hence $0 \leq v(\frac{a_\alpha}{x})$ or $a_\alpha \in xR$. Thus $M = xR$ and $x$ is a regular prime element. □

In Theorem 1, if $R$ is quasi-local with maximal ideal $M$, then (3) implies (1) since an invertible ideal in a quasi-local ring is principal [3, Proposition 7.4]. This cannot be generalized to a Krull ring with the unique regular maximal ideal. We next use the concept of "idealization" to give a rank one DVR (and hence a Krull ring) with the unique regular maximal ideal $M$ such that $M$ is invertible but not principal.

Let $R$ be a ring, and let $B$ be an $R$-module. Consider the set $R \times B = \{(r, b) | r \in R, b \in B\}$. For any $(r, b), (s, c) \in R \times B$, define

1. $(r, b) = (s, c)$ if $r = s$ and $b = c$,
2. $(r, b) + (s, c) = (r + s, b + c)$, and
3. $(r, b)(s, c) = (rs, rc + sb)$.

Then $R \times B$ becomes a commutative ring with identity under these definitions. This ring, denoted by $R(+)B$, is called the *idealization* of $B$ in $R$.

Example 2. Let $\mathbb{Q}$ be the field of rational numbers, and let $v$ be the rank one valuation on $\mathbb{Q}(X)$ given by $v(\frac{f}{g}) = \deg g - \deg f$. The valuation ring associated with $v$ is $\mathbb{Q}[X^{-1}]_{(X^{-1})}$. Let $p$ be a prime polynomial in $\mathbb{Q}[X]$ of degree $> 1$. Restricting $v$ to $\mathbb{Q}[X][\frac{1}{p}]$ gives a valuation $v_0 : \mathbb{Q}[X][\frac{1}{p}] \longrightarrow \mathbb{Z} \cup \{\infty\}$. The valuation ring of $v_0$ is $\mathbb{Q}[X][\frac{1}{p}] \cap \mathbb{Q}[X^{-1}]_{(X^{-1})}$. Let $B = \sum(\mathbb{Q}[X]/P)$ (direct), where $P \in Spec(\mathbb{Q}[X]) \setminus \{(p)\}$, and let $R_0 = \mathbb{Q}[X](+)B$. Then the total quotient ring $T(R_0)$ of $R_0$ is $\mathbb{Q}[X]_S(+)B_S$, where $S = \{(cp^m, b) | 0 \neq c \in Q$ and $m \geq 0\}$. Thus

$T(R_0) = \mathbb{Q}[X][\frac{1}{p}](+)B$. Moreover, $w : \mathbb{Q}[X][\frac{1}{p}](+)B \longrightarrow \mathbb{Z} \cup \{\infty\}$ given by $w((a,b)) = v_0(a)$ is a rank one discrete valuation whose associated valuation ring is $V = \mathbb{Q}[X][\frac{1}{p}] \cap \mathbb{Q}[X^{-1}]_{(X^{-1})}(+)B_S$. Let $M = \{(a,b) \in V | v_0(a) > 0\}$, then $M$ is the unique regular maximal ideal of $V$ such that $M$ is invertible but not principal.

**Remark.** See [4, Examples 9 and 10, p.183] for more on the ring $V = \mathbb{Q}[X][\frac{1}{p}] \cap \mathbb{Q}[X^{-1}]_{(X^{-1})}(+)B_S$ of Example 2.

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