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GENERALIZED JENSEN'S EQUATIONS IN A HILBERT MODULE

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Abstract. We prove the stability of generalized Jensen's equations in a Hilbert module over a unital C^*-algebra. This is applied to show the stability of a projection, a unitary operator, a self-adjoint operator, a normal operator, and an invertible operator in a Hilbert module over a unital C^*-algebra.

1. Introduction

Let E_1 and E_2 be Banach spaces, and $f : E_1 \to E_2$ a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in E_1$. Th.M. Rassias [14] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}||x||^p$$

for all $x \in E_1$.

Throughout this paper, let A be a unital C^*-algebra with a norm $|\cdot|$, $A_{\frac{1}{2}} = \{a \in A \mid |a| = \frac{1}{2}\}$, $A^+_{\frac{1}{2}}$ the set of positive elements in $A_{\frac{1}{2}}$, A_{in} the set of invertible elements in A, A_{sa} the set of self-adjoint elements in A, $\mathbb{R}^+$ the set of nonnegative real numbers, and ${}_A\mathcal{H}$ a left Hilbert A-module with a norm $\|\cdot\|$.

We prove the Hyers-Ulam-Rassias stability of generalized Jensen's equations in a Hilbert module over a unital C^*-algebra.

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2. Stability of generalized Jensen's equations in a Hilbert module over a C^*-algebra

In this section, let $\varphi : {}_A\mathcal{H} \setminus \{0\} \times {}_A\mathcal{H} \setminus \{0\} \to [0, \infty)$ be a function such that

$$\widetilde{\varphi}(x, y) := \sum_{k=0}^{\infty} 3^{-k} \varphi(3^k x, 3^k y) < \infty.$$

LEMMA 2.1. *Let $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be a mapping such that*

$$\|2F(ax + \frac{1}{2}y) - 2aF(x) - F(y)\| \leq \varphi(x, y)$$

for all $a \in A^+_{\frac{1}{2}} \cup \{i\}$ and all $x, y \in {}_A\mathcal{H} \setminus \{0\}$. If $F(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{H}$, then there exists a unique A-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ such that

$$\text{(i)} \qquad \|F(x) - F(0) - T(x)\| \leq \frac{1}{3}(\widetilde{\varphi}(x, -x) + \widetilde{\varphi}(-x, 3x))$$

for all $x \in {}_A\mathcal{H} \setminus \{0\}$.

Proof. Let $a = \frac{1}{2} \in A^+_{\frac{1}{2}}$ in the statement. By [11, Theorem 1], there exists a unique additive mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (i). The mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ was given by

$$T(x) = \lim_{n \to \infty} \frac{F(3^n x)}{3^n}.$$

The additive mapping T given in the proof of [11, Theorem 1] is similar to the additive mapping T given in the proof of [14, Theorem]. By the same reasoning as in the proof of [14, Theorem], it follows from the assumption that $F(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{H}$ that the additive mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is $\mathbb{R}$-linear.

Since $a = \frac{1}{2} \in A^+_{\frac{1}{2}}$,

$$\|2F(\frac{1}{2}x + \frac{1}{2}y) - F(x) - F(y)\| \leq \varphi(x, y)$$

for all $x, y \in {}_A\mathcal{H} \setminus \{0\}$. For each $a \in A^+_{\frac{1}{2}} \cup \{i\}$,

$$\|2F(\frac{1}{2}2ax + \frac{1}{2}x) - F(2ax) - F(x)\| \le \varphi(2ax, x)$$

for all $x \in {}_A\mathcal{H} \setminus \{0\}$. So

$$\begin{aligned}\|F(2ax) - 2aF(x)\| \le & \|2F(ax + \frac{1}{2}x) - 2aF(x) - F(x)\| \\ & + \|2F(ax + \frac{1}{2}x) - F(2ax) - F(x)\| \\ \le & \varphi(x, x) + \varphi(2ax, x)\end{aligned}$$

for all $a \in A^+_{\frac{1}{2}} \cup \{i\}$ and all $x \in {}_A\mathcal{H} \setminus \{0\}$. Thus $3^{-n}\|F(3^n 2ax) - 2aF(3^n x)\| \to 0$ as $n \to \infty$ for all $a \in A^+_{\frac{1}{2}} \cup \{i\}$ and all $x \in {}_A\mathcal{H} \setminus \{0\}$. Hence

$$T(2ax) = \lim_{n\to\infty} 3^{-n} F(3^n 2ax) = \lim_{n\to\infty} 3^{-n} 2aF(3^n x) = 2aT(x)$$

for all $a \in A^+_{\frac{1}{2}} \cup \{i\}$ and all $x \in {}_A\mathcal{H} \setminus \{0\}$. But $T(2ax) = T(ax + ax) = 2T(ax)$ since T is additive. So $T(ax) = aT(x)$ for all $a \in A^+_{\frac{1}{2}} \cup \{i\}$ and all $x \in {}_A\mathcal{H} \setminus \{0\}$. Since T is $\mathbb{R}$-linear and $T(ax) = aT(x)$ for each $a \in A^+_{\frac{1}{2}} \cup \{i\}$,

$$T(ax) = T(2|a| \cdot \frac{a}{2|a|}x) = 2|a| \cdot T(\frac{a}{2|a|}x) = 2|a| \cdot \frac{a}{2|a|} \cdot T(x) = aT(x)$$

for all positive elements $a \in A \setminus \{0\}$ and all $x \in {}_A\mathcal{H}$. Thus

$$\begin{aligned}T(ax) = & aT(x), \\ T(ix) = & iT(x)\end{aligned}$$

for all positive elements $a \in A$ and all $x \in {}_A\mathcal{H}$.

For any element $a \in A$, $a = a_1 + ia_2$, where $a_1 = \frac{a+a^*}{2}$ and $a_2 = \frac{a-a^*}{2i}$ are self-adjoint elements, furthermore, $a = a_1{}^+ - a_1{}^- + ia_2{}^+ - ia_2{}^-$, where ${a_1}^+$, ${a_1}^-$, ${a_2}^+$, and ${a_2}^-$ are positive elements (see [3, Lemma 38.8]). So

$$\begin{aligned} T(ax) =& T({a_1}^+x - {a_1}^-x + i{a_2}^+x - i{a_2}^-x) \\ =& {a_1}^+T(x) - {a_1}^-T(x) + {a_2}^+T(ix) - {a_2}^-T(ix) \\ =& {a_1}^+T(x) - {a_1}^-T(x) + i{a_2}^+T(x) - i{a_2}^-T(x) \\ =& ({a_1}^+ - {a_1}^- + i{a_2}^+ - i{a_2}^-)T(x) = aT(x) \end{aligned}$$

for all $a \in A$ and all $x \in {}_A\mathcal{H}$. Hence

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in {}_A\mathcal{H}$. So the unique $\mathbb{R}$-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is an A-linear operator satisfying (i), as desired. □

Corollary 2.2. *Let $p < 1$ and $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ a mapping such that*

$$\|2F(ax + \frac{1}{2}y) - 2aF(x) - F(y)\| \le ||x||^p + ||y||^p$$

for all $a \in A^+_{\frac{1}{2}} \cup \{i\}$ and all $x, y \in {}_A\mathcal{H} \setminus \{0\}$. If $F(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{H}$, then there exists a unique A-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ such that

$$\|F(x) - F(0) - T(x)\| \le \frac{3 + 3^p}{3 - 3^p}||x||^p$$

for all $x \in {}_A\mathcal{H} \setminus \{0\}$.

Proof. Define $\varphi : {}_A\mathcal{H} \setminus \{0\} \times {}_A\mathcal{H} \setminus \{0\} \to [0, \infty)$ by $\varphi(x, y) = ||x||^p + ||y||^p$, and apply Lemma 2.1. □

From now on, let $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be mappings such that $F(3^nx) = 3^nF(x)$ and $G(3^nx) = 3^nG(x)$ for all positive integers n and all $x \in {}_A\mathcal{H}$.

Theorem 2.3. *Let $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be mappings such that*

$$\|2F(ax + \frac{1}{2}y) - 2aF(x) - F(y)\| \le \varphi(x, y),$$
$$\|2G(ax + \frac{1}{2}y) - 2aG(x) - G(y)\| \le \varphi(x, y)$$

for all $a \in A^+_{\frac{1}{2}} \cup \{i\}$ *and all* $x, y \in {}_A\mathcal{H} \setminus \{0\}$. *Assume that* $F(tx)$ *and* $G(tx)$ *are continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in {}_A\mathcal{H}$. *Then the mappings* $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *are* A*-linear operators. Furthermore,*

(1) *if the mappings* $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *satisfy the inequalities*

$$\|F \circ G(x) - x\| \le \varphi(x, x),$$
$$\|G \circ F(x) - x\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, *then the mapping* G *is the inverse of the mapping* F,

(2) *if the mapping* $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *satisfies the inequality*

$$\|F(x) - F^*(x)\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, *then the mapping* $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *is a self-adjoint operator,*

(3) *if the mapping* $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *satisfies the inequality*

$$\|F \circ F^*(x) - F^* \circ F(x)\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, *then the mapping* $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *is a normal operator,*

(4) *if the mapping* $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *satisfies the inequalities*

$$\|F \circ F^*(x) - x\| \le \varphi(x, x),$$
$$\|F^* \circ F(x) - x\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, *then the mapping* $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *is a unitary operator, and*

(5) *if the mapping* $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *satisfies the inequalities*

$$\|F \circ F(x) - F(x)\| \le \varphi(x, x),$$
$$\|F^*(x) - F(x)\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, *then the mapping* $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *is a projection.*

Proof. Let $a = \frac{1}{2} \in A^+_{\frac{1}{2}}$ in the statement. By the same reasoning as in the proof of Lemma 2.1, there exists a unique A-linear operator $L : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying

$$\text{(ii)} \qquad \|G(x) - G(0) - L(x)\| \leq \frac{1}{3}(\widetilde{\varphi}(x, -x) + \widetilde{\varphi}(-x, 3x))$$

for all $x \in {}_A\mathcal{H} \setminus \{0\}$. The A-linear operator $L : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is given by

$$L(x) = \lim_{n\to\infty} \frac{G(3^n x)}{3^n}.$$

By the assumption,

$$\begin{aligned} T(x) &= \lim_{n\to\infty} \frac{F(3^n x)}{3^n} = \lim_{n\to\infty} \frac{3^n F(x)}{3^n} = F(x), \\ L(x) &= \lim_{n\to\infty} \frac{G(3^n x)}{3^n} = \lim_{n\to\infty} \frac{3^n G(x)}{3^n} = G(x) \end{aligned}$$

for all $x \in {}_A\mathcal{H}$, where the mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is given in the proof of Lemma 2.1. Hence the A-linear operators T and L are the mappings F and G, respectively. So the mappings $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are A-linear operators.

(1) By the assumption,

$$\begin{aligned} \|F \circ G(3^n x) - 3^n x\| &\leq \varphi(3^n x, 3^n x), \\ \|G \circ F(3^n x) - 3^n x\| &\leq \varphi(3^n x, 3^n x) \end{aligned}$$

for all positive integers n and all $x \in {}_A\mathcal{H}$. Thus

$$\begin{aligned} 3^{-n}\|F \circ G(3^n x) - 3^n x\| &\to 0, \\ 3^{-n}\|G \circ F(3^n x) - 3^n x\| &\to 0 \end{aligned}$$

as $n \to \infty$ for all $x \in {}_A\mathcal{H}$. Hence

$$\begin{aligned} F \circ G(x) &= \lim_{n\to\infty} \frac{F \circ G(3^n x)}{3^n} = x, \\ G \circ F(x) &= \lim_{n\to\infty} \frac{G \circ F(3^n x)}{3^n} = x \end{aligned}$$

for all $x \in {}_A\mathcal{H}$. So the mapping G is the inverse of the mapping F.

(2) By the assumption,

$$\|F(3^n x) - F^*(3^n x)\| \leq \varphi(3^n x, 3^n x)$$

for all positive integers n and all $x \in {}_A\mathcal{H}$. Thus $3^{-n}\|F(3^n x) - F^*(3^n x)\| \to 0$ as $n \to \infty$ for all $x \in {}_A\mathcal{H}$. Hence

$$F(x) = \lim_{n\to\infty} \frac{F(3^n x)}{3^n} = \lim_{n\to\infty} \frac{F^*(3^n x)}{3^n} = F^*(x)$$

for all $x \in {}_A\mathcal{H}$. So the mapping F is a self-adjoint operator.

The proofs of the others are similar to the proofs of (1) and (2). □

Given a locally compact abelian group G and a multiplier ω on G, one can associate to them the twisted group C^*-algebra $C^*(G, \omega)$. $C^*(\mathbb{Z}^m, \omega)$ is said to be a *noncommutative torus of rank* m and denoted by A_ω. The multiplier ω determines a subgroup S_ω of G, called its *symmetry group*, and the multiplier is called *totally skew* if the symmetry group S_ω is trivial. And A_ω is called *completely irrational* if ω is totally skew (see [1]). It was shown in [1] that if G is a locally compact abelian group and ω is a totally skew multiplier on G, then $C^*(G, \omega)$ is a simple C^*-algebra. It was shown in [2, Theorem 1.5] that if A_ω is a completely irrational noncommutative torus, then A_ω has real rank 0, where *"real rank* 0" means that the set of invertible self-adjoint elements is dense in the set of self-adjoint elements (see [4, 6]).

We prove the Hyers-Ulam-Rassias stability of a generalized Jensen's equation in Hilbert module over a unital C^*-algebra of real rank 0.

Theorem 2.4. *Let A have real rank 0, and $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ mappings such that*

$$\|2F(ax + \frac{1}{2}y) - 2aF(x) - F(y)\| \leq \varphi(x, y),$$
$$\|2G(ax + \frac{1}{2}y) - 2aG(x) - G(y)\| \leq \varphi(x, y)$$

for all $a \in (A_{in} \cap A^+_{\frac{1}{2}}) \cup \{i\}$ and all $x, y \in {}_A\mathcal{H} \setminus \{0\}$. Assume that (iii) $F(ax)$ and $G(ax)$ are continuous in $a \in A^+_{\frac{1}{2}} \cup \mathbb{R}$ for each fixed

$x \in {}_A\mathcal{H}$. Then the mappings $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are A-linear operators. Furthermore, the properties, given in the statement of Theorem 2.3, hold.

Proof. By the same reasoning as in the proof of Lemma 2.1, there exists a unique $\mathbb{R}$-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (i), and

$$T(ax) = aT(x) \tag{1}$$

for all $a \in (A_{in} \cap A^{+}_{\frac{1}{2}}) \cup \{i\}$ and all $x \in {}_A\mathcal{H} \setminus \{0\}$.

Let $b \in A^{+}_{\frac{1}{2}} \setminus A_{in}$. Since $A_{in} \cap A_{sa}$ is dense in A_{sa}, there exists a sequence $\{b_m\}$ in $A_{in} \cap A_{sa}$ such that $b_m \to b$ as $m \to \infty$. Put $c_m = \frac{1}{2|b_m|} b_m$. Then $c_m \to \frac{1}{2|b|} b = b$ as $m \to \infty$ and $c_m \in A_{in} \cap A_{sa} \cap A_{\frac{1}{2}}$. Put $a_m = \sqrt{{c_m}^* c_m}$. Then $a_m \to \frac{1}{2|b|} b = b$ as $m \to \infty$ and $a_m \in A_{in} \cap A^{+}_{\frac{1}{2}}$. Thus there exists a sequence $\{a_m\}$ in $A_{in} \cap A^{+}_{\frac{1}{2}}$ such that $a_m \to b$ as $m \to \infty$, and by (iii)

$$\lim_{m\to\infty} T(a_m x) = \lim_{m\to\infty} F(a_m x) = F(\lim_{m\to\infty} a_m x) = F(bx) = T(bx) \tag{2}$$

for all $x \in {}_A\mathcal{H}$. By (1),

$$\|T(a_m x) - bT(x)\| = \|a_m T(x) - bT(x)\| \to \|bT(x) - bT(x)\| = 0 \tag{3}$$

as $m \to \infty$. By (2) and (3),

$$\begin{aligned} \|T(bx) - bT(x)\| \leq & \|T(bx) - T(a_m x)\| + \|T(a_m x) - bT(x)\| \\ \to & 0 \text{ as } m \to \infty \end{aligned} \tag{4}$$

for all $x \in {}_A\mathcal{H}$. By (1) and (4), $T(ax) = aT(x)$ for all $a \in A^{+}_{\frac{1}{2}} \cup \{i\}$ and all $x \in {}_A\mathcal{H}$.

Similarly, one can show that there exists a unique $\mathbb{R}$-linear operator $L : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (ii) such that $L(ax) = aL(x)$ for all $a \in A^{+}_{\frac{1}{2}} \cup \{i\}$ and all $x \in {}_A\mathcal{H}$.

The rest of the proof is the same as in the proofs of Lemma 2.1 and Theorem 2.3. So the mappings $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are A-linear operators, and the properties, given in the statement of Theorem 2.3, hold. □

We prove the Hyers-Ulam-Rassias stability of another generalized Jensen's equation in a Hilbert module over a unital C^*-algebra.

THEOREM 2.5. *Let $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be mappings such that*

$$\|2F(ax + \frac{1}{2}y) - 2aF(x) - F(y)\| \le \varphi(x, y),$$
$$\|2G(ax + \frac{1}{2}y) - 2aG(x) - G(y)\| \le \varphi(x, y)$$

for all $a \in A^+_{\frac{1}{2}} \cup \{i\}$ and all $x, y \in {}_A\mathcal{H}\backslash\{0\}$. Assume that $F(x)$ and $G(x)$ are continuous. Then the mappings $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are bounded A-linear operators. Furthermore, the properties, given in the statement of Theorem 2.3, hold.

Proof. By the same reasoning as in the proof of Theorem 2.3, the mappings $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are A-linear operators.

Since the A-linear operators $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are continuous, the A-linear operators $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are bounded (see [5, Proposition II.1.1]). By the same reasoning as the proof of Theorem 2.3, the properties, given in the statement of Theorem 2.3, hold, as desired. □

COROLLARY 2.6. *Let A have real rank 0, and $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ mappings such that*

$$\|2F(ax + \frac{1}{2}y) - 2aF(x) - F(y)\| \le \varphi(x, y),$$
$$\|2G(ax + \frac{1}{2}y) - 2aG(x) - G(y)\| \le \varphi(x, y)$$

for all $a \in (A_{in} \cap A^+_{\frac{1}{2}}) \cup \{i\}$ and all $x, y \in {}_A\mathcal{H} \setminus \{0\}$. Assume that $F(x)$ and $G(x)$ are continuous. Then the mappings $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are bounded A-linear operators. Furthermore, the properties, given in the statement of Theorem 2.3, hold.

Proof. By the same reasoning as in the proof of Theorem 2.4, the mappings $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are A-linear operators.

By the same reasoning as in the proof of Theorem 2.5, the A-linear operators $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are bounded, and the properties, given in the statement of Theorem 2.3, hold. □

Now we prove the Hyers-Ulam-Rassias stability of another generalized Jensen's equation in a Hilbert module over a unital C^*-algebra.

THEOREM 2.7. *Let $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be mappings such that*

$$\|2F(ax + \frac{1}{2}y) - 2aF(x) - F(y)\| \le \varphi(x, y),$$
$$\|2G(ax + \frac{1}{2}y) - 2aG(x) - G(y)\| \le \varphi(x, y)$$

for all $a \in A^+_{\frac{1}{2}} \cup \{i\} \cup \mathbb{R}^+$ and all $x, y \in {}_A\mathcal{H} \setminus \{0\}$. Then the mappings $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are A-linear operators. Furthermore, the properties, given in the statement of Theorem 2.3, hold.

Proof. By the same reasoning as in the proof of Lemma 2.1, there exist unique additive mappings $T, L : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (i) and (ii), respectively.

By the same method as in the proof of Lemma 2.1, one can show that

$$T(ax) = aT(x)$$

for all $a \in A^+_{\frac{1}{2}} \cup \{i\} \cup \mathbb{R}^+$ and all $x \in {}_A\mathcal{H} \setminus \{0\}$. So $T(ax) = aT(x)$ for all $a \in (A^+ \setminus \{0\}) \cup \{i\}$ and all $x \in {}_A\mathcal{H}$. Since T is additive, $T(x) = T(x - y + y) = T(x - y) + T(y)$ and $T(x - y) = T(x) - T(y)$ for all $x, y \in {}_A\mathcal{H}$. So

$$\begin{aligned} T(ax) &= T({a_1}^+x - {a_1}^-x + i{a_2}^+x - i{a_2}^-x) \\ &= ({a_1}^+ - {a_1}^- + i{a_2}^+ - i{a_2}^-)T(x) \\ &= aT(x) \end{aligned}$$

for all $a \in A$ and all $x \in {}_A\mathcal{H}$, where ${a_1}^+$, ${a_1}^-$, ${a_2}^+$, and ${a_2}^-$ are as defined in the proof of Theorem 2.3. Hence

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in {}_A\mathcal{H}$. So the unique additive mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is an A-linear operator satisfying (i).

Similarly, one can show that the unique additive mapping $L : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is an A-linear operator satisfying (ii).

The rest of the proof is the same as in the proofs of Lemma 2.1 and Theorem 2.3. So the mappings $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are A-linear operators, and the properties, given in the statement of Theorem 2.3, hold. □

Combining Theorem 2.7 and Theorem 2.4 yields the following.

Corollary 2.8. *Let A have real rank 0, and $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ mappings such that*

$$\|2F(ax + \frac{1}{2}y) - 2aF(x) - F(y)\| \le \varphi(x, y),$$
$$\|2G(ax + \frac{1}{2}y) - 2aG(x) - G(y)\| \le \varphi(x, y)$$

for all $a \in (A_{in} \cap A^+_{\frac{1}{2}}) \cup \{i\} \cup \mathbb{R}^+$ and all $x, y \in {}_A\mathcal{H} \setminus \{0\}$. Assume that $F(ax)$ and $G(ax)$ are continuous in $a \in A^+_{\frac{1}{2}}$ for each fixed $x \in {}_A\mathcal{H}$. Then the mappings $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are A-linear operators. Furthermore, the properties, given in the statement of Theorem 2.3, hold.

Proof. By the same method as in the proof of Lemma 2.1, one can show that there exist unique additive mappings $T, L : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (i) and (ii), respectively, and that

$$T(ax) = aT(x)$$

for all $a \in \mathbb{R}^+$ and all $x \in {}_A\mathcal{H} \setminus \{0\}$. Since T is additive, $T(x) = T(x - y + y) = T(x - y) + T(y)$ and $T(x - y) = T(x) - T(y)$ for all $x, y \in {}_A\mathcal{H}$. So the unique additive mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is $\mathbb{R}$-linear.

Similarly, one can show that the unique additive mapping $L : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is $\mathbb{R}$-linear.

The rest of the proof is similar to the proof of Theorem 2.4. So the mappings $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are A-linear operators, and the properties, given in the statement of Theorem 2.3, hold. □

3. Stability of other generalized Jensen's equations in a Hilbert module over a C^*-algebra

In this section, let $\varphi : {}_A\mathcal{H} \setminus \{0\} \times {}_A\mathcal{H} \setminus \{0\} \to [0, \infty)$ be a function such that

$$\widetilde{\varphi}(x, y) := \sum_{k=0}^{\infty} 3^k \varphi(3^{-k}x, 3^{-k}y) < \infty.$$

Lemma 3.1. *Let $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be a mapping such that*

$$\|2F(ax + \frac{1}{2}y) - 2aF(x) - F(y)\| \le \varphi(x, y)$$

for all $a \in A^+_{\frac{1}{2}} \cup \{i\}$ and all $x, y \in {}_A\mathcal{H} \setminus \{0\}$. Assume that $F(x)$ is continuous, and that $\lim_{n\to\infty} 3^n F(3^{-n}x)$ converges uniformly on ${}_A\mathcal{H}$. Then there exists a unique bounded A-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ such that

$$\text{(iv)} \qquad \|F(x) - F(0) - T(x)\| \le \widetilde{\varphi}(\frac{x}{3}, \frac{-x}{3}) + \widetilde{\varphi}(\frac{-x}{3}, x)$$

for all $x \in {}_A\mathcal{H} \setminus \{0\}$.

Proof. Let $a = \frac{1}{2} \in A^+_{\frac{1}{2}}$ in the statement. By [11, Theorem 6], there exists a unique additive mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (iv). The additive mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ was given by

$$T(x) = \lim_{n\to\infty} 3^n F(3^{-n}x)$$

for all $x \in {}_A\mathcal{H} \setminus \{0\}$. The additive mapping T given in the proof of [11, Theorem 6] is similar to the additive mapping T given in the proof of [14, Theorem]. By the same reasoning as the proof of [14, Theorem], it follows from the assumption that $F(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{H}$ that the additive mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is $\mathbb{R}$-linear.

By the same method as in the proof of Lemma 2.1, one can show that

$$T(2ax) = \lim_{n\to\infty} 3^n F(3^{-n}2ax) = \lim_{n\to\infty} 3^n 2aF(3^{-n}x) = 2aT(x)$$

for all $a \in A^+_{\frac{1}{2}} \cup \{i\}$ and all $x \in {}_A\mathcal{H} \setminus \{0\}$, and that the unique $\mathbb{R}$-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is an A-linear operator satisfying (iv). But by the assumption, the A-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is continuous. So the A-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is bounded (see [5, Proposition II.1.1]), as desired. □

Corollary 3.2. *Let $p > 1$ and $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ a mapping such that*

$$\|2F(ax + \frac{1}{2}y) - 2aF(x) - F(y)\| \le ||x||^p + ||y||^p$$

for all $a \in A^+_{\frac{1}{2}} \cup \{i\}$ and all $x, y \in {}_A\mathcal{H} \setminus \{0\}$. Assume that $F(x)$ is continuous, and that $\lim_{n\to\infty} 3^n F(3^{-n}x)$ converges uniformly on ${}_A\mathcal{H}$. Then there exists a unique bounded A-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ such that

$$\|F(x) - F(0) - T(x)\| \le \frac{3^p + 3}{3^p - 3}||x||^p$$

for all $x \in {}_A\mathcal{H} \setminus \{0\}$.

Proof. Define $\varphi : {}_A\mathcal{H} \setminus \{0\} \times {}_A\mathcal{H} \setminus \{0\} \to [0, \infty)$ by $\varphi(x, y) = ||x||^p + ||y||^p$, and apply Lemma 3.1. □

Under the assumption that $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are mappings such that $F(3^{-n}x) = \frac{F(x)}{3^n}$ and $G(3^{-n}x) = \frac{G(x)}{3^n}$ for all positive integers n and all $x \in {}_A\mathcal{H}$, one can obtain similar results to Theorems 2.3, 2.4, 2.5, 2.7 and Corollaries 2.6, 2.8.

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