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THE HAMILTONIAN SYSTEM WITH THE NONLINEAR PERTURBED POTENTIAL

Tacksun Jung and Q-Heung Choi*

Abstract. We investigate the multiplicity of $2\pi-$periodic solutions of the nonlinear Hamiltonian system with perturbed polynomial and exponential potentials, $\dot{z} = JG'(z)$, where $z : R \to R^{2n}$, $\dot{z} = \frac{dz}{dt}$, $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, I is the identity matrix on R^n, $G : R^{2n} \to R$, $G(0,0) = 0$ and G' is the gradient of G. We look for the weak solutions $z = (p,q) \in E$ of the nonlinear Hamiltonian system.

1. Introduction

We consider the classical Hamiltonian dynamics with n degrees of freedom, the phase space is $R^{2n} = \{(p,q)\}$ endowed with the canonical symplectic form

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$$

and the Hamiltonian $H(p,q)$. A typical Hamiltonian

$$H(q,p) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(q)$$

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*Corresponding author.

was studied by many (Hamiltonian) physicists. The Hamiltonian equations of motion are

$$\dot{q}_i = \frac{\partial H}{\partial p_i},$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = -\frac{\partial V}{\partial q_i}.$$

A function $F(q,p)$ is an integral if $\dot{F} = 0$. A Hamiltonian system with n degrees of freedom having n independent integrals $F_i(q,p)$ is called (completely) integrable (for brevity we say that a Hamiltonian is integrable) if these integrals are in involution, i.e. that the Poisson bracket

$$\{F_i, F_j\} = \sum_{\alpha=1}^{n} \frac{\partial F_i}{\partial q_\alpha}\frac{\partial F_j}{\partial p_\alpha} - \frac{\partial F_i}{\partial p_\alpha}\frac{\partial F_j}{\partial q_\alpha}$$

vanishes for any i, j.

By the Liouville-Arnold theorem (cf. [4]), equations of motion of such systems can be in principle explicitly integrated which justifies the terminology and the interest in integrable Hamiltonians. They have been much studied by physicists because of exciting connections between them and the seemingly unrelated subjects such as algebraic geometry and theory of Lie groups.

Specialists in integrable Hamiltonian had shown a lot of ingenuity constructing integrals of motion. To illustrate this we have the nonperiodic Toda lattice

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + e^{q_1 - q_2} + \cdots + e^{q_{n-1} - q_n}.$$

Toda [17] introduced and studied the infinite version of the nonperiodic Toda lattice (i runs from $-\infty$ to ∞).

Bogoyablensky [8] put Toda lattice into the frame work of Hamiltonians with exponential potential. More precisely let $f_1, \cdots, f_N$ be arbitrary vectors in R^n endowed with a Euclidean inner product and let let $c_1, \cdots, c_N$ be arbitrary constants (N and n are not related). The corresponding exponential potential is

$$V(q) = \sum_{k=1}^{N} c_k e^{f_k \cdot q}$$

and the Hamiltonian is

$$H(p,q) = \frac{1}{2} p \cdot p + V(q).$$

Bogoyablensky [8] studied the above Hamiltonians satisfying certain additional conditions which appear in the theory of Einstein cosmological models. The Bogoyablensky Hamiltonian can be thought of as perturbations of the Toda lattice. The many old (Hamiltonian) physicists had studied the integrability of the Hamiltonian and had many results for the integrability of the Hamiltonian satisfying certain additional conditions. However, the the integrable Hamiltonians are the particular phenomena among the Hamiltonian equations of motion.

Therefore we are concern with the generalized perturbed Hamiltonian satisfying certain additional conditions.

In this paper we are concerned with the multiple periodic solutions of the Hamiltonian system with perturbed polynomial and exponential potentials

$$\dot{z} = JG'(z), \tag{1.1}$$

where $z : R \to R^{2n}$, $\dot{z} = \frac{dz}{dt}$, $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, I is the identity matrix on R^n, $G : R^{2n} \to R$, and G' is its gradient. Let $a \cdot b$ and $|\cdot|$ denote the usual inner product and norm on R^{2n}. G will be required to satisfy the following conditions:

$(G1)$ $G \in C^1(R^{2n}, R)$, $G(0,0,\ldots,0) = 0$ and G is even function.

$(G2)$ There exist $\mu > 2$, $r_0 \geq 0$ such that

$$0 < \mu G(z) \leq G'(z) \cdot z \qquad \text{for every } |z| \geq r_0.$$

$(G3)$ There exist $1 < p_1 \leq p_2 < 2p_1 + 1$, $\alpha_i > 0$, $\beta_i \geq 0$, for $i = 1, 2$ such that

$$\alpha_1 |z|^{p_1+1} - \beta_1 \leq G(z) \leq \alpha_2 |z|^{p_2+1} + \beta_2, \qquad \text{for every } z \in R^{2n}.$$

$(G4)$ There exist $0 < \frac{q_2}{2} < q_1 \leq q_2 < 2$ and α_i, $\tau_i > 0$, $\beta_i \geq 0$, for $i = 1, 2$, such that

$$\alpha_1 e^{\tau_1 |z|^{q_1}} - \beta_1 \leq G(z) \leq \alpha_2 e^{\tau_2 |z|^{q_2}} + \beta_2, \qquad \text{for every } z \in R^{2n}.$$

Our main results are

THEOREM 1.1. *Assume that G satisfies the conditions $(G1) - (G3)$. Then (1.1) possesses an unbounded sequence of 2π-periodic solutions.*

Theorem 1.2. *Assume that G satisfies the conditions $(G1)$, $(G2)$ and $(G4)$. Then (1.1) possesses an unbounded sequence of 2π-periodic solutions.*

Such global existence problems have been studied extensively in recent years. For the autonomous case of (1.1), the result was proved by Rabinowitz under the conditions $(G1)$ and $(G2)$ ([13], [15]). His proof is based on a group symmetry possessed by the corresponding variational formulation. To prove the main results we use the ideas of critical point theory. In section 2, we define a space X which is invariant under A ($Az = \int_0^{2\pi} \dot{z} \cdot J(z)\, dt$), G' and shift, and we introduce an invariant subspace V and a sequence of invariant subspace $H^-_{m,\infty} \oplus H^+_{m,0} \oplus H^-_{0,m}$ of X to apply to Theorem 2.2. In section 3, we prove Theorem 1.1. In section 4, we prove Theorem 1.2.

2. Spaces invariant under A, G, and shifts

Let X be a real Hilbert space on which the compact Lie group S^1 acts by means of time-translations, hence by orthogonal transformations; for $z \in X$ and $\theta \in [0, 2\pi]$, we define an S^1-action on X by

$$(T_\theta z)(t) = z(t+\theta), \qquad \text{for all } t \in [0, 2\pi].$$

Let $Fix\{T_\theta\}$ be the set of fixed points of the action, i. e.,

$$Fix\{T_\theta\} = \{z \in X |\ T_\theta z = z, \qquad \forall \theta \in [0, 2\pi]\}.$$

We say a subset B of X is S^1-invariant if $T_\theta z \in B$, for all $z \in B$, $\theta \in [0, 2\pi]$. A function $f : X \to R^1$ is called S^1-invariant, if $f(T_\theta z) = f(z)$, $\forall z \in X$, for all $\theta \in [0, 2\pi]$. Let $C(B, X)$ be the set of continuous functions from B into X. If B is an invariant set we say $h \in C(B, X)$ is an equivariant map if $h(T_\theta z) = T_\theta h(z)$ for all $\theta \in [0, 2\pi]$ and $z \in B$. Let S_ρ be the sphere centered at the origin, of radius r.
We need the following Theorem 2.1 in [10].

Theorem 2.1. *Assume that*
(f_1) $f \in C^1(X, R)$ *is* S^1*-invariant,*
(f_2) *There is a sequence of invariant finite dimensional vector subspaces*

$$X^1 \subset X^2 \subset \ldots \subset X^n, \qquad \dim X^n = 2n,$$

with $\overline{\cup_{n=1}^\infty X^n} = X$*, satisfying*
(1) $((PS)^n$*condition): There exists* $n_0 \in N$ *such that* $\forall n \geq n_0$*, the*

restriction $f^n = f|_{X^n}$ *satisfies the* $(PS)_c$ *condition* $\forall c \in [a,b]$.
(2) $((PS)^*$*condition): For any sequence* $x_n \in X^n$, $n = 1, 2, \ldots$, $a \leq f(x_n) \leq b$ *and* $df^n(x_n) \to 0$ *imply a convergent subsequence.*
$(f3)'$ *There exist invariant subspaces* X_+ *and* X_- *with*

$$j = \frac{1}{2} codim X_+ < m = \frac{1}{2} \dim X_- < +\infty,$$

where j *and* m *are integers such that*
(1) $Fix\{T_\theta\} \subset X_-$, $Fix\{T_\theta\} \cap X_+ = \{0\}$,
(2) $f(x) > a$, $\forall x \in X_+ \cap S_\rho$ *for some* $\rho > 0$,
(3) $f(x) < b$, $\forall x \in X_-$, a, b *are regular values,*
(4) $Fix\{T_\theta\} \cap f^{-1}[a,b] = \phi$.
Then f *has at least* $m - j$ *distinct critical orbits.*

We obtain the following theorem which is the crucial role for the proof of our main result.

THEOREM 2.2. *Suppose that* $f \in C^1(X, R)$ *is even with* $f(0) = 0$, *and that*
(1) *there exist an invariant subspace* V *and* α *such that*

$$f|_{V^{\perp} \cap S_\rho} \geq \alpha,$$

(2) *there exist two sequences of linear subspaces* $H^+_{0,m}$, $H^-_{0,m}$ *with* $\dim H^+_{0,m} = H^-_{0,m} = m$ *and* $R_m > 0$ *such that*

$$f(x) \leq 0, \qquad \forall x \in H^-_{m,\infty} \oplus H_0 \oplus H^+_{0,m} \oplus H^-_{0,m}, |x| \leq R_m, \qquad m = 1, 2, \cdots.$$

If f *satisfies* $(PS)^*$ *condition with respect to* $\{H^-_{m,\infty} \oplus H^+_{0,m} \oplus H^-_{0,m} | m = 1, 2, \cdots\}$, *then* f *possesses infinitely many distinct critical points.*

Proof. If not, f has at most l critical points. We choose m, j such that $H^-_{j,\infty} \oplus H^+_{0,j} \subset H^-_{m,\infty} \oplus H^+_{0,m} \oplus H^-_{0,m}$. We choose $m - j > l$. Now, (f_1) is obviously true, with $Fix\{T_\theta\} = \{0\}$. Let $X^n = H^+_{0,n} \oplus H^-_{0,n}$. Then $(PS)^n_c$ condition holds, by the condition (2). So (f_2) is satisfied. Let $X_+ = V^{\perp}$, $X_- = H^-_{m,\infty} \oplus H_0 \oplus H^+_{0,m} \oplus H^-_{0,m}$, $a = \alpha$, and $b = \sup_{x \in H^-_{m,\infty} \oplus H^+_{0,m} \oplus H^-_{0,m}} f(x) + 1$. Then (f_3) holds. Applying Theorem 2.1., there are at least $m - j$ pairs of critical points. This is a contradiction. $\square$

3. Proof of Theorem 1.1

Let $H = W^{\frac{1}{2},2}(S^1, R^{2n})$. The scalar product in L^2 naturally extends as the duality pairing between H and $H' = W^{-\frac{1}{2},2}(S^1, R^{2n})$. Thus for $z \in H$, the actional integral $\frac{1}{2}A(z)$ is well defined, where

$$A(z) = \int_0^{2\pi} \dot{z} \cdot J(z)dt. \tag{3.1}$$

The basic idea we use in trying to find periodic solutions of (1.1) is to obtain them as critical points of the corresponding functional

$$f(z) = \frac{1}{2}A(z) - \int_0^{2\pi} G(z) + dt. \tag{3.2}$$

By the following proposition, $f(z)$ is C^1, even, and the critical point z of f in H will be a weak solution of (1.1).

Proposition 3.1. *Assume that G satisfy the assumptions of Theorem 1.1 or Theorem 1.2. Then $f(z)$ is continuous and Fréchet differentiable in H with Fréchet derivative*

$$\begin{aligned} Df(z)w &= \int_0^{2\pi} (\dot{z} - JG'(z)) \cdot Jw \\ &= \int_0^{2\pi} [(-\dot{p} - G_q(z)) \cdot \psi + (\dot{q} - G_p(z)) \cdot \phi]\, dt, \end{aligned}$$

$z \in (p, q)$ and $w = (\phi, \psi) \in H$. Moreover $\int_0^{2\pi} G(z)dt \in C^1$. Thus f is C^1.

Proof. For $z, w \in H$,

$$\begin{aligned} & |f(z+w) - f(z) - Df(z)w| \\ = & \ |\frac{1}{2}\int_0^{2\pi} (\dot{z} + \dot{w}) \cdot J(z+w) - \int_0^{2\pi} G(z+w) \\ - & \ \frac{1}{2}\int_0^{2\pi} \dot{z} \cdot Jz + \int_0^{2\pi} G(z) - \int_0^{2\pi} (\dot{z} - JG'(z)) \cdot Jw| \\ = & \ |\frac{1}{2}\int_0^{2\pi} [\dot{z} \cdot Jw + \dot{w} \cdot Jz + \dot{w} \cdot Jw] - \int_0^{2\pi} [G(z+w) - G(z)] \\ & - \int_0^{2\pi} [(\dot{z} - JG'(z)) \cdot Jw]|. \end{aligned}$$

We have

$$|\int_0^{2\pi}[G(z+w)-G(z)]| \leq |\int_0^{2\pi}[G'(z)\cdot w+o(|w|)]dt| \leq o(|w|^2).$$

Thus we have

$$|f(z+w)-f(z)-Df(z)w| \leq o(|w|^2).$$

Next we will prove that $f(z)$ is continuous. For $z, w \in H$,

$$\begin{aligned}
& |f(z+w)-f(z)| \\
= & |\frac{1}{2}\int_0^{2\pi}(\dot{z}+\dot{w})\cdot J(z+w)-\int_0^{2\pi}G(z+w) \\
- & \frac{1}{2}\int_0^{2\pi}\dot{z}\cdot Jz+\int_0^{2\pi}G(z)| \\
= & |\frac{1}{2}\int_0^{2\pi}[\dot{z}\cdot Jw+\dot{w}\cdot Jz+\dot{w}\cdot Jw]-\int_0^{2\pi}[G(z+w)-G(z)]| \\
\leq & o(|w|).
\end{aligned}$$

□

Let z be a function of $W^{\frac{1}{2},2}([0,2\pi],R^{2n})$. Then there exists one and only one function of $W^{\frac{1}{2},2}(R,R^{2n})$ which is 2π periodic in t and equals to z on $[0,2\pi]$; we shall denote this function by z. We set

$$\begin{aligned}
& H_0 = span\{e_1,\cdots,e_{2n}\}, \\
& H_+ = span\{(\cos jt)e_k+(\sin jt)e_{k+n}, \\
& \quad (\sin jt)e_k-(\cos jt)e_{k+n}|\ j\in N, 1\leq k\leq n\}, \\
& H_- = span\{(\sin jt)e_k+(\cos jt)e_{k+n}, \\
& \quad (\cos jt)e_k-(\sin jt)e_{k+n}|\ j\in N, 1\leq k\leq n\}.
\end{aligned}$$

and let

$$\begin{aligned}
& H^+_{m,l} = span\{(\cos jt)e_k+(\sin jt)e_{k+n}, \\
& \quad (\sin jt)e_k-(\cos jt)e_{k+n}|\ m\leq j\leq l, 1\leq k\leq n\}, \\
& H^-_{m,l} = span\{(\sin jt)e_k+(\cos jt)e_{k+n}, \\
& \quad (\cos jt)e_k-(\sin jt)e_{k+n}|\ m\leq j\leq l, 1\leq k\leq n\}.
\end{aligned}$$

Then $H = H_+ \oplus H_- \oplus H_0$ and $A(z)$ is positive definite, negative definite, and null on H_+, H_-, and H_0, respectively. For

$$z = z^+ + z^- + z^0 \in H_+ \oplus H_- \oplus H_0 = H,$$

we take a norm for H

$$\|z\|_H^2 = A(z^+) - A(z^-) + |z^0|^2.$$

Under this norm, H becomes a Hilbert space and H_+, H_-, H_0 are orthogonal subspaces of H with respect to the inner product associated with this norm, as well as the L^2 inner product. One further analytical fact about H is needed which is proved in [13].

PROPOSITION 3.2. *For each $p \in [1, \infty)$, H is compactly embedded in $L^p(S^1, R^{2n})$. In particular there is an $\alpha_p > 0$ such that*

$$\|z\|_{L^p} \leq \alpha_p \|z\|_H$$

for all $z \in H$

Let X be the closed subspace of H defined by

$$\begin{aligned} X &= \{z \in H |\ a_j = 0, \qquad \text{if } j \text{ is odd }\} \\ &= \{z \in H |\ z = \sum [a_{2k} \cos 2kt + b_{2k} \sin 2kt]\}, \end{aligned}$$

where $z = \sum_{j \in Z} a_j e^{ijt}$, $a_j = \frac{1}{2\pi} \int_0^{2\pi} z(t) e^{-ijt} dt$. Then X is a closed invariant linear subspace of H and invariant under shifts; let $z \in X$ and τ be a real number, if $v(t) = z(t + \tau)$, then $v \in X$. X is invariant under G'; let $z \in X$ such that $G'(z) \in W^{\frac{1}{2},2}([0, 2\pi], R^{2n})$, then $G'(z) \in X$. Moreover $A(X) \subseteq X$, $A : X \to X$ is an isomorphism and $\nabla f(X) \subseteq X$. Therefore constrained critical points on X are in fact free critical points on H. Moreover, distinct critical orbits give rise to geometrically distinct solutions. From now on f will denote the restriction of f to X. Let $X_{m,l}^+ = X \cap H_{m,l}^+$, $X_{m,l}^- = X \cap H_{m,l}^-$, $X_+ = X \cap H_+$, $X_- = X \cap H_-$, and $X_0 = X \cap H_0$. Let P^+ be the orthogonal projection on X_+ and P^- be the orthogonal projection on X_-. The norm of $z = P^+ z + P^- z + z^0 \in H$ is

$$\|z\|^2 = \frac{1}{2}\|P^+ z\|^2 - \frac{1}{2}\|P^- z\|^2 + |z^0|^2.$$

The space X_m is a $2m$ dimensional subspace of X. By $(G2)$, without loss of generality, we assume that $r_0 \geq 1$. Set $\alpha_0 = \min_{|z| = r_0} G(z)$, $\beta_0 =$

$\beta_1 + \max_{|z|\le r_0} |G(z)|$, where β_1 is given by $(G3)$ or $(G4)$. Conditions $(G1)$ and $(G2)$ imply that, for some $\beta_3 \ge 0$,

$$\alpha_0|z|^\mu \le G(z) + \beta_0 \le \frac{1}{\mu}(G'(z)\cdot z + \beta_3) \qquad \text{for all } z \in R^{2n}. \tag{3.4}$$

Thus we have

$$\begin{aligned} f(z) &\le \frac{1}{2}\|z\|_H^2 - \int_0^{2\pi} G(z)dt \\ &\le \frac{1}{2}\|z\|_H^2 - \alpha_0\int_0^{2\pi} |z|^\mu\, dt + 2\beta_0\pi. \end{aligned}$$

Since $\mu > 2$, for any $m \in N$ there exists an $R_m > 0$ such that

$$f(z) \le 0, \qquad \forall z \in X^-_{m,\infty}\oplus X_0 \oplus X^+_{m,0}\oplus X^-_{0,m} \qquad \text{and } \|z\|_H \ge R_m. \tag{3.5}$$

PROOF OF THEOREM 1.1. Since $G \in C^1(R^{2n}, R)$, $G(0,0) = 0$ and G is even, $f \in C^1(X, R)$ and even with $f(0) = 0$. From (3.5), there exists an $R_m > 0$ such that $f(z) \le 0$, $\forall z \in X^-_{m,\infty} \cup X_0 \cup X^+_{0,m} \cup X^-_{0,m}$ and $\|z\|_H \ge R_m$. Thus condition (2) of Theorem 2.2 holds. If $z \in (X^-_{j,\infty} \cup X^+_{0,j} \cup X^-_{0,j})^\perp$, then there exists ϵ_j with

$$\lim_{j\to+\infty} \epsilon_j = 0$$

such that

$$\|z\|_{L^p} \le \epsilon_j \|z\|_H, \qquad p \in [1,\infty).$$

Then if $z \in (X^-_{j,\infty} \cup X^+_{0,j} \cup X^-_{0,j})^\perp$ and $\|z\|_H = 1$, then $P^- z = 0$. So we have

$$\begin{aligned} f(\rho z) &= \frac{1}{2}\int_0^{2\pi} (\rho\dot{z})\cdot J(\rho z)dt - \int_0^{2\pi} G(\rho z)dt \\ &= \frac{1}{2}\|P^+(\rho z)\|^2 + \frac{1}{2}|(\rho z)^0|^2 - \int_0^{2\pi} G(\rho z)dt \\ &\ge \frac{1}{2}\rho^2 - \epsilon_j \frac{\alpha_2}{p_2+1}|\rho|^{p_2+1} - 2\beta_2\pi. \end{aligned}$$

Thus there exists $\rho > 0$ such that

$$f(\rho z) \ge 0, \qquad \text{for } z \in (X^-_{j,\infty}\oplus X^+_{0,j}\oplus X^-_{0,j})^\perp \cap S_\rho \qquad \text{and } j \text{ large enough}. \tag{3.6}$$

Set $V = (X^-_{j,\infty} \oplus X^+_{0,j} \oplus X^-_{0,j})^\perp$. For $z \in V^\perp \cap S_\rho$ and large enough j, $f(\rho z) \ge 0$. Thus condition (1) of Theorem 2.2 holds. By (3.5) and (3.6), $f^j = f|_{X^-_{j,\infty}\oplus X_0 \oplus X^+_{0,j}\oplus X^-_{0,j}}$ satisfies $(PS)^*$ condition with respect to

$\{X^-_{j,\infty} \oplus X_0 \oplus X^+_{0,j} \oplus X^-_{0,j} | j = 1, 2, \ldots\}$. Thus we proved Theorem 1.1. □

4. Proof of Theorem 1.2

We need the following lemma which was proved in [9].

LEMMA 4.1. *For $\tau > 0$, $0 < q < 2$, there exist constant C_1, $C_2 > 0$, depending only on τ and q, such that*

$$\int_0^{2\pi} \exp(\sigma\tau|z|^q)dt \leq C_1 \exp(C_2(\sigma\|z\|_H)^{\frac{q}{2-q}}),$$

for every $\sigma > 0$, $z \in H$.

Proof. We use the notations in [9]. We prove the lemma for $H = W^{\frac{1}{2},2}(S^1, C)$. For $z \in H$, write

$$z(t) = C_0 + \sum_{n\neq 0} \frac{C_n}{\sqrt{|n|}} e^{int},$$

where $c_n \in C$. Then $z = k * g$, where

$$k(t) = \sum_{n\neq 0} \frac{1}{\sqrt{|n|}} e^{int} \in L(2, \infty),$$

$$g(t) = \sum_{n\in Z} c_n e^{int} \in L(2, 2).$$

By (8) of [9],

$$\begin{aligned} \sigma\tau|z^*|^q &\leq (\epsilon\sigma\tau)^{\frac{2}{q}}|z^*|^2 + \epsilon^{-\frac{2}{2-q}} \\ &\leq (\epsilon\sigma\tau)^{\frac{2}{q}} C[1 + |\log t|]\|k\|^2_{L^\infty}\|g\|^2_{L^2} + \epsilon^{-\frac{2}{2-q}} \end{aligned}$$

for $0 < t < 1$. Choose $\epsilon = \frac{1}{\sigma\tau}(qC\|k\|^2_{L^\infty}\|g\|^2_{L^2})^{-\frac{q}{2}}$, then from $\|g\|_{L^2} \leq C\|z\|_H$ and $\int_0^{2\pi} e^{\sigma\tau|z|^q}dt = \int_0^{2\pi} e^{\sigma\tau|z^*|^q}dt$, we get the lemma. □

PROOF OF THEOREM 1.2 From (3.5), there exists $R_m > 0$ such that $f(z) \leq 0$, $\forall z \in X^-_{m,\infty} \oplus X^+_{0,m} \oplus X^-_{0,m}$ and $\|z\|_H \geq R_m$ for any $m \in N$. Thus condition (2) of Theorem 2.2 holds. If $z \in (X^-_{j,\infty} \oplus X^+_{0,j} \oplus X^-_{0,j})^\perp$, then there exists ϵ_j with

$$\lim_{j\to+\infty} \epsilon_j = 0$$

such that

$$\|z\|_{L^p} \le \epsilon_j \|z\|_H, \qquad p \in [1, \infty).$$

Thus, if $z \in (X^-_{j,\infty} \oplus X^+_{0,j} \oplus X^-_{0,j})^\perp$ and $\|z\|_H = 1$, then $P^- z = 0$. By Lemma 4.1,

$$f(\rho z) = \frac{1}{2}\|P^+ \rho z\|^2 + \frac{1}{2}|(\rho z)^0|^2 - \int_0^{2\pi} G(z)dt.$$

Thus we have

$$f(\rho z) \ge \frac{1}{2}\rho^2 - C_1 \alpha_2 e^{C_2 (\sigma \epsilon_j \rho^2 \|z\|_H)^{\frac{q}{2-q}}} - 2\beta_2 \pi.$$

Thus there exists $\rho > 0$ such that

$$f(\rho z) \ge 0 \qquad \text{for } z \in (X^-_{j,\infty} \oplus X^+_{0,j} \oplus X^-_{0,j})^\perp \cap S_\rho \qquad \text{and } j \text{ large enough}. \tag{4.1}$$

By (3.5) and (4.1), $f^j = f|_{(X^-_{j,\infty} \oplus X^+_{0,j} \oplus X^-_{0,j})^\perp}$ satisfies $(PS)^*$ condition. Applying Theorem 2.2, we have that $f(z)$ has infinitely many distinct critical points. This proved Theorem 2.2. □

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Department of Mathematics
Kunsan National University
Kunsan 573-701 Korea
E-mail: `tsjung@kunsan.ac.kr`

Department of Mathematics Education
Inha University
Incheon 402-751 Korea
E-mail: `qheung@inha.ac.kr`