Kangweon-Kyungki Math. Jour. **15** (2007), No. 2, pp. 149–165

AN APPLICATION OF CRITICAL POINT THEORY TO THE NONLINEAR HYPERBOLIC SYSTEM

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Abstract. We investigate the existence of multiple nontrivial solutions $u(x,t)$ for a perturbation $b[(\xi-\eta+2)^+-2]$ of the hyperbolic system with Dirichlet boundary condition

$$\begin{aligned} &L\xi=\mu[(\xi-\eta+2)^+-2] \quad \text{in} \quad (-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R},\\ &L\eta=\nu[(\xi-\eta+2)^+-2] \quad \text{in} \quad (-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}, \end{aligned} \tag{0.1}$$

where $u^+=\max\{u,0\}$, μ,ν are nonzero constants. Here L is the wave operator in $\mathbb{R}^2$ and the nonlinearity $(\mu-\nu)[(\xi-\eta+2)^+-2]$ crosses the eigenvalues of the wave operator.

1. Introduction

Let L be the wave operator in $\mathbb{R}^2$, $Lu=u_{tt}-u_{xx}$. In this paper we investigate the existence of solutions $u(x,t)$ for a perturbation $b[(\xi+\eta+1)^+-1]$ of the hyperbolic system with Dirichlet boundary condition

$$\begin{aligned} &L\xi=\mu[(\xi-\eta+2)^+-2] \quad \text{in} \quad (-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R},\\ &L\eta=\nu[(\xi-\eta+2)^+-2] \quad \text{in} \quad (-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R},\\ &\xi(\pm\frac{\pi}{2},t)=0, \quad \xi(x,t+\pi)=\xi(x,t)=\xi(-x,t),\\ &\eta(\pm\frac{\pi}{2},t)=0, \quad \eta(x,t+\pi)=\eta(x,t)=\eta(-x,t), \end{aligned} \tag{1.1}$$

where $u^+=\max\{u,0\}$, μ,ν are nonzero constants and the nonlinearity $(\mu-\nu)[(\xi-\eta+2)^+-2]$ crosses the eigenvalues of the wave operator.

Received September 10, 2007.

2000 Mathematics Subject Classification: 35B10, 35L05, 35L20.

Key words and phrases: Hyperbolic system, eigenvalue problem, Dirichlet boundary condition, uniqueness.

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The following type nonlinear equation with Dirichlet boundary condition was studied by many authors.

$$
\begin{aligned}
&u_{tt} - u_{xx} = b[(u+2)^+ - 2] \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}, \\
&u(\pm\frac{\pi}{2}, t) = 0, \quad u(x, t+\pi) = u(x,t) = u(-x,t).
\end{aligned} \tag{1.2}
$$

In [6] Lazer and McKenna point out that this kind of nonlinearity $b[(u+2)^+ - 2]$ can furnish a model to study traveling waves in suspension bridges. So the nonlinear equation with jumping nonlinearity have been extensively studied by many authors. For fourth elliptic equation Tarantello [11] , Micheletti and Pistoia [8][9] proved the existence of nontrivial solutions used degree theory and critical points theory separately. For one-dimensional case Lazer and McKenna [7] proved the existence of nontrivial solution by the global bifurcation method. For this jumping nonlinearity we are interest in the multiple nontrivial solutions of the equation. Here we used variational reduction method to find the nontrivial solutions of problem (1.2).

The organization of this paper is as following. In section 2, we investigate some properties of the Hilbert space spanned by eigenfunctions of the wave operator. We show that only the trivial solution exists for the steady state problem of (1.2) when $-3 < b < 1$. In section 3, we investigate the existence of multiple solutions of (1.2), by using critical point theory, when $-7 < b < -3$. In section 4, we investigate the existence of multiple nontrivial solutions $u(x,t)$ for the hyperbolic system with Dirichlet boundary condition when the nonlinearity $(\mu - \nu)[(\xi - \eta + 2)^+ - 2]$ crosses the eigenvalues of the wave operator.

2. Trivial solution problem

Let L be the wave operator in $\mathbb{R}^2$, i.e., $Lu = u_{tt} - u_{xx}$. The eigenvalue problem

$$
\begin{aligned}
&Lu = \lambda u \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}, \\
&u(\pm\frac{\pi}{2}, t) = 0, \quad u(x, t+\pi) = u(x,t) = u(-x,t),
\end{aligned} \tag{2.1}
$$

has infinitely many eigenvalues $\lambda_{mn} = (2n+1)^2 - 4m^2$ $(m, n = 0, 1, 2, ...)$ and corresponding normalized eigenfunctions $\phi_{mn}, \psi_{mn} (m, n \geq 0)$ given by

$$\begin{aligned}
\phi_{0n} &= \frac{\sqrt{2}}{\pi}\cos(2n+1)x \quad \text{for} \quad n \geq 0,\\
\phi_{mn} &= \frac{2}{\pi}\cos 2mt \cdot \cos(2n+1)x \quad \text{for} \quad m > 0, \quad n \geq 0,\\
\psi_{mn} &= \frac{2}{\pi}\sin 2mt \cdot \cos(2n+1)x \quad \text{for} \quad m > 0, \quad n \geq 0.
\end{aligned}$$

Let n be fixed and define

$$\begin{aligned}
\lambda_n^+ &= \inf_m\{\lambda_{mn} : \lambda_{mn} > 0\} = 4n+1,\\
\lambda_n^- &= \sup_m\{\lambda_{mn} : \lambda_{mn} < 0\} = -4n-3.
\end{aligned}$$

Then we obtain that $\lim_{n\to\infty} \lambda_n^+ = +\infty$, $\lim_{n\to-\infty} \lambda_n^- = -\infty$. Thus it is easy to check that the only eigenvalues in the interval (-15, 9) are given by

$$\lambda_{32} = -11 < \lambda_{21} = -7 < \lambda_{10} = -3 < \lambda_{00} = 1 < \lambda_{11} = 5.$$

Let Ω be the square $(-\pi/2, \pi/2) \times (-\pi/2, \pi/2)$ and H_0 the Hilbert space defined by

$$H_0 = \{u \in L^2(\Omega) : u \text{ is even in } x\}.$$

The set of functions $\{\phi_{mn}, \psi_{mn}\}$ is an orthonormal basis in H_0. Let us denote an element u in H_0 as

$$u = \sum (h_{mn}\phi_{mn} + k_{mn}\psi_{mn}),$$

and we define a subspace H of H_0 as

$$H = \{u \in H_0 : \sum |\lambda_{mn}|(h_{mn}^2 + k_{mn}^2) < \infty\}.$$

Then this is a complete normed space with a norm

$$\|u\|_H = [\sum |\lambda_{mn}|(h_{mn}^2 + k_{mn}^2)]^{\frac{1}{2}}.$$

Since $|\lambda_{mn}| \geq 1$ for all m, n, we have that

(i) $\|u\|_H \geq \|u\|$, where $\|u\|$ denotes the L^2 norm of u,

(ii) $\|u\| = 0$ if and only if $\|u\|_H = 0$.

Define $L_\beta u = Lu + \beta u$. Then we have the following lemma.

LEMMA 2.1. *Let $\beta \in \mathbb{R}$, $\beta \neq -\lambda_{mn}$ $(m, n \geq 0)$. Then we have:*

L_β^{-1} is a bounded linear operator from H_0 into H.

Proof. Suppose that $\beta \neq -\lambda_{mn}$. Since $\lim_{n\to\infty} \lambda_n^+ = +\infty$, $\lim_{n\to-\infty} \lambda_n^- = -\infty$, we know that the number of elements in the set $\{\lambda_{mn} : |\lambda_{mn}| < |\beta|\}$ is finite, where λ_{mn} is an eigenvalue of L. Let

$$u = \sum(h_{mn}\phi_{mn} + k_{mn}\psi_{mn}).$$

Then

$$L_\beta^{-1}u = \sum(\frac{1}{\lambda_{mn}+\beta}h_{mn}\phi_{mn} + \frac{1}{\lambda_{mn}+\beta}k_{mn}\psi_{mn}).$$

Hence we have the inequality

$$\|L_\beta^{-1}u\|_H = \sum \frac{|\lambda_{mn}|}{|\lambda_{mn}+\beta|^2}(h_{mn}^2 + k_{mn}^2) \leq C\sum(h_{mn}^2 + k_{mn}^2)$$

for some $C > 0$, which means that

$$\|L_\beta^{-1}u\|_H \leq C_1\|u\|, C_1 = \sqrt{C}.$$

So L_β^{-1} is a bounded linear operator from H_0 to H. □

Theorem 2.1. *Let $-3 < b < 1$. Then equation*

$$Lu = b[(u+2)^+ - 2]$$

has only the trivial solution in H_0.

Proof. Since $\lambda_{10} = -3$ and $\lambda_{00} = 1$, let $\beta = -\frac{1}{2}(\lambda_{00}+\lambda_{10}) = -\frac{1}{2}(-3+1) = 1$. The equation is equivalent to

$$u = (L+\beta)^{-1}[(b+\beta)(u+2)^+ - \beta(u+2)^- - (b+\beta)], \tag{2.2}$$

where we use the equality $u = u^+ - u^-$.

By lemma 2.1 $(L+\beta)^{-1}$ is a compact linear map from H_0 into H_0. Therefore its L^2 norm $\frac{1}{2}$. We note that

$$\begin{aligned} & \|(b+\beta)[(u_1+2)^+ - (u_2+2)^+] - \beta[(u_1+2)^- - (u_2+2)^-]\| \\ \leq\ & \max\{|b+\beta|, |\beta|\}\|u_1 - u_2\| \\ <\ & \frac{1}{2}(\lambda_{00} - \lambda_{10})\|u_1 - u_2\| \\ =\ & 2\|u_1 - u_2\| \end{aligned}$$

where we used the inequality $|s_1^+ - s_2^+| + |s_1^- - s_2^-| \leq |s_1 + s_2|$.

So the right hand side of (2.2) defines a Lipschitz mapping of H_0 into H_0 with Lipschitz constant $\gamma < 1$. Therefore, by the contraction mapping principle, there exists a unique solution $u \in H_0$. Since $u \equiv 0$ is a solution of equation (2.2), $u \equiv 0$ is the unique solution. □

3. Critical point theory and nontrivial solutions

In this section we investigate the existence of multiple solutions of (1.2) when $-7 < b < -3$. Then we define a functional on H by

$$J(u) = \int_\Omega [\frac{1}{2}(-|u_t|^2 + |u_x|^2) - \frac{b}{2}|(u+2)^+|^2 + bu]dxdt.$$

So J is well-defined in H and the solutions of (1.2) coincide with the critical points of $J(u)$. Now we investigate the property of functional J.

Lemma 3.1. **(cf.**[4]**)** *$J(u)$ is continuous and Frechet differentiable at each $u \in H$ with*

$$DJ(u)v = \int_\Omega (Lu - b(u+2)^+ + b)vdxdt,\ v \in H.$$

We shall use a variational reduction method to apply the mountain pass theorem.

Let $V =$ closure of *span*$\{\phi_{10}, \psi_{10}\}$ be the two-dimensional subspace of H. Both of them have the same eigenvalue λ_{10}. Then $\|v\|_H = \sqrt{3}\|v\|$ for $v \in V$. Let W be the orthogonal complement of V in H. Let $P : H \to V$ denote that of H onto V and $I - P : H \to W$ denote that of H onto W. Then every element $u \in H$ is expressed by

$$u = v + w,$$

where $v = Pu$, $w = (I-P)u$.

Lemma 3.2. *Let $-7 < b < -3$ and let $v \in V$ be given. Then we have: there exists a unique solution $z \in W$ of equation*

$$Lz + (I-P)[-b(v+z+2)^+ + b] = 0 \text{ in } W. \tag{3.1}$$

Let $z = \theta(v)$, then θ satisfies a uniform Lipschitz continuous on V with respect to the L^2 norm(also the norm $\|\cdot\|_H$).

Proof. Choose $\beta = 3$ and let $g(\xi) = (b+\beta)(\xi+2)^+ - \beta(\xi+2)^-$. Then equation (3.1) can be written as

$$z = (L+\beta)^{-1}(I-P)[g(v+z) - (b+\beta)]. \tag{3.2}$$

Since $(L+\beta)^{-1}(I-P)$ is a self-adjoint, compact, linear map from $(I-P)H$ into itself, the eigenvalues of $(L+\beta)^{-1}(I-P)$ in W are $(\lambda+\beta)^{-1}$, where $\lambda_{mn} > 1$ or $\lambda_{mn} \le -7$. Therefore $\|(L+\beta)^{-1}(I-P)\|$is $\frac{1}{4}$. Since

$$g(\xi_1) - g(\xi_2) \le \max\{|b+\beta|, |\beta|\}|\xi_1 - \xi_2| < 4|\xi_1 - \xi_2|,$$

the right-hand side of equation (3.2) defines a Lipschitz mapping if $(I-P)H_0$ into itself for fixed $v \in V$. By the contraction mapping principle there exists a unique $z \in (I-P)H_0$ (also $z \in (I-P)H$) for fixed $v \in V$. Since $(L+\beta)^{-1}$ is bounded from H to W there exsits a unique solution $z \in W$ of (3.1) for given $v \in V$.

Let

$$\gamma = \frac{\max\{|b+\beta|, |\beta|\}}{4}.$$

Then $0 < \gamma < 1$. If $z_1 = \theta(v_1)$ and $z_2 = \theta(v_2)$ for any v_1, $v_2 \in V$, then

$$\begin{aligned} \|z_1 - z_2\| &\leq \|(L+\beta)^{-1}(I-P)\| \, \|(g(v_1+z_1) - g(v_2+z_2))\| \\ &\leq \frac{1}{4} \cdot 4\gamma \|(v_1 + z_1 + 2 - (v_2 + z_2 + 2)\| \\ &\leq \gamma(\|v_1 - v_2\| + \|z_1 - z_2\|). \end{aligned}$$

Hence

$$\|z_1 - z_2\| \leq \frac{\gamma}{1-\gamma}\|v_1 - v_2\|.$$

Since $\|(L+\beta)^{-1}(I-P)\|_H \leq \frac{1}{\sqrt{2}}\|u\|$,

$$\begin{aligned} \|z_1 - z_2\|_H &= \|(L+\beta)^{-1}(I-P)(g(v_1+z_1) - g(v_2+z_2))\|_H \\ &\leq \frac{4}{\sqrt{2}}(\|z_1 - z_2\| + \|v_1 - v_2\|) \\ &\leq \frac{4}{\sqrt{6}}(\frac{1}{1-\gamma})\|v_1 - v_2\|_H. \end{aligned}$$

Therefore θ is continuous on V with norm $\|\cdot\|$ and $\|\cdot\|_H$. □

LEMMA 3.3. *If $\tilde{J} : V \to \mathbb{R}$ is defined by $\tilde{J}(v) = J(v + \theta(v))$, then $\tilde{J}$ is a continuous Frechet derivative $D\tilde{J}$ with respect to V and*

$$D\tilde{J}(v)s = DJ(v + \theta(v))(s) \text{ for all } s \in V.$$

If v_0 is a critical point of $\tilde{J}$, then $v_0 + \theta(v_0)$ is a solution of (1.2) and conversely every solution of (1.2) is of this form.

Proof. Let $v \in V$ and set $z = \theta(v)$. If $w \in W$, then from (3.1)

$$\int_\Omega -\theta(v)_t w_t + \theta(v)_x w_x - b(v + \theta(v) + 2)^+ w + bwdtdx = 0.$$

Since $\int_\Omega v_t w_t = 0$ and $\int_\Omega v_x w_x = 0$,

$$DJ(v + \theta(v))(w) = 0 \quad \text{for all} \quad w \in W.$$

Let W_1, W_2 be the two subspaces of H as defining following:

$$\begin{aligned} W_1 &= \text{closure of span}\{\phi_{mn}, \psi_{mn} | \lambda_{mn} \leq -7\}, \\ W_2 &= \text{closure of span}\{\phi_{mn}, \psi_{mn} | \lambda_{mn} \geq 1\}. \end{aligned}$$

Given $v \in V$ and consider the function $h : W_1 \times W_2 \rightarrow$ defined by

$$h(w_1, w_2) = J(v + w_1 + w_2).$$

The function h has continuous partial Fréchet derivatives $D_1 h$ and $D_2 h$ with respect to its first and second variables given by

$$D_1 h(w_1, w_2)(y_1) = DJ(v + w_1 + w_2)(y_1) \text{ for } y_1 \in W_1,$$

$$D_2 h(w_1, w_2)(y_2) = DJ(v + w_1 + w_2)(y_2) \text{ for } y_2 \in W_2.$$

Therefore let $\theta(v) = \theta_1(v) + \theta_2(v)$ with $\theta_1(v) \in W_1$ and $\theta_2(v) \in W_2$. Then by Lemma 3.2

$$\begin{aligned} D_1 h(\theta_1(v), \theta_2(v))(y_1) = 0, &\quad \text{for } y_1 \in W_1 \\ D_2 h(\theta_1(v), \theta_2(v))(y_2) = 0, &\quad \text{for } y_2 \in W_2. \end{aligned} \tag{3.3}$$

If $w_2, y_2 \in W_2$ and $w_1 \in W_1$, then

$$\begin{aligned} & [Dh(w_1, w_2) - Dh(w_1, y_2)](w_2 - y_2) \\ = \; & (DJ(v + w_1 + w_2) - DJ(v + w_1 + y_2))(w_2 - y_2) \\ = \; & \int_\Omega -|(w_2 - y_2)_t|^2 + |(w_2 - y_2)_x^2| - b[(v + w_1 + w_2 + 2)^+ \\ & -(v + w_1 + y_2 + 2)^+](w_2 - y_2) dt dx. \end{aligned}$$

Since $(s^+ - t^+))(s - t) \geq 0$ for any $s, t \in \mathbb{R}$ and $-7 < b < -3$, it is easy to know that

$$\int_\Omega -b[(v + w_1 + w_2 + 2)^+ - (v + w_1 + y_2 + 2)^+](w_2 - y_2) dx dt \geq 0.$$

And

$$\int_\Omega [-|(w_2 - y_2)_t|^2 + (w_2 - y_2)_x^2] dt dx = \|w_2 - y_2\|_H^2,$$

it follows that

$$(Dh(w_1, w_2) - Dh(w_1, y_2))(w_2 - y_2) \geq \|w_2 - y_2\|_H^2.$$

Therefore, h is strictly convex with respect to the second variable. Similarly, using the fact that $-b(s^+ - t^+)(s - t) \leq -b(s - t)^2$ for any $s, t \in \mathbb{R}$,if

w_1 and y_1 are in W_1 and $w_2 \in W_2$, then

$$\begin{aligned} & (D_1h(w_1, w_2) - D_1h(y_1, w_2))(w_1 - y_1) \\ \le\ & -\|w_1 - y_1\|_H^2 + b\|w_1 - y_1\|^2 \\ \le\ & (-1 - \frac{b}{7})\|w_1 - y_1\|_H^2, \end{aligned}$$

where $-7 < b < -3$. Therefore, h is strictly concave with respect to the first variable. From equation (3.3) it follows that

$$J(v + \theta_1(v) + \theta_2(v)) \le J(v + \theta_1(v) + y_2) \text{ for any } y_2 \in W_2,$$

$$J(v + \theta_1(v) + \theta_2(v)) \ge J(v + y_1 + \theta_2(v)) \text{ for any } y_1 \in W_1,$$

with equality if and only if $y_1 = \theta_1(v), y_2 = \theta_2(v)$.

Since h is strictly concave (convex) with respect to its first (second) variable, Theorem 2.3 of [1] implies that $\tilde{J}$ is C^1 with respect to v and

$$D\tilde{J}(v)(s) = DJ(v + \theta(v))(s), \quad \text{any} \quad s \in V. \tag{3.4}$$

Suppose that there exists $v_0 \in V$ such that $D\tilde{J}(v_0) = 0$. From (3.4) it follows that $DJ(v_0 + \theta(v_0))(v) = 0$ for all $v \in V$. Then by Lemma 3.2 it follows that $DJ(v_0+\theta(v_0))v = 0$ for any $v \in H$. Therefore, $u = v_0+\theta(v_0)$ is a solution of (1.2).

Conversely if u is a solution of (1.2) and $v_0 = Pu$, then $D\tilde{J}(v_0)v = 0$ for any $v \in H$. □

Lemma 3.4. *Let $-7 < b < -3$. Then there exists a small open neighborhood B of 0 in V such that $v = 0$ is a strict local minimum of $\tilde{J}$.*

Proof. Since $-7 < b < -3$, problem (1.2) has a trivial solution $u_0 = 0$ by theorem 2.1. Then we have $0 = u_0 = v + \theta(v)$. Since the subspace W is orthogonal complement of subspace V, we get $v = 0$ and $\theta(v) = 0$. Furthermore $\theta(0)$ is the unique solution of equation (3.1) in W for $v = 0$. Thus the trivial solution u_0 is of the form $u_0 = 0+\theta(0)$ and $I+\theta$, where I is an identity map on V, is continuous, it follows that there exists a small open neighborhood B of 0 in V such that if $v \in B$ then $v+\theta(v)+2 > 0$. By Lemma 3.2, $\theta(0) = 0$ is the solution of (3.2) for any $v \in B$ Therefore, if $v \in B$, then for $z = \theta(v)$ we have $z = 0$. Thus

$$\begin{aligned} & \tilde{J}(v) = J(v+z) \\ = & \int_\Omega [\frac{1}{2}(-|(v+z)_t|^2 + |(v+z)_x|^2) - \frac{b}{2}|(v+z+2)^+|^2 \\ & +b(v+z)]dtdx \\ = & \int_\Omega [\frac{1}{2}(-|v_t|^2 + |v_x|^2) - \frac{b}{2}(v+2)^2 + bv]dtdx \\ = & \int_\Omega [\frac{1}{2}(-|v_t|^2 + |v_x|^2) - \frac{b}{2}v^2 - \frac{b}{2}]dtdx, \end{aligned}$$

If $v \in V$, then $Lv = -3v$.Therefore in B,

$$\begin{aligned} \tilde{J}(v) &= \tilde{J}(v) - \tilde{J}(0) \\ &= \int_\Omega [\frac{1}{2}(-|v_t|^2 + |v_x|^2) - \frac{b}{2}v^2]dtdx \\ &= \frac{1}{2}(-3-b)\int_\Omega v^2 dtdx \geq 0, \end{aligned}$$

where $\tilde{J}(0) = \int_\Omega -\frac{b}{2}dtdx$and $-7 < b < -3$. It follows that $v = 0$ is a strict local point of minimum of $\tilde{J}$. □

PROPOSITION 1. *If $-7 < b < 1$, then the equation $Lu - bu^+ = 0$ admits only the trivial solution $u = 0$ in H_0.*

Proof. $H_1 = span\{\cos x \cos 2mt, m \geq 0\}$ is invariant under L and under the map $u \mapsto bu^+$. So the spectrum σ_1 of L retracted to H_1 contains $\lambda_{10} = -3$ in $(-7, 1)$. the spectrum σ_2 of L retracted to $H_2 = H_1^\perp$ contains $\lambda_{10} = -3$ in $(-7, 1)$. From the symmetry theorem in [5] , any solution $y(t)cosx$ of this equation satisfies $y'' + y - by^+ = 0$.This nontrivial periodic solution is periodic with periodic $\pi + \frac{\pi}{\sqrt{-b+1}} \neq \pi$. This shows that there is no nontrivial solution of $Lv - bv^+ = 0$. □

LEMMA 3.5. *Let $-7 < b < -3$. Then function $\tilde{J}$, defined on V, satisfies the Palais-Smale condition.*

Proof. Let $\{v_n\} \subset V$ be a Palais-Smale sequence that is $\tilde{J}(v_n)$ is bounded and $D\tilde{J}(v_n) \to 0$ in V. since V is two-dimensional it is enough to prove that $\{v_n\}$ is bounded in V.

Let u_n be the solution of (1.2) with $u_n = v_n + \theta(v_n)$ where $v_n \in V$. So

$$Lu_n - b(u_n + 2)^+ + b = DJ(u_n) \quad \text{in} \quad H.$$

By contradiction we suppose that $\|v_n\| \to +\infty$, also $\|u_n\| \to +\infty$. Dividing by $\|u_n\|$ and taking $w_n = \frac{u_n}{\|u_n\|}$ we get

$$Lw_n - b(w_n + \frac{1}{\|u_n\|})^+ + \frac{b}{\|u_n\|} = \frac{(DJ(u_n))}{\|u_n\|} \to 0. \tag{3.5}$$

Since $\|w_n\| = 1$ we get : $w_n \to w_0$ weakly in H_0. By L^{-1} is a compact operator, passing to a subsequence we get : $w_n \to w_0$ strongly in H_0. Taking the limit of both sides of (3.5), it follows

$$Lw_0 - bw_0^+ = 0,$$

with $\|w_0\| \neq 0$. This contradicts to the fact that for $-7 < b < -3$ the following equation

$$Lu - bu^+ = 0 \quad \text{in} \quad H_0$$

has only the trivial solution by Proposition 1. Hence$\{v_n\}$ is bounded in V. □

We now define the functional on H

$$J^*(u) = \int_\Omega [-\frac{1}{2}(-|u_t|^2 + |u_x|^2) - \frac{b}{2}|u^+|^2 dxdt.$$

The critical points of $J^*(u)$ coincide with solutions of the equation

$$Lu - bu^+ = 0 \quad \text{in} \quad H_0$$

The above equation has only the trivial solution and hence $J^*(u)$ has only one critical point $u = 0$.

Given $v \in V$, let $\theta^*(v) = \theta(v) \in W$ be the unique solution of the equation

$$Lz + (I - P)[-b(v + z + 2)^+ + b] = 0 \text{ in } W,$$

where $-7 < b < -3$. Let us define the reduced functional $\tilde{J}^*(v)$ on V by $J(v + \theta^*(v))$. We note that we can obtain the same results as Lemma 3.1 and Lemma 3.2 when we replace $\theta(v)$ and $\tilde{J}(v)$ by $\theta^*(v)$ and $\tilde{J}^*(v)$. We also note that, for $-7 < b < -3$, $\tilde{J}^*(v)$ has only one critical point $v = 0$.

LEMMA 3.6. *Let $-7 < b < -3$. Then we have: $\tilde{J}^*(u) < 0$ for all $v \in V$ with $v \neq 0$.*

The proof of this lemma can be found in [4].

Lemma 3.7. *Let $-7 < b < -3$. Then we have*

$$\lim_{\|v\|\to\infty} \tilde{J}(v) \to -\infty$$

for all $v \in V$ (certainly for also the norm $\|\cdot\|_H$).

Proof. Suppose that it is not true that

$$\lim_{\|v\|\to\infty} \tilde{J}(v) \to -\infty.$$

Then there exists a sequence (v_n) in V and a constant C such that

$$\lim_{n\to\infty} \|v_n\| \to \infty$$

and

$$\begin{aligned} \tilde{J}(v_n) \ &= \ \int_\Omega (\frac{1}{2}L(v_n+\theta(v_n))\cdot(v_n+\theta(v_n)) - \frac{b}{2}|(v_n+\theta(v_n)+1)^+|^2 \\ &\quad +b(v_n+\theta(v_n))dtdx \geq C. \end{aligned}$$

For given $v_n \in V$ let $w_n = \theta(v_n)$ be the unique solution of the equation

$$Lw + (I-P)[-b(v_n+w+2)^+ + b] = 0 \qquad \text{in} \quad W. \tag{3.6}$$

Let $z_n = v_n + w_n$, $v_n^* = \dfrac{v_n}{\|v_n\|}$, $w_n^* = \dfrac{w_n}{\|v_n\|}$. Then $z_n^* = v_n^* + w_n^*$. By dividing $\|v_n\|$ we have

$$w_n^* = L^{-1}(I-P)\left(b\left(\frac{v_n+w_n+2}{\|v_n\|}\right)^+ - \frac{b}{\|v_n\|}\right) \quad \text{in} \quad W.$$

By lemma 3.2 $w_n = \theta(v_n)$ is Lipschitz continuous on V. So sequence $\left\{\dfrac{w_n+v_n}{\|v_n\|}\right\}$ is bounded in H. Since $\lim\limits_{n\to\infty}\dfrac{1}{\|v_n\|} = 0, \lim\limits_{n\to\infty}\dfrac{b}{\|v_n\|} = 0$, it follows that $b\left(\dfrac{v_n+w_n+2}{\|v_n\|}\right)^+ - \dfrac{b}{\|v_n\|}$ is bounded in H. Since L^{-1} is a compact operator there is a subsequence of w_n^* converge to some w^* in W, denote by itself. Since V is two-dimensional space, assume that sequence (v_n^*) converges to $v^* \in V$ with $\|v^*\| = 1$. Therefore, we can get that sequence (z_n^*) converges to an element z^* in H.

On the other hand, since $\tilde{J}(v_n) \geq C$, dividing this inequality by $\|v_n\|^2$, we get

$$\int_\Omega \frac{1}{2}L(z_n^*)\cdot z_n^* - \frac{b}{2}((z_n^* + \frac{2}{\|v_n\|})^+)^2 + b\frac{z_n^*}{\|v_n\|}dtdx \geq \frac{C}{\|v_n\|^2}. \tag{3.7}$$

By lemma 3.2 it follows that for any $y \in W$

$$\int_\Omega [-(z_n)_t y_t + (z_n)_x y_x - b(z_n+2)^+ y + by]dtdx = 0. \tag{3.8}$$

If we set $y = w_n$ in (3.8) and divide by $\|v_n\|^2$, then we obtain

$$\int_\Omega [-|(w_n^*)_t|^2 + |(w_n^*)_x|^2 - b(z_n^*)^+ w_n^* + \frac{b}{\|v_n\|} w_n^*]dtdx = 0. \tag{3.9}$$

Let $y \in W$ be arbitrary. Dividing (3.8) by $\|v_n\|$ and letting $n \to \infty$, we obtain

$$\int_\Omega [-(z^*)_t y_t + (z^*)_x y_x - b(z^*)^+ y]dtdx = 0. \tag{3.10}$$

Then (3.10) can be written in the form $D\tilde{J}^*(v^* + w^*)(y) = 0$ for all $y \in W$. Hence by $w^* = \theta(v^*)$. Letting $n \to \infty$ in (3.9), We obtain

$$\begin{aligned}
& \lim_{n\to\infty} \int_\Omega (-|(w_n^*)_t|^2 + |(w_n^*)_x|^2)dtdx \\
= & \lim_{n\to\infty} \int_\Omega b(z_n^*)^+ w_n^* - \frac{b}{\|v_n\|} w_n^* dtdx \\
= & \int_\Omega b(z^*)^+ w^* dtdz \\
= & \int_\Omega (-(z^*)_t (w^*)_t + (z^*)_x (w^*)_x)dtdx \\
= & \int_\Omega (-|(w^*)_t|^2 + |(w^*)_x|^2)dtdx,
\end{aligned}$$

where we have used (3.10). Hence

$$\lim_{n\to\infty} \int_\Omega [-|(z_n^*)_t|^2 + |(z_n^*)_x|^2]dtdx = \int_\Omega [-|(z^*)_t|^2 + |(z^*)_x|^2]dtdx.$$

Letting $n \to \infty$ in (3.7), we obtain

$$\tilde{J}^*(v^*) = \int_\Omega [\frac{1}{2}(-|(z^*)_t|^2 + |(z^*)_x|^2) + \frac{b}{2}|(z^*)^+|^2]dtdx \geq 0.$$

Since $\|v^*\| = 1$, this contradicts to the fact that $\tilde{J}^*(v) < 0$ for all $v \neq 0$. This proves that $\lim_{\|v\|\to\infty} \tilde{J}(v) \to -\infty$. □

Now we state the main result in this paper:

THEOREM 3.1. *Let $-7 < b < -3$. Then the equation*

$$\begin{aligned} u_{tt} - u_{xx} &= b[(u+2)^+ - 2] \quad in \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}, \\ u(\pm\frac{\pi}{2}, t) &= 0, \quad u(x, t+\pi) = u(x,t). \end{aligned}$$

has at least three solutions, two of which are nontrivial solutions.

Proof. We know that $u = 0$ is the trivial solution of problem (1.2). Then $v = 0$ is a critical point of functional $\tilde{J}$. Next we want to find others critical points of $\tilde{J}$ which are corresponding to the solutions of problem (1.2).

By Lemma 3.4, there exists a small open neighborhood B of 0 in V such that $v = 0$ is a strict local point of minimum of $\tilde{J}$. Since $\lim_{\|v\|_H \to \infty} \tilde{J}(v) \to -\infty$ from lemma 3.7 and V is a two-dimensional space, there exists a critical point $v_0 \in V$ of $\tilde{J}$ such that

$$\tilde{J}(v_0) = \max_{v \in V} \tilde{J}(v).$$

Let B_{v_0} be an open neighborhood of v_0 in V such that $B \cap B_{v_0} = \varnothing$. Since $\lim_{\|v\|_H \to \infty} \tilde{J}(v) \to -\infty$, we can choose $v_1 \in V \backslash (B \cup B_{v_0})$ such that $\tilde{J}(v_1) < \tilde{J}(0)$. Since $\tilde{J}$ satisfies the Palais-Smale condition, by the Mountain Pass Theorem, there is a critical value

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} \tilde{J}(v)$$

where $\Gamma = \{\gamma \in C([0,1], E) | \gamma(0) = 0, \gamma(1) = v_0\}$.

If $\tilde{J}(v_0) \neq c$, then there exists a critical point v of $\tilde{J}$ at level c such that $v \neq v_0$, 0 (since $c \neq \tilde{J}(v_0)$ and $c > \tilde{J}(0)$). Therefore, in case $\tilde{J}(v_0) \neq c$, equation (1.2) has also at least 3 critical points $0, v_0, v$.

If $\tilde{J}(v_0) = c$, then define

$$c' = \inf_{\gamma \in \Gamma'} \sup_{\gamma} \tilde{J}(v)$$

where $\Gamma' = \{\gamma \in \Gamma : \gamma \cap B_{v_0} = \varnothing\}$. Hence

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} \tilde{J}(v) \leq \inf_{\gamma \in \Gamma'} \sup_{\gamma} \tilde{J}(v) \leq \max_{v \in V} \tilde{J}(v) = c.$$

That is $c = c'$. By contradiction assume $K_c = \{v \in V | \tilde{J}(v) = c, D\tilde{J}(v) = 0\} = \{v_0\}$. Use the functional $\tilde{J}$ for the deformation theorem (theorem

4.1) and taking $\epsilon < \frac{1}{2}(c - \tilde{J}(0))$. We choose $\gamma \in \Gamma'$ such that $\sup_\gamma \tilde{J} \leq c$. From the deformation theorem (theorem 4.1) $\eta(1,\cdot) \circ \gamma \in \Gamma$ and

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} \tilde{J}(v) \leq \sup_{\eta(1,\cdot)\circ\gamma} \tilde{J}(v) \leq c - \epsilon,$$

which is a contradiction. Therefore, there exists a critical point v of $\tilde{J}$ at level c such that $v \neq v_0$, 0, which means that the equation (1.2) has at least three critical points. Since $\|v\|_H, \|v_0\|_H \neq 0$, these two critical points coincide with two nontrivial period solutions of problem (1.2). $\square$

4. Multiple nontrivial solutions for the system

In this section we investigate the existence of multiple nontrivial solutions (ξ, η) for a perturbation $b[(\xi-\eta+2)^+-2]$ of the hyperbolic system with Dirichlet boundary condition

$$\begin{aligned}
&L\xi = \mu[(\xi-\eta+2)^+ - 2] \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R},\\
&L\eta = \nu[(\xi-\eta+2)^+ - 2] \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R},\\
&\xi(\pm\frac{\pi}{2}, t) = 0, \quad \xi(x, t+\pi) = \xi(x,t) = \xi(-x,t),\\
&\eta(\pm\frac{\pi}{2}, t) = 0, \quad \eta(x, t+\pi) = \eta(x,t) = \eta(-x,t),
\end{aligned} \tag{4.1}$$

where $u^+ = \max\{u, 0\}$, μ, ν are nonzero constants. Here we assume that $-7 < \mu - \nu < -3$.

Theorem 4.1. *Let μ, ν be nonzero constants. Assume that $-7 < \mu - \nu < -3$. Then hyperbolic system (4.1) has at least three solutions (ξ, η), two of which are nontrivial solutions.*

Proof. From problem (4.1) we get that $L\xi = \frac{\mu}{\nu} L\eta$. By Theorem 2.1, the problem

$$\begin{aligned}
&Lu = 0 \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R},\\
&u(\pm\frac{\pi}{2}, t) = 0, \quad u(x, t+\pi) = u(x,t) = u(-x,t),
\end{aligned} \tag{4.2}$$

has only the trivial solution. So the solution (ξ, η) of problem (4.1) satisfies $\xi = \frac{\mu}{\nu}\eta$. On the other hand, from problem (4.1) we get the equation

$$\begin{aligned} & L(\xi-\eta) = (\mu-\nu)[(\xi-\eta+2)^+ - 2] \quad \text{in} \quad (-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}, \\ & \xi(\pm\frac{\pi}{2},t) = 0, \quad \xi(x,t+\pi) = \xi(x,t) = \xi(-x,t), \\ & \eta(\pm\frac{\pi}{2},t) = 0, \quad \eta(x,t+\pi) = \eta(x,t) = \eta(-x,t). \end{aligned} \tag{4.3}$$

Let $w = \xi - \eta$. Then the above equation is equivalent to

$$\begin{aligned} & Lw = (\mu-\nu)[(w+2)^+ - 2] \quad \text{in} \quad (-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}, \\ & w(\pm\frac{\pi}{2},t) = 0, \quad w(x,t+\pi) = w(x,t) = w(-x,t). \end{aligned} \tag{4.4}$$

When $-7 < \mu - \nu < -3$, the above equation has at least three solutions , two of which are nontrivial solutions, say w_1, w_2. Hence we get the solutions (ξ,η) of problem (4.1) from the following systems:

$$\begin{aligned} & \xi - \eta = 0 \quad \text{in} \quad (-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}, \\ & \xi = \frac{\mu}{\nu}\eta \quad \text{in} \quad (-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}, \\ & \xi(\pm\frac{\pi}{2},t) = 0, \quad \xi(x,t+\pi) = \xi(x,t) = \xi(-x,t), \\ & \eta(\pm\frac{\pi}{2},t) = 0, \quad \eta(x,t+\pi) = \eta(x,t) = \eta(-x,t), \end{aligned} \tag{4.5}$$

$$\begin{aligned} & \xi - \eta = w_1 \quad \text{in} \quad (-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}, \\ & \xi = \frac{\mu}{\nu}\eta \quad \text{in} \quad (-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}, \\ & \xi(\pm\frac{\pi}{2},t) = 0, \quad \xi(x,t+\pi) = \xi(x,t) = \xi(-x,t), \\ & \eta(\pm\frac{\pi}{2},t) = 0, \quad \eta(x,t+\pi) = \eta(x,t) = \eta(-x,t), \end{aligned} \tag{4.6}$$

$$\begin{aligned} & \xi - \eta = w_2 \quad \text{in} \quad (-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}, \\ & \xi = \frac{\mu}{\nu}\eta \quad \text{in} \quad (-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}, \\ & \xi(\pm\frac{\pi}{2},t) = 0, \quad \xi(x,t+\pi) = \xi(x,t) = \xi(-x,t), \\ & \eta(\pm\frac{\pi}{2},t) = 0, \quad \eta(x,t+\pi) = \eta(x,t) = \eta(-x,t). \end{aligned} \tag{4.7}$$

From (4.5) we get the trivial solution $(\xi, \eta) = (0, 0)$. From (4.6), (4.7) we get the nontrivial solutions (ξ, η).

Therefore system(4.1) has at least three solutions (ξ, η), two of which are nontrivial solutions. □

By using the similar method as in the proof of Theorem 4.1, we have the following corollary.

Corollary 1. *Let* μ, ν *be nonzero constants and* $1 - \frac{\mu}{\nu} \neq 0$. *Assume that* $-7 < \mu + \nu < -3$. *Then the hyperbolic system*

$$\begin{aligned} &L\xi = \mu[(\xi + \eta + 2)^+ - 2] \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}, \\ &L\eta = \nu[(\xi + \eta + 2)^+ - 2] \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}, \\ &\xi(\pm\frac{\pi}{2}, t) = 0, \quad \xi(x, t + \pi) = \xi(x, t) = \xi(-x, t), \\ &\eta(\pm\frac{\pi}{2}, t) = 0, \quad \eta(x, t + \pi) = \eta(x, t) = \eta(-x, t), \end{aligned} \tag{4.8}$$

has at least three solutions (ξ, η), *two of which are nontrivial solutions.*

References

[1] H.Amann, *Saddle points and multiple solutions of differential equation*, Math.Z.(1979), 27-166.

[2] A. Ambrosetti and P. H. Rabinowitz, *Dual variation methods in critical point theory and applications*, J. Functional analysis, **14**(1973), 349-381.

[3] Q. H. Choi, T. Jung and P. J. McKenna, *The study of a nonlinear suspension bridge equation by a variational reduction mehtod*, Appl. Anal., **50**(1993),73-92.

[4] Q. H. Choi and T. Jung, *An application of a variational reduction method to a nonlinear wave equation*,
J. Differential equations, **117**(1995), 390-410.

[5] A. C. Lazer and P. J. McKenna, *A symmetry theorem and applications to nonlinear partial differential equations*,J. Differential Equations **72**(1988), 95-106.

[6] A. C. Lazer and P. J. McKenna, *Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis*, SIAM Review, **32**(1990), 537-578.

[7] A. C. Lazer and P. J. McKenna, *Global bifurcation and a theorem of Tarantello*, J. Math. Anal. Appl, **181**(1994), 648-655.

[8] A.M. Micheletti and A.Pistoia, *Multiplicity results for a fourth-order semilinear elliptic problem*, Nonlinear Analysis TMA, **31**(1998), 895-908.

[9] A. M. Micheletti, A.Pistoia, *Nontrivial solutions for some fourth order semilinear elliptic problems*, Nonlinear Analysis, **34**(1998), 509-523.

[10] Paul H. Rabinobitz, *Minimax methods in critical point theory with applications to differential equations*,Mathematical Science regional conference series, **No. 65**, AMS (1984).

[11] G. Tarantello, *A note on a semilinear elliptic problem*, Diff. Integ. Equations. **5**(3)(1992), 561-565.

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