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SOME APRIORI ESTIMATES FOR THE QUASI-GEOSTROPHIC EQUATION

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Abstract. We present a new apriori estimates for the surface quasi-geostrophic equation. This apriori estimates give a new blow-up criterion which is different from the known Beale-Kato-Majda type criterion.

1. Introduction

We consider the quasi-geostrophic equations in the whole 2-dimensional domain $\Omega = \mathbb{R}^2$,

$$(1.1) \quad (QG)\begin{cases} \dfrac{\partial\theta}{\partial t} + (v\cdot\nabla)\theta = 0, \quad v = \nabla^{\perp}\Lambda^{-1}\theta & \text{in } \Omega\times\mathbb{R}_+ \\ \theta(x,0) = \theta_0(x) & \text{in } \Omega \end{cases}$$

where θ and v, respectively, are the surface temperature and the velocity of the flow. $\Lambda = (-\Delta)^{\frac{1}{2}}$ is the pseudo-differential operator defined in Fourier space by $\widehat{(-\Delta)^{\frac{1}{2}}u}(\mathbf{k}) = |\mathbf{k}|\hat{u}(\mathbf{k})$ and $\nabla^{\perp}$ is the orthogonal derivative operator defined by $(-\partial_2, \partial_1)$. The surface quasi-geostrophic equation describes the dynamics of large eddies in the atmosphere and ocean. For the geophysical meaning of the surface quasi-geostrophic equation, see [9]. The main mathematical interest in the surface quasi-geostrophic equation lies in the similarities with the 3D Euler equations. $\nabla^{\perp}\theta$ plays the similar role of the vorticity for the 3D Euler equations. This direction of the research was first initiated by Constantin, Majda and Tabak[5]. Weak solutions have been constructed by Resnick[10]. The following Beale-Kato-Majda[1] type blow up criterion for the quasi-geostrophic

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equation has been proved by Constantin, Majda and Tabak[5].

$$\limsup_{t\to T^*} \|\theta(t)\|_{H^m} = \infty \quad \text{if and only if} \quad \int_0^{T^*} \|\nabla^{\perp}\theta(t)\|_{L^\infty} dt = \infty.$$

This criterion has been refined in [2] using Triebel-Lizorkin space. Hyperbolic saddle collapse blow-up was one of the possible singularity formation scenario for the solutions of the quasi-geostrophic equation. The hyperbolic saddle type scenario for the quasi-geostrophic equation has been excluded by Cordoba[6](see [4] for numerical simulations).

Following the method presented in [3], we have

THEOREM 1. *Let $\theta \in C([0, T); H^m(\mathbb{R}^2))$ be a classical solution to the 2D quasi-geostrophic equations with $m > 2$. Suppose that there exists an absolute constant $\epsilon_0 > 0$ such that for some t_0 with $0 \le t_0 < T$,*

$$\sup_{t_0 \le t < T} (T - t) \left\|\nabla\theta(t)\right\|_{L^\infty(\mathbb{R}^2)} < \epsilon_0. \tag{1.2}$$

Then $\theta \in C([0, T+\delta); H^m(\mathbb{R}^2))$ for some $\delta > 0$.

REMARK 1. *The Theorem 1 implies that if T^* is the first time of singularity, then we have the following blow-up rate*

$$\limsup_{t\nearrow T^*} \left\|\nabla\theta(t)\right\|_{L^\infty(\mathbb{R}^2)} \ge \frac{\epsilon_0}{T^* - t}.$$

Our blow-up estimate has an advantage over BKM type criterion[5] *in the sense that their estimates cannot exclude the possibility that blow-up rate behaves like $o((T^* - t)^{-1})$, e.g.,*

$$\left\|\nabla\theta(t)\right\|_{L^\infty(\mathbb{R}^2)} \sim O\left(1/\left((T^* - t)|\log(T^* - t)|\right)\right),$$

since $1/(t\log t)$ is not integrable near origin. In contrast, our estimate (1.2) *does not allow such blow-up rate.*

2. Proof of Theorem 1

In this section we present the proofs of Theorem 1. The following commutator estimate is useful for the proof of Theorem 1 and the proof of the following proposition can be found in [8](see also [7]). The space $H^{s,p}$ denotes a subspace of $L^p(\Omega)$, equipped with the norm $\|f\|_{H^{s,p}} = \|\Lambda^s f\|_p$.

Proposition 1. *Suppose that $s > 0$ and $p \in (1, \infty)$. If $f, g \in \mathcal{S}$, then*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_p \le C\left(\|\nabla f\|_{p_1}\|g\|_{H^{s-1,p_2}} + \|f\|_{H^{s,p_3}}\|g\|_{p_4}\right),$$

where $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{p}$.

Proof of Theorem 1. We first take ∇^α operator on the both sides of quasi-geostrophic equation, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index with $|\alpha| = \sum_{i=1}^3 \alpha_i \le m$. Let $t < T$. Multiplying (1.1) by $(T-t)\nabla^\alpha\theta(t)$, integrating over $\mathbb{R}^2$, and summing over α for $|\alpha| \le m$, we have

$$\frac{1}{2}\frac{d}{dt}\Big((T-t)\|\theta\|_{H^m}^2\Big) + \frac{1}{2}\|\theta\|_{H^m}^2 = -(T-t)\sum_{|\alpha|\le m}\int_{\mathbb{R}^2}\nabla^\alpha((v\cdot\nabla)\theta)\nabla^\alpha\theta dx := RHS.$$

Due to the commutator estimates Proposition 1, the righthand side is estimated as follows:

(2.3)
$$RHS \le C(T-t)\left(\|\Lambda v(t)\|_\infty \|\theta(t)\|_{H^m}^2 + \|\nabla\theta(t)\|_\infty \|v(t)\|_{H^m}\|\theta(t)\|_{H^m}\right).$$

Since we have $\Lambda v(t) = \nabla^\perp\theta(t)$ and $\|v(t)\|_{H^m} = \|\theta(t)\|_{H^m}$, (2.3) reduces to the following

$$RHS \le C(T-t)\|\nabla\theta(t)\|_\infty \|\theta(t)\|_{H^m}^2. \tag{2.4}$$

Thus we have

$$\frac{1}{2}\frac{d}{dt}\Big((T-t)\|\theta\|_{H^m}^2\Big) + \left(\frac{1}{2} - C(T-t)\|\nabla\theta(t)\|_\infty\right)\|\theta(t)\|_{H^m}^2 \le 0.$$

We choose $\epsilon_0 = \frac{1}{4C}$, where C is the absolute constant in (2.4). It is straightforward, from the assumption (1.2), that

$$\frac{1}{2}\frac{d}{dt}\Big((T-t)\|\theta\|_{H^m}^2\Big) + \frac{1}{4}\|\theta\|_{H^m}^2 \le 0.$$

Therefore, integrating in time from t_0 to τ for any $\tau \in (t_0, T)$, we obtain

$$\sup_{t_0\le t<T}(T-t)\|\theta\|_{H^m}^2 + \frac{1}{2}\int_{t_0}^T \|\theta\|_{H^m}^2 dt \le (T-t_0)\|\theta(t_0)\|_{H^m}^2. \tag{2.5}$$

Since $\int_{t_0}^T \|\theta\|_{H^m}^2 dt$ is finite, the conclusion is immediate from BKM type criterion. □

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