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HYPONORMALITY OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE.

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Abstract. In this paper we consider the hyponormality of Toeplitz operators T_φ on the Bergman space $L_a^2(\mathbb{D})$ with symbol in the case of function $f+\overline{g}$ with polynomials f and g. We present some necessary conditions for the hyponormality of T_φ under certain assumptions about the coefficients of φ.

1. Introduction

A bounded linear operator A on a Hilbert space is said to be hyponormal if its selfcommutator $[A^*, A] := A^*A - AA^*$ is positive semidefinite. Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$, dA the area measure on the complex plane $\mathbb{C}$. The space $L^2(\mathbb{D})$ is a Hilbert space with the inner product

$$\langle f,\ g\rangle = \frac{1}{\pi}\int_{\mathbb{D}} f(z)\overline{g(z)}dA(z).$$

The Bergman space $L_a^2(\mathbb{D})$ is the subspace of $L^2(\mathbb{D})$ consisting of all analytic functions on $\mathbb{D}$. Let $L^\infty(\mathbb{D})$ be the space of bounded area measureable function on $\mathbb{D}$. For $\varphi \in L^\infty(\mathbb{D})$, the multiplication operator M_φ on the Bergman space are defined by $M_\varphi(f) = \varphi \cdot f$, where f is in L_a^2. If P denotes the orthogonal projection of $L^2(\mathbb{D})$ onto the Bergman space L_a^2, the Toeplitz operator T_φ on the Bergman space is defined by

$$T_\varphi(f) = P(\varphi \cdot f),$$

where φ is measurable and f is in L_a^2. It is clear that those operators are bounded if φ is in $L^\infty(\mathbb{D})$. The Hankel operator $H_\varphi : L_a^2 \longrightarrow {L_a^2}^\perp$ is defined by

$$H_\varphi(f) = (I - P)(\varphi \cdot f).$$

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Let $H^2(\mathbb{T})$ denote the Hardy space of the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. For a given $\psi \in L^\infty(\mathbb{T})$, the Toeplitz operator on the Hardy space is the operator T_ψ on $H^2(\mathbb{T})$ defined by $T_\psi f = P_+(\psi \cdot f)$, where f is in $H^2(\mathbb{T})$ and P_+ denotes the orthogonal projection that maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$.

Basic properties of the Bergman space and the Hardy space can be found in [1]and [3]. The hyponormality of Toeplitz operators on the Hardy space has been studied by C. Cowen [2], T. Nakazi and K. Takahashi [7], K. Zhu [9], W.Y. Lee [4], [5], [6] and others. In [2], Cowen characterized the hyponormality of Toeplitz operator T_φ on $H^2(\mathbb{T})$ by properties of the symbol $\varphi \in L^\infty(\mathbb{T})$. It also exploited the fact that functions in ${H^2}^\perp$ are conjugates of functions in zH^2. For the Bergman space, ${L^2_a}^\perp$ is much larger than the conjugates of functions in zL^2_a, and no dilation theorem(similar to Sarason's theorem) is available. Indeed it is quite difficult to determine the hyponormality of T_φ. In fact the study of hyponormal Toeplitz operators on the Bergman space seems to be scarce from the literature. In [10], it was shown that if $\varphi(z) = a_{-m}\overline{z}^m + a_{-N}\overline{z}^N + a_m z^m + a_N z^N \quad (0 < m < N)$ and if $a_m\overline{a_N} = a_{-m}\overline{a_{-N}}$, then

$$T_\varphi \text{ is hyponormal} \iff$$
$$\begin{cases} \frac{1}{N+1}(|a_N|^2 - |a_{-N}|^2) \geq \frac{1}{m+1}(|a_{-m}|^2 - |a_m|^2) & \text{if } |a_{-N}| \leq |a_N| \\ N^2(|a_{-N}|^2 - |a_N|^2) \leq m^2(|a_m|^2 - |a_{-m}|^2) & \text{if } |a_N| \leq |a_{-N}|. \end{cases}$$

In this paper we study the hyponormality of Toeplitz operators T_φ on the Bergman space $L^2_a(\mathbb{D})$ with symbols in the case of function $\varphi = \overline{g} + f$ with polynomial f and g. We will now consider the hyponormality of Toeplitz operators on the Bergman space with a symbol in the class of functions $\overline{g} + f$, where f and g are polynomials. Since the hyponormality of operators is translation invariant we may assume that $f(0) = g(0) = 0$. We shall list the well-known properties of Toeplitz operators T_φ on the Bergman space.

If f, g are in $L^\infty(\mathbb{D})$ then we can easily check that

(i) $T_{f+g} = T_f + T_g$

(ii) ${T_f}^* = T_{\overline{f}}$

(iii) $T_{\overline{f}} T_g = T_{\overline{f}g}$ if f or g is analytic.

These properties enable to us establish several consequences of hyponormality.

Proposition 1.1. ([**8**]) Let f, g be bounded and analytic in the $\overline{g} + f \in L^{\infty}(\mathbb{D})$. Then the followings are equivalent.
(i) $T_{\overline{g}+f}$ is hyponormal.
(ii) $H^*_{\overline{g}} H_{\overline{g}} \leq H^*_{\overline{f}} H_{\overline{f}}$.
(iii) $||(I-P)(\overline{g}k)|| \leq ||(I-P)(\overline{f}k)||$ for any k in L^2_a.
(iv) $||\overline{g}k||^2 - ||P(\overline{g}k)||^2 \leq ||\overline{f}k||^2 - ||P(\overline{f}k)||^2$ for any k in L^2_a.
(v) $H_{\overline{g}} = C H_{\overline{f}}$ where C is of norm less than or equal to one.

Let s, t be nonnegative integers and P be the orthogonal projection. Then we have

$$P(\overline{z}^t z^s) = \begin{cases} \frac{s-t+1}{s+1} z^{s-t}, & \text{if } s \geq t \\ 0, & \text{if } s < t. \end{cases} \tag{1.1}$$

2. Main results

In this section we establish some necessary conditions for the hyponormality of Toeplitz operator T_φ on the Bergman space L^2_a under certain additional assumption concerning the polynomial symbol φ.

In [11], it was shown that if $f(z) = a_m z^m + a_N z^N$, $g(z) = a_{-m} z^m + a_{-N} z^N$ $(0 < m < N)$ and $|a_N| < |a_{-N}|$, then the hyponormality of $T_{\overline{g}+f}$ implies that

$$N^2(|a_{-N}|^2 - |a_N|^2) \leq m^2(|a_m|^2 - |a_{-m}|^2).$$

The following Theorem gives a necessary condition for hyponormality of $T_{\overline{g}+f}$ when $m = 1$ and $N = 2$.

THEOREM 2.1. *Let* $\varphi(z) = \overline{g(z)} + f(z)$, *where*

$$f(z) = a_1 z + a_2 z^2 \quad \text{and} \quad g(z) = a_{-1} z + a_{-2} z^2.$$

If T_φ *is hyponormal, then*

(i) $2(|a_2|^2 - |a_{-2}|^2) \geq 3(|a_{-1}|^2 - |a_1|^2)$.

(ii) $\Big(\frac{1}{2}(|a_1|^2 - |a_{-1}|^2) + \frac{1}{3}(|a_2|^2 - |a_{-2}|^2)\Big)\Big(\frac{1}{12}(|a_1|^2 - |a_{-1}|^2) + \frac{1}{4}(|a_2|^2 - |a_{-2}|^2)\Big) \geq \frac{1}{9}|\overline{a_1} a_2 - \overline{a_{-1}} a_{-2}|^2$.

Proof. Let T_φ is hyponormal operator. Observe that

$$\begin{aligned}\big\langle M_{\overline{f}}(c_0 + c_1 z), M_{\overline{f}}(c_0 + c_1 z)\big\rangle = |a_1|^2\Big(\frac{1}{2}|c_0|^2 + \frac{1}{3}|c_1|^2\Big) + \\ |a_2|^2\Big(\frac{1}{3}|c_0|^2 + \frac{1}{4}|c_1|^2\Big) + \frac{2}{3} Re(\overline{a_1} a_2 c_0 \overline{c_1}).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\big\langle M_{\overline{g}}(c_0 + c_1 z), M_{\overline{g}}(c_0 + c_1 z)\big\rangle = |a_{-1}|^2\Big(\frac{1}{2}|c_0|^2 + \frac{1}{3}|c_1|^2\Big) + \\ |a_{-2}|^2\Big(\frac{1}{3}|c_0|^2 + \frac{1}{4}|c_1|^2\Big) + \frac{2}{3} Re(\overline{a_{-1}} a_{-2} c_0 \overline{c_1}).\end{aligned}$$

It follows that

$$\big\langle T_{\overline{f}}(c_0 + c_1 z), T_{\overline{f}}(c_0 + c_1 z)\big\rangle = \frac{1}{4}|a_1|^2 |c_1|^2.$$

Similarly, we have

$$\big\langle T_{\overline{g}}(c_0 + c_1 z), T_{\overline{g}}(c_0 + c_1 z)\big\rangle = \frac{1}{4}|a_{-1}|^2 |c_1|^2.$$

Hence

$$\left\langle H_{\overline{f}}(c_0+c_1z), H_{\overline{f}}(c_0+c_1z)\right\rangle = |a_1|^2\Big(\frac{1}{2}|c_0|^2+\frac{1}{12}|c_1|^2\Big)+ \\ |a_2|^2\Big(\frac{1}{3}|c_0|^2+\frac{1}{4}|c_1|^2\Big)+\frac{2}{3}Re(\overline{a_1}a_2c_0\overline{c_1}).$$

Similarly, we have

$$\left\langle H_{\overline{g}}(c_0+c_1z), H_{\overline{g}}(c_0+c_1z)\right\rangle = |a_{-1}|^2\Big(\frac{1}{2}|c_0|^2+\frac{1}{12}|c_1|^2\Big)+ \\ |a_{-2}|^2\Big(\frac{1}{3}|c_0|^2+\frac{1}{4}|c_1|^2\Big)+\frac{2}{3}Re(\overline{a_{-1}}a_{-2}c_0\overline{c_1}).$$

Hence if T_φ is hyponormal, then

(2.1)

$$\left\langle (H^*_{\overline{f}}H_{\overline{f}}-H^*_{\overline{g}}H_{\overline{g}})(c_0+c_1z),(c_0+c_1z)\right\rangle \\ =(|a_1|^2-|a_{-1}|^2)\Big(\frac{1}{2}|c_0|^2+\frac{1}{12}|c_1|^2\Big)+(|a_2|^2-|a_{-2}|^2)\Big(\frac{1}{3}|c_0|^2+\frac{1}{4}|c_1|^2\Big) \\ +\frac{2}{3}Re(\overline{a_1}a_2c_0\overline{c_1}-\overline{a_{-1}}a_{-2}c_0\overline{c_1})\geq 0.$$

Since $Re(\overline{a_1}a_2c_0\overline{c_1}-\overline{a_{-1}}a_{-2}c_0\overline{c_1})\geq -|\overline{a_1}a_2-\overline{a_{-1}}a_{-2}||c_0\overline{c_1}|$, it follows from (2.1) that

$$(|a_1|^2-|a_{-1}|^2)\Big(\frac{1}{2}|c_0|^2+\frac{1}{12}|c_1|^2\Big)+(|a_2|^2-|a_{-2}|^2)\Big(\frac{1}{3}|c_0|^2+\frac{1}{4}|c_1|^2\Big) \\ -\frac{2}{3}|\overline{a_1}a_2-\overline{a_{-1}}a_{-2}||c_0\overline{c_1}|\geq 0.$$

There are two cases to consider.

Case 1) If $c_1=0$, then

$$3(|a_1|^2-|a_{-1}|^2)+2(|a_2|^2-|a_{-2}|^2)\geq 0.$$

Case 2) If $c_1\neq 0$, then we have that

$$\Big|\frac{c_0}{c_1}\Big|^2\Big(\frac{1}{2}(|a_1|^2-|a_{-1}|^2)+\frac{1}{3}(|a_2|^2-|a_{-2}|^2)\Big)-\frac{2}{3}\Big|\frac{c_0}{c_1}\Big||\overline{a_1}a_2-\overline{a_{-1}}a_{-2}| \\ +\frac{1}{12}(|a_1|^2-|a_{-1}|^2)+\frac{1}{4}(|a_2|^2-|a_{-2}|^2)\geq 0.$$

Since $\frac{1}{2}(|a_1|^2 - |a_{-1}|^2) + \frac{1}{3}(|a_2|^2 - |a_{-2}|^2) \geq 0$, it follows that

$$\Big(\frac{1}{2}(|a_1|^2 - |a_{-1}|^2) + \frac{1}{3}(|a_2|^2 - |a_{-2}|^2)\Big)\Big(\frac{1}{12}(|a_1|^2 - |a_{-1}|^2) + \frac{1}{4}(|a_2|^2 - |a_{-2}|^2)\Big) \geq \frac{1}{9}|\overline{a_1}a_2 - \overline{a_{-1}}a_{-2}|^2.$$

This completes the proof. □

The following Theorem gives a necessary condition for hyponormality of $T_{\overline{g}+f}$ when $m > 1$ and $N = m + 1$.

Theorem 2.2. *Let* $\varphi(z) = \overline{g(z)} + f(z)$, *where*

$$f(z) = a_m z^m + a_{m+1} z^{m+1} \quad \text{and} \quad g(z) = a_{-m} z^m + a_{-(m+1)} z^{m+1}.$$

If T_φ *is hyponormal, then* (i) $(m+1)(|a_{m+1}|^2 - |a_{-(m+1)}|^2) \geq (m+2)(|a_{-m}|^2 - |a_m|^2)$.

(ii) $\Big(\frac{1}{m+1}(|a_m|^2 - |a_{-m}|^2) + \frac{1}{m+2}(|a_{m+1}|^2 - |a_{-(m+1)}|^2)\Big)\Big(\frac{1}{m+2}(|a_m|^2 - |a_{-m}|^2) + \frac{1}{m+3}(|a_{m+1}|^2 - |a_{-(m+1)}|^2)\Big) \geq \frac{1}{(m+2)^2}|\overline{a_m}a_{m+1} - \overline{a_{-m}}a_{-(m+1)}|^2.$

Proof. Let T_φ be a hyponormal operator. Observe that

$$\begin{aligned}&\big\langle M_{\overline{f}}(c_0 + c_1 z), M_{\overline{f}}(c_0 + c_1 z)\big\rangle \\ &= |a_m|^2\Big(\frac{1}{m+1}|c_0|^2 + \frac{1}{m+2}|c_1|^2\Big) + \\ &|a_{m+1}|^2\Big(\frac{1}{m+2}|c_0|^2 + \frac{1}{m+3}|c_1|^2\Big) + \frac{2}{m+2}Re(\overline{a_m}a_{m+1}c_0\overline{c_1}).\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\big\langle M_{\overline{g}}(c_0 + c_1 z), M_{\overline{g}}(c_0 + c_1 z)\big\rangle \\ &= |a_{-m}|^2\Big(\frac{1}{m+1}|c_0|^2 + \frac{1}{m+2}|c_1|^2\Big) + \\ &|a_{-(m+1)}|^2\Big(\frac{1}{m+2}|c_0|^2 + \frac{1}{m+3}|c_1|^2\Big) + \frac{2}{m+2}Re(\overline{a_{-m}}a_{-(m+1)}c_0\overline{c_1}).\end{aligned}$$

It follows that

$$\left\langle T_{\overline{f}}(c_0 + c_1 z), T_{\overline{f}}(c_0 + c_1 z)\right\rangle = 0.$$

Similarly, we have

$$\left\langle T_{\overline{g}}(c_0 + c_1 z), T_{\overline{g}}(c_0 + c_1 z)\right\rangle = 0.$$

Hence

$$\begin{aligned}&\left\langle H_{\overline{f}}(c_0 + c_1 z), H_{\overline{f}}(c_0 + c_1 z)\right\rangle \\ &\quad = |a_m|^2\Big(\frac{1}{m+1}|c_0|^2 + \frac{1}{m+2}|c_1|^2\Big) + \\ &\quad |a_{m+1}|^2\Big(\frac{1}{m+2}|c_0|^2 + \frac{1}{m+3}|c_1|^2\Big) + \frac{2}{m+2}Re(\overline{a_m}a_{m+1}c_0\overline{c_1}).\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\left\langle H_{\overline{g}}(c_0 + c_1 z), H_{\overline{g}}(c_0 + c_1 z)\right\rangle \\ &\quad = |a_{-m}|^2\Big(\frac{1}{m+1}|c_0|^2 + \frac{1}{m+2}|c_1|^2\Big) + \\ &\quad |a_{-(m+1)}|^2\Big(\frac{1}{m+2}|c_0|^2 + \frac{1}{m+3}|c_1|^2\Big) + \frac{2}{m+2}Re(\overline{a_{-m}}a_{-(m+1)}c_0\overline{c_1}).\end{aligned}$$

Hence if T_φ is hyponormal, then

$$\begin{aligned}&\left\langle (H^*_{\overline{f}}H_{\overline{f}} - H^*_{\overline{g}}H_{\overline{g}})(c_0 + c_1 z), (c_0 + c_1 z)\right\rangle \\ &= (|a_m|^2 - |a_{-m}|^2)\Big(\frac{1}{m+1}|c_0|^2 + \frac{1}{m+2}|c_1|^2\Big) \\ &+ (|a_{m+1}|^2 - |a_{-(m+1)}|^2)\Big(\frac{1}{m+2}|c_0|^2 + \frac{1}{m+3}|c_1|^2\Big) \\ &+ \frac{2}{m+2}Re(\overline{a_m}a_{m+1}c_0\overline{c_1} - \overline{a_{-m}}a_{-(m+1)}c_0\overline{c_1}) \geq 0.\end{aligned} \tag{2.2}$$

Since

$$Re(\overline{a_m}a_{m+1}c_0\overline{c_1} - \overline{a_{-m}}a_{-(m+1)}c_0\overline{c_1}) \geq -|\overline{a_m}a_{m+1} - \overline{a_{-m}}a_{-(m+1)}||c_0\overline{c_1}|,$$

it follow from (2.2) that

$$(|a_m|^2 - |a_{-m}|^2)\Big(\frac{1}{m+1}|c_0|^2 + \frac{1}{m+2}|c_1|^2\Big) + (|a_{m+1}|^2 - |a_{-(m+1)}|^2)$$
$$\Big(\frac{1}{m+2}|c_0|^2 + \frac{1}{m+3}|c_1|^2\Big) - \frac{2}{m+2}|\overline{a_m}a_{m+1} - \overline{a_{-m}}a_{-(m+1)}||c_0\overline{c_1}|$$
$$\geq 0.$$

There are two cases to consider.

Case 1) If $c_1 = 0$, then

$$(m+2)(|a_m|^2 - |a_{-m}|^2) + (m+1)(|a_{m+1}|^2 - |a_{-(m+1)}|^2) \geq 0.$$

Cases 2) If $c_1 \neq 0$, then we have that

$$\Big|\frac{c_0}{c_1}^2\Big|\Big(\frac{1}{m+1}(|a_m|^2 - |a_{-m}|^2) + \frac{1}{m+2}(|a_{m+1}|^2 - |a_{-(m+1)}|^2)\Big)$$
$$- \frac{2}{m+2}\Big|\frac{c_0}{c_1}\Big||\overline{a_m}a_{m+1} - \overline{a_{-m}}a_{-(m+1)}| + \frac{1}{m+2}(|a_m|^2 - |a_{-m}|^2)$$
$$+ \frac{1}{m+3}(|a_{m+1}|^2 - |a_{-(m+1)}|^2) \geq 0.$$

Since $\frac{1}{m+1}(|a_m|^2 - |a_{-m}|^2) + \frac{1}{m+2}(|a_{m+1}|^2 - |a_{-(m+1)}|^2) > 0$, it follows that

$$\Big(\frac{1}{m+1}(|a_m|^2 - |a_{-m}|^2) + \frac{1}{m+2}(|a_{m+1}|^2 - |a_{-(m+1)}|^2)\Big)$$
$$\Big(\frac{1}{m+2}(|a_m|^2 - |a_{-m}|^2) + \frac{1}{m+3}(|a_{m+1}|^2 - |a_{-(m+1)}|^2)\Big)$$
$$\geq \frac{1}{(m+2)^2}|\overline{a_m}a_{m+1} - \overline{a_{-m}}a_{-(m+1)}|^2.$$

This completes the proof. □

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