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SEQUENCES IN THE RANGE OF A VECTOR MEASURE

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Abstract. We prove that every strong null sequence in a Banach space X lies inside the range of a vector measure of bounded variation if and only if the condition $\mathcal{N}_1(X,\ell_1) = \Pi_1(X,\ell_1)$ holds. We also prove that for $1 \leq p < \infty$ every strong ℓ_p sequence in a Banach space X lies inside the range of an X-valued measure of bounded variation if and only if the identity operator of the dual Banach space X^* is $(p',1)$-summing, where p' is the conjugate exponent of p. Finally we prove that a Banach space X has the property that any sequence lying in the range of an X-valued measure actually lies in the range of a vector measure of bounded variation if and only if the condition $\Pi_1(X,\ell_1) = \Pi_2(X,\ell_1)$ holds.

1. Introduction

The intriguing connection between the geometry of subsets of Banach spaces and vector measure theory is not confined to Radon-Nikodym considerations. Questions regarding the finer structure of the range of a vector measure have found interest since Liapounoff's discovery of his everintriguing convexity theorem which states that the range of a nonatomic vector measure with values in a finite dimensional space is compact and convex. The infinite dimensional version of Liapounoff's theorem remained resistant to analysis for a long time. It is an important fact, first established by Bartle, Dunford and Schwartz in the early fifties, that the range of a vector measure is always relatively weakly compact.

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Among the relatively weakly compact subsets of Banach spaces, those that are the range of a vector measure occupy a special place ; a remarkable similarity to the relatively norm compact sets is evidenced. For instance, Diestel and Seifert [3] proved that any sequence in the range of a vector measure admits a subsequence with norm convergent arithmetic means, a phenomenon not shared by all weakly compact sets.

Any intuition gained by noting the similarities between relatively norm compact sets and sets arising as ranges of vector measures must be tempered by the fact that the closed unit ball of an infinite dimensional Banach space can be the range of a vector measure.

Anantharaman and Garg [1] proved that the closed unit ball of a Banach space X is the range of a vector measure if and only if the dual of a Banach space X is isometrically isomorphic to a reflexive subspace of $L^1(\mu)$ for some probability measure μ.

Anantharaman and Diestel [2] found that every weakly compact subset of $BD1$ (the separable $\mathcal{L}_\infty$ space of Bourgain and Delbaen that has the weakly compact extension property) lies inside the range of a $BD1$-valued measure. They also gave some necessary and some sufficient conditions for a sequence in a Banach space X to lie in the range of an X-valued measure.

Piñeiro and Rodriguez-Piazza [7] showed that the compact subset of a Banach space X lies inside the range of an X-valued measure if and only if the dual of a Banach space X can be embedded into an $L^1(\mu)$-space for a suitable measure μ.

It is an easy consequence of the celebrated Dvoretsky-Rogers theorem that given an infinite dimensional Banach space X, there is an X-valued measure that does not have finite variation [12]. Thus the question arose : Which Banach spaces X have the property that every compact subset of X lies inside the range of an X-valued measure of bounded variation ? This was settled by Piñeiro and Rodriguez-Piazza [7]: Only finite-dimensional Banach spaces have this property.

Piñeiro [9] treated the same problem in which case the measure is not X-valued. He proved that every strong null sequence in a Banach space X lies inside the range of a vector measure of bounded variation if and only if X is isomorphic to a Hilbert space. He also showed that every bounded sequence in X that lies in the range of a vector measure of bounded variation actually lies inside the range of an X^{**}-valued measure of bounded variation if and only if X^* satisfies Grothendieck's theorem.

In this paper we survey geometric structures of the ranges of vector measures.

We first characterize Banach spaces X having the property that every strong null sequence in X lies inside the range of a vector measure of bounded variation in terms of operators.

Next we give descriptions of Banach spaces X for which every strong ℓ_p sequence in X lies inside the range of an X-valued measure of bounded variation. And then we find usable sufficient condition for a sequence lying in the range of an X-valued measure to lie inside the range of an X^{**}-valued measure of bounded variation.

Finally we give a characterization of Banach spaces X with the property that every sequence lying in the range of an X-valued measure actually lies in the range of a vector measure of bounded variation.

2. Definitions and Notations

We give some definitions and notation to be used. Throughout this paper X and Y denote Banach spaces.

A function μ from a σ-field Σ of subsets of a set Ω to a Banach space X is called a countably additive vector measure if $\mu(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$ in the norm topology of X for all sequences (E_n) of pairwise disjoint members of Σ such that $\cup_{n=1}^{\infty} E_n \in \Sigma$. The range of μ will be denoted by $\operatorname{rg}\mu$. The variation of μ is the extended nonnegative function $|\mu|$ whose value on a set $E \in \Sigma$ is given by $|\mu|(E) = \sup_\pi \sum_{A\in\pi} \|\mu(A)\|$, where the supremum is taken over all partitions π of E into a finite number of pairwise disjoint members of Σ. If $|\mu|(\Omega) = \operatorname{tv}(\mu) < \infty$ then μ will be called a measure of bounded variation. The semivariation of μ is the extended nonnegative function $\|\mu\|$ whose value on a set $E \in \Sigma$ is given by $\|\mu\|(E) = \sup\{|x^* \circ \mu|(E) : x^* \in X^*, \|x^*\| \le 1\}$, where $|x^* \circ \mu|$ is the variation of the real-valued measure $x^* \circ \mu$. If $\|\mu\|(\Omega) = \operatorname{tsv}(\mu) < \infty$, then μ will be called a measure of bounded semivariation.

Notations. (1) The dual of a Banach space X is denoted by X^*.

(2) The closed unit ball of a Banach space X is denoted by B_X.
(3) The dual operator of an operator T is denoted by T^*.
(4) $\mathcal{B}(X,Y)$ denotes the set of all bounded linear operators from X into Y.
(5) $\mathcal{K}(X,Y)$ denotes the set of all compact operators from X into Y.
(6) The identity operator of a Banach spaces X is denoted by id_X.
(7) The canonical isometric embedding from a Banach space X into the bidual of X is denoted by κ_X.
(8) The natural isometric embedding from X into $\ell_\infty(B_{X^*})$ is denoted by i_X.
(9) For $1 < p < \infty$, the conjugate exponent of p is denoted by p', i.e. $1/p + 1/p' = 1$.

The space $\mathcal{R}(X)$ is defined to consist of all sequences (x_n) in X such that there exists an X-valued measure μ satisfying $\{x_n : n \in \mathbb{N}\} \subset \operatorname{rg}\mu$. For each $(x_n) \in \mathcal{R}(X)$, define $\|(x_n)\|_r = \inf \operatorname{tsv}(\mu)$, where the infimum is taken over all vector measures μ as above.

The space $\mathcal{R}_c(X)$ consists of all sequences in X that lie inside the range of an X-valued measure with relatively compact range. If (x_n) belongs to $\mathcal{R}_c(X)$ then proposition 1.4 of [7] ensures that there exists an unconditionally convergent series $\sum_{k=1}^\infty y_k$ in X for which $\{x_n : n \in \mathbb{N}\} \subset \{\sum_{k=1}^\infty \alpha_k y_k : (\alpha_k) \in \ell_\infty, \|(\alpha_k)\|_\infty \le 1\}$. For each $(x_n) \in \mathcal{R}_c(X)$, define $\|(x_n)\|_{rc} = \inf \sup\{\sum_{k=1}^\infty |\langle x^*, y_k\rangle| : x^* \in B_{X^*}\}$, where the infimum is taken over all unconditionally convergent series $\sum_{k=1}^\infty y_k$ of the kind described above.

The space $\mathcal{R}_{bv}(X)$ is defined to consist of all sequences (x_n) in X such that there exists an X-valued measure μ with bounded variation satisfying $\{x_n : n \in \mathbb{N}\} \subset \operatorname{rg}\mu$. For each $(x_n) \in \mathcal{R}_{bv}(X)$, set $\|(x_n)\|_{bv} = \inf \operatorname{tv}(\mu)$, where the infimum is taken over all vector measures μ as above.

The space $\mathcal{R}_{iv}(X)$ is defined to consist of all sequences (x_n) in X such that there exists an isometric $J : X \to X_0$ and an X_0-valued measure μ with bounded variation satisfying $\{Jx_n : n \in \mathbb{N}\} \subset \operatorname{rg}\mu$.

Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. We say that the operator $T : X \to Y$ belongs to $\mathcal{A}^*(X,Y)$ provided there is a constant $C \ge 0$ such that regardless of the finite dimensional normed spaces E and F and operators $a \in \mathcal{B}(E,X), b \in \mathcal{B}(Y,F)$ and $U \in \mathcal{B}(F,E)$, the composition $E \xrightarrow{a} X \xrightarrow{T} Y \xrightarrow{b} F \xrightarrow{U} E$ satisfies $|\operatorname{tr}(UbTa)| \leq C \cdot \|a\| \cdot \|b\| \cdot \alpha(U)$. The collection of all such C has an infimum, which is denoted by $\alpha^*(T)$. The Banach operator ideal $[\mathcal{A}^*, \alpha^*]$ is called the adjoint operator ideal of $[\mathcal{A}, \alpha]$.

For $1 \leq p \leq \infty$, an operator $T \in \mathcal{B}(X,Y)$ is called p-integral if there are a probability measure μ and operators $A \in \mathcal{B}(L^p(\mu), Y^{**})$ and $B \in \mathcal{B}(X, L^\infty(\mu))$ such that $\kappa_Y \circ T = A \circ i_p \circ B$, where $i_p : L^\infty(\mu) \to L^p(\mu)$ is the formal identity. The p-integral norm of T is defined by $\imath_p(T) = \inf\{\|A\|\|B\|\}$, where the infimum is extended over all measures μ and operators A and B as above. The collection of all p-integral operators from X into Y is denoted by $\mathcal{I}_p(X,Y)$.

We denote by $C_0(X)$ the space of all sequences (x_n) in X with $\lim_{n\to\infty} \|x_n\| = 0$.

Let $1 \leq p < \infty$. The vector sequence (x_n) in X is strongly p-summable if the corresponding scalar sequence $(\|x_n\|)$ is in ℓ_p. We denote by $\ell_p^{\mathrm{strong}}(X)$ the set of all such sequences in X. This is a Banach space with respect to the norm $\|(x_n)\|_p^{\mathrm{strong}} = (\sum_n \|x_n\|^p)^{1/p}$.

Let $1 \leq p < \infty$. The vector sequence (x_n) in X is weakly p-summable if the scalar sequences $(\langle x^*, x_n\rangle)$ are in ℓ_p for every $x^* \in X^*$. We denote by $\ell_p^{\mathrm{weak}}(X)$ the set of all such sequences in X. This is a Banach space under the norm

$$\|(x_n)\|_p^{\mathrm{weak}} = \sup\big\{(\sum_n |\langle x^*, x_n\rangle|^p)^{1/p} : x^* \in X^*, \ \|x^*\| \leq 1\big\}.$$

For $1 \leq p \leq \infty$, an operator $T \in \mathcal{B}(X,Y)$ is said to be p-nuclear if it can be written in the form $T = \sum_{i=1}^\infty x_i^* \otimes y_i$, where (x_i^*) in X^* and (y_i) in Y satisfy $N_p((x_i^*),(y_i)) < \infty$. Here

$$N_p((x_i^*),(y_i)) = \begin{cases} \|(x_i^*)\|_1^{\mathrm{strong}} \cdot (\sup_i \|y_i\|) & \text{for } p = 1, \\ \|(x_i^*)\|_p^{\mathrm{strong}} \cdot \|(y_i)\|_{p'}^{\mathrm{weak}} & \text{for } 1 < p < \infty, \\ (\sup_i \|x_i\|) \cdot \|(y_i)\|_1^{\mathrm{weak}} & \text{for } p = \infty. \end{cases}$$

Each such representation of T is called a p-nuclear representation. The set of all p-nuclear operators from X into Y is denoted by $\mathcal{N}_p(X,Y)$.

With each $T \in \mathcal{N}_p(X,Y)$ we associate its p-nuclear norm, $\nu_p(T) = \inf N_p((x_i^*),(y_i))$, where the infimum is taken over all p-nuclear representations of T.

Let $1 \leq p \leq q < \infty$. An operator $T \in \mathcal{B}(X,Y)$ is called absolutely (q,p)-summing if there exists a constant $C \geq 0$ such that for all finite subsets $\{x_i\}_{i=1}^n \subset X$, we have

$$(\sum_{i=1}^n \|Tx_i\|^q)^{1/q} \leq C \cdot \sup\{(\sum_{i=1}^n |\langle x^*, x_i\rangle|^p)^{1/p} : x^* \in B_{X^*}\}.$$

The infimum of such C is the absolutely (q,p)-summing norm of T and denoted by $\pi_{q,p}(T)$. We write $\Pi_{q,p}(X,Y)$ for the set of all absolutely (q,p)-summing operators from X into Y. If $p = q$ we say absolutely p-summing operator instead of absolutely (p,p)-summing operator, and we use the notation $\pi_p(T)$ instead of $\pi_{p,p}(T)$ and $\Pi_p(X,Y)$instead of $\Pi_{p,p}(X,Y)$.

An operator $T \in \mathcal{B}(X,Y)$ is called Pietsch integral if there exists a Y-valued countably additive vector measure μ of bounded variation defined on the Borel sets of B_{X^*} such that for each $x \in X$, $T(x) = \int_{B_{X^*}} \langle x^*, x\rangle \, d\mu(x^*)$. The set of Pietsch integral operators from X into Y becomes a Banach space under the norm $\|T\|_{\text{pint}} = \inf\{|\mu|(B_{X^*})\}$,where the infimum is taken over all measures μ that satisfy the above definition.

A Banach space X has the Radon-Nikodym property with respect to (Ω, Σ, μ) if for each μ-continuous vector measure $G : \Sigma \to X$ of bounded variation there exists $g \in L_1(\mu, X)$ such that $G(E) = \int_E g \, d\mu$ for all $E \in \Sigma$. A Banach space X has the Radon-Nikodym property if X has the Radon-Nikodym property with respect to every finite measure space.

We will say that a Banach space X satisfies Grothendieck's theorem if every operator from X into a Hilbert space is absolutely 1-summing.

3. Results

We begin by describing Banach spaces X for which any strong null sequence in X lies inside the range of a vector measure of bounded variation.

Theorem 1. *The following statements are equivalent.*

(i) $C_0(X) \subset \mathcal{R}_{iv}(X)$.

(ii) $\mathcal{N}_1(X, \ell_1) = \Pi_2(X, \ell_1)$.

(iii) *X is isomorphic to a Hilbert space.*

Proof. (i)⇒(ii). The hypothesis (i) leads us to have that there exist an isometric $J : X \to X_0$ and a vector measure $F : \sum \to X_0$ with bounded variation such that $\{Jx_n : n \in \mathbb{N}\} \subset \operatorname{rg} F$ for any sequence $(x_n) \in C_0(X)$. We invoke theorem 2.4 of [8] to infer that J^* is 1-summing and so 2-summing. Then J^* is factorizable through a Hilbert space H as follows : $J^* : X_0^* \xrightarrow{v} H \xrightarrow{u} X^*$. As J^* is onto, so is u. Injectivity of u can be arranged by replacing H by the orthogonal complement of $\ker u$. Then the open mapping theorem tells us that u is an isomorphism and hence X is isomorphic to a Hilbert space. This allows us to create a continuous map $\Phi : C_0(X) \to \Pi_1(\ell_1, X)$ through $\Phi(x_n) = \sum_n e_n \otimes x_n$. Taking account of the fact that $\Pi_1(\ell_1, X) = \Pi_2(\ell_1, X)$ and $\nu_2(S) = \pi_2(S)$ for any finite rank operator S, we derive that there exists a constant $C > 0$ such that $\nu_2(\sum_{k=1}^n e_k \otimes x_k) \leq C \sup\{\|x_k\| : 1 \leq k \leq n\}$ for all finite sets $\{x_1, \cdots x_n\}$ in X and $n \in \mathbb{N}$. Using the trace duality we have

$$(*) \qquad \sum_{k=1}^n \|x_k^*\| \leq C\, \pi_2(\sum_{k=1}^n x_k^* \otimes e_k : X \to \ell_1^n).$$

Now let us take operator $T = \sum_k x_k^* \otimes e_k \in \Pi_2(X, \ell_1)$. Then it follows from $(*)$ that $\sum_{k=1}^n \|x_k^*\| \leq C\, \pi_2(T)$ for all n and hence $T \in \mathcal{N}_1(X, \ell_1)$. As $\mathcal{N}_1(X, \ell_1) \subset \Pi_2(X, \ell_1)$ we end up with the desired equality $\mathcal{N}_1(X, \ell_1) = \Pi_2(X, \ell_1)$.

(ii)⇒(iii). The hypothesis (ii) assures us that there exists a constant $C > 0$ such that $\nu_1(U) = \sum \|x_k^*\| \leq C\, \pi_2(U)$ for any operator $U = \sum_k x_k^* \otimes e_k \in \mathcal{N}_1(X, \ell_1)$. We make use of the fact that $\nu_2(S) = \pi_2(S)$ for any finite rank operator S to obtain that

$$\sum_{k=1}^n \|x_k^*\| \leq C\, \nu_2(\sum_{k=1}^n x_k^* \otimes e_k : X \to \ell_1^n).$$

Using the trace duality we obtain

$$\pi_2(\sum_{k=1}^n e_k \otimes x_k : \ell_1^n \to X) \leq C \sup\{\|x_k\| : 1 \leq k \leq n\} \quad \text{for all } n.$$

Since $\Pi_1(\ell_1, X) = \Pi_2(\ell_1, X)$, it follows that the operator $\sum_n e_n \otimes x_n \in \Pi_1(\ell_1, X)$ for any sequence $(x_n) \in C_0(X)$. The injectivity of $\ell_\infty(B_{X^*})$ ensures that $i_X \circ \sum_n e_n \otimes x_n \in \mathcal{I}_1(\ell_1, \ell_\infty(B_{X^*}))$. This permits us to define a continuous map $\Psi : C_0(X) \to \mathcal{I}_1(\ell_1, \ell_\infty(B_{X^*}))$ by $\Psi(x_n) = \sum_n e_n \otimes i_X x_n$. Notice that Ψ maps each finite sequence into a nuclear operator and $\mathcal{N}_1(\ell_1, \ell_\infty(B_{X^*}))$ is a normed subspace of $\mathcal{I}_1(\ell_1, \ell_\infty(B_{X^*}))$. From this we find that the range of Ψ is contained in $\mathcal{N}_1(\ell_1, \ell_\infty(B_{X^*}))$.
Given $S = \sum_n a_n \otimes e_n \in \mathcal{B}(\ell_\infty(B_{X^*}), \ell_1)$, with $a_n \in \ell_1^{\text{weak}}(\ell_\infty(B_{X^*})^*)$, and $(x_n) \in C_0(X)$, we use the trace duality to obtain the following :

$$\begin{aligned}\langle \Psi^* S, (x_n)\rangle &= \langle S, \Psi(x_n)\rangle = \operatorname{tr}(S \circ \sum_n e_n \otimes i_X x_n) \\ &= \sum_k \langle S \circ \sum_n e_n \otimes i_X x_n(e_k),\, e_k\rangle = \sum_k \langle \sum_n \langle a_n,\, i_X x_k\rangle e_n,\, e_k\rangle \\ &= \sum_n \langle a_n,\, i_X x_n\rangle = \sum_n \langle x_n,\, {i_X}^* a_n\rangle.\end{aligned}$$

Thus $\Psi^* : \mathcal{B}(\ell_\infty(B_{X^*}), \ell_1) \to \ell_1(X^*) : \sum_n a_n \otimes e_n \mapsto ({i_X}^* a_n)_n$.
As a result ${i_X}^*$ is 1-summing and thus 2-summing. Then ${i_X}^*$ admits a factorization ${i_X}^* : \ell_\infty(B_{X^*})^* \xrightarrow{b} H \xrightarrow{a} X^*$, where H is a Hilbert space. The surjectivity of ${i_X}^*$ transfers to a. Summoning the open mapping theorem we discover that X is isomorphic to a Hilbert space.
(iii)⇒(i). The hypothesis (iii) informs us that the identity operator id_X of X factors through a Hilbert space H as follows : $id_X : X \xrightarrow{v} H \xrightarrow{u} X$. Let us take any sequence $(x_n) \in C_0(X)$. As we know, an operator $T : \ell_1 \to H : (\alpha_n) \mapsto \sum_n \alpha_n v x_n$ is 1-summing. Consequently $u \circ T : \ell_1 \to X : (\alpha_n) \mapsto \sum_n \alpha_n u \circ v x_n = \sum_n \alpha_n x_n$ is 1-summing. Let $i_X \circ u \circ T : \ell_1 \xrightarrow{i} C(K) \xrightarrow{j} \ell_\infty(B_{X^*})$ be a 1-summing factorization, where K is a weak* compact norming subset of B_{ℓ_∞} and j is 1-summing. We apply proposition 1.3 of [7] to deduce that there exists an $\ell_\infty(B_{X^*})$-valued measure ν with bounded variation satisfying $\overline{j(B_{C(K)})} = \operatorname{rg}(\nu)$ and so $\{i_X(x_n) : n \in \mathbb{N}\} \subset \operatorname{rg}(\|i\|\nu)$, that is $(x_n) \in \mathcal{R}_{iv}(X)$. □

Using the same method as in the proof of theorem 4 in [9], we establish the following useful characterization of Banach spaces X with the property that every strong ℓ_p sequence in X lies inside the range of an X-valued measure of bounded variation.

THEOREM 2. *Let $1 < p < \infty$. The following are equivalent statements.*

(i) $\ell_p^{\text{strong}}(X) \subset \mathcal{R}_{bv}(X)$.

(ii) $\ell_p^{\text{strong}}(X) \subset \mathcal{R}_{bv}(X^{**})$.

(iii) *For every sequence (x_n^*) in X^* such that the operator $\sum_n x_n^* \otimes e_n \in \mathcal{K}(X, \ell_1)$, we have $\sum_n \|x_n^*\|^{p'} < \infty$.*

(iv) *id_{X^*} is $(p', 1)$-summing.*

Proof. (i)⇒(ii) is obvious.
(ii)⇒(iii). Let us take any sequence $(x_n) \in \ell_p^{\text{strong}}(X)$. The hypothesis (ii) provides a vector measure $\mu : \sum \to X^{**}$ with bounded variation for which $\{x_n : n \in \mathbb{N}\} \subset \operatorname{rg} \mu$. Choose $A_n \in \sum$ so that $\mu(A_n) = x_n$ for each $n \in \mathbb{N}$. Define operators $u : L_\infty(|\mu|) \to X^{**}$ and $v : \ell_1 \to L_\infty(|\mu|)$ via $u(f) = \int f d\mu$ and $v(\alpha_n) = \sum_n \alpha_n \chi_{A_n}$, respectively. These operators validate the following :

$$uv(\alpha_n) = u(\sum_n \alpha_n \chi_{A_n}) = \int \sum_n \alpha_n \chi_{A_n} d\mu = \sum_n \alpha_n x_n.$$

Since u is a 1-summing operator with domain $L_\infty(|\mu|)$, it follows that u is Pietsch-integral, so integral. Consequently $uv \in \mathcal{I}_1(\ell_1, X^{**})$ and thus $\sum_n e_n \otimes x_n \in \mathcal{I}_1(\ell_1, X)$. This allows us to define a continuous map $\Phi : \ell_p^{\text{strong}}(X) \to \mathcal{I}_1(\ell_1, X)$ by $\Phi(x_n) = \sum_n e_n \otimes x_n$. Note that Φ maps each finite sequence into a nuclear operator and $\mathcal{N}_1(\ell_1, X)$ is a normed subspace of $\mathcal{I}_1(\ell_1, X)$. This implies that the range of Φ is contained in $\mathcal{N}_1(\ell_1, X)$.
Given $S = \sum_n x_n^* \otimes e_n \in \mathcal{B}(\ell_1, X)$ and $(x_n) \in \ell_p^{\text{strong}}(X)$, we use the trace duality to get the following :

$$\langle \Phi^* S, (x_n) \rangle = \langle S, \Phi(x_n) \rangle = \operatorname{tr}(S \circ \sum_n e_n \otimes x_n)$$

$$\sum_k \langle S \circ \sum_n e_n \otimes x_n(e_k), e_k \rangle = \sum_k \langle \sum_n \langle x_n^*, x_k \rangle e_n, e_k \rangle = \sum_n \langle x_n^*, x_n \rangle.$$

Thus $\Phi^* : \mathcal{B}(X, \ell_1) \to \ell_{p'}^{\text{strong}}(X^*) : \sum_n x_n^* \otimes e_n \mapsto (x_n^*)_n$. In particular Φ^* takes $\mathcal{K}(X, \ell_1)$ into $\ell_{p'}^{\text{strong}}(X^*)$.
(iii)⇒(iv). The hypothesis (iii) guarantees that every weak ℓ_1 sequence in X^* is a strong $\ell_{p'}$ sequence. In other words, id_{X^*} is $(p', 1)$-summing.

(iv)⇒(i). The hypothesis (iv) enables us to define a map $\phi : \mathcal{B}(X, \ell_1) \to \ell_{p'}^{\text{strong}}(X^*)$ by $\phi(\sum_n x_n^* \otimes e_n) = (x_n^*)_n$. Using the trace duality we have that ϕ^* takes $\ell_p^{\text{strong}}(X)$ into $\mathcal{N}_1(\ell_1, X)$.

Now let us select any sequence $(x_n) \in \ell_p^{\text{strong}}(X)$. Then the map $T : \ell_1 \to X$ given by $T(\alpha_n) = \sum_n \alpha_n x_n$ is nuclear, so Pietsch-integral. Hence T admits a factorization $T : \ell_1 \xrightarrow{a} C(\Omega) \xrightarrow{b} X$, where Ω is some compact Hausdorff space, a is bounded and b is 1-summing. An appeal to proposition 1.3 of [7] reveals that there exists an X-valued measure ν with bounded variation for which $\overline{b(B_{C(\Omega)})} = \text{rg}(\nu)$ and hence $\{x_n : n \in \mathbb{N}\} \subset \text{rg}(\|a\|\nu)$, that is $(x_n) \in \mathcal{R}_{bv}(X)$. □

As a corollary we drive another usable sufficient condition which implies that every strong ℓ_2 sequence in a Banach space X lies inside the range of an X-valued measure of bounded variation.

COROLLARY 3. *If* $\Pi_1(X^*, Y) = \Pi_2(X^*, Y)$ *for every Banach space* Y, *then* $\ell_2^{\text{strong}}(X) \subset \mathcal{R}_{bv}(X)$.

Proof. For each $n \in \mathbb{N}$, take any operator $T : \ell_\infty^n \to X^*$. Consider a composition $E \xrightarrow{a} \ell_\infty^n \xrightarrow{T} X^* \xrightarrow{b} F \xrightarrow{U} E$ of operators with E and F finite dimensional. In view of our hypothesis, we have

$$|\operatorname{tr}(UbTa)| \le \imath_1(UbT)\|a\| = \pi_1(UbT)\|a\| \le C\|b\|\,\|T\|\pi_2(U)\|a\|.$$

This ensures that $\pi_2(T) = \pi_2^*(T) \le C\|T\|$. By localization we get that

$$(**) \qquad \mathcal{I}_\infty(Y, X^*) = \Pi_2(Y, X^*) \qquad \text{for every Banach space } Y.$$

Now we select a sequence $(x_n^*) \in \ell_1^{\text{weak}}(X^*)$. Let $v : C_0 \to X^*$ be the operator defined by $v\,e_n = x_n^*$ for all n, so that $\|v\| = \|(x_n^*)\|_1^{\text{weak}}$. Then it follows from $(**)$ that v is 2-summing and thus $(x_n^*) \in \ell_2^{\text{strong}}(X^*)$. Theorem 2 steps in to conclude that $\ell_2^{\text{strong}}(X) \subset \mathcal{R}_{bv}(X)$. □

Next we find a special kind of Banach space X with the property that any sequence lying in the range of an X-valued measure actually lies in the range of an X^{**}-valued measure of bounded variation.

Proposition 4. *Suppose X and X^* satisfy Grothendieck's theorem. Then $\mathcal{R}(X) \subset \mathcal{R}_{bv}(X^{**})$.*

Proof. Let $S = \sum_n x_n^* \otimes e_n \in \mathcal{B}(X, \ell_1)$. As X^* satisfies Grothendieck's theorem, we have that $S \in \Pi_2(X, \ell_1)$ and hence there exists a factorization $S : X \xrightarrow{v} \ell_2 \xrightarrow{u} \ell_1$. Since X satisfies Grothendieck's theorem, it follows that $v \in \Pi_1(X, \ell_2)$ and so $S \in \Pi_1(X, \ell_1)$. This forces that there is a constant C such that $\pi_1(S) \leq C\|S\|$. We consider the map $\Psi_S : \mathcal{R}(X) \to \mathbb{R}$ which is defined by $\Psi_S(z_n) = \sum_n \langle z_n, x_n^* \rangle$. If $(z_n) \in \mathcal{R}(X)$, given $\epsilon > 0$ there exists a vector measure $\mu : \sum \to X$ such that $\{z_n : n \in \mathbb{N}\} \subset \operatorname{rg}\mu$ and $\operatorname{tsv}(\mu) \leq \epsilon + \|(z_n)\|_r$. Let λ be a control measure for μ and let $I : L_\infty(\lambda) \to X : f \mapsto \int f d\mu$ be the integration operator. Then $S \circ I \in \Pi_1(L_\infty(\lambda), \ell_1)$. The Radon-Nikodym property of ℓ_1 indicates that $S \circ I$ is nuclear and so is $(S \circ I)^*$. Notice that

$$(\cdot) \quad \|(S \circ I)^* e_n\| = \sup\{|\langle (S \circ I)^* e_n, f\rangle| : \|f\|_\infty \leq 1\}$$
$$\sup\{|\langle \int f d\mu, x_n^* \rangle| : \|f\|_\infty \leq 1\} = \sup\{|\int f d(x_n^* \circ \mu)| : \|f\|_\infty \leq 1\}.$$

Choose $A_n \in \sum$ such that $\mu(A_n) = z_n$ for each $n \in \mathbb{N}$. Then it follows from $(\cdot)$ that

$$\begin{aligned}\sum_n |\int \chi_{A_n} d(x_n^* \circ \mu)| &= \sum_n |x_n^* \circ \mu(A_n)| = \sum_n |\langle x_n^*, z_n \rangle| \\ &\leq \sum_n \|(S \circ I)^* e_n\| = \nu_1((S \circ I)^*) \leq \nu_1(S \circ I) = \pi_1(S \circ I) \\ &= \operatorname{tv}(S \circ \mu) \leq \pi_1(S) \operatorname{tsv}(\mu) \leq \pi_1(S)(\epsilon + \|(z_n)\|_r).\end{aligned}$$

This gives that $\|\Psi_S\| \leq \pi_1(S) \leq C\,\|S\|$.

Now take any sequence $(x_n) \in \mathcal{R}(X)$. Taking note of the fact that the closed convex hull of the range of a vector measure is the range of another vector measure, we deduce that the map $U : C_0 \to \mathcal{R}(X) : (\alpha_n) \mapsto (\alpha_n x_n)$ is continuous. The upshot of all this is that the map $\Phi : \mathcal{B}(X, \ell_1) \to \ell_1 : S \mapsto U^*(\Psi_S)$ is continuous.
Observe that

$$\langle U^*(\Psi_S), (\alpha_n) \rangle = \langle \Psi_S, U(\alpha_n) \rangle = \Psi_S(\alpha_n x_n) = \sum_n \langle \alpha_n x_n, x_n^* \rangle$$

and so $U^*(\Psi_S) = (\langle x_n, x_n^*\rangle)_n$.
Given $S = \sum_n x_n^* \otimes e_n$ and $(\beta_n) \in \ell_\infty$, we use the trace duality to obtain the following :

$$\langle \Phi^*(\beta_n), S\rangle = \langle (\beta_n), \Phi(S)\rangle = \sum_n \langle \beta_n x_n, x_n^*\rangle = \mathrm{tr}(S \circ \Phi^*(\beta_n))$$

$$\sum_k \langle S \circ \Phi^*(\beta_n) e_k,\, e_k\rangle = \sum_k \langle \sum_n \langle x_n^*,\, \Phi^*(\beta_n) e_k\rangle e_n,\, e_k\rangle$$

$$= \sum_n \langle x_n^*,\, \Phi^*(\beta_n) e_n\rangle.$$

Therefore $\Phi^* : \ell_\infty \to \mathcal{I}(\ell_1, X) : (\beta_n) \mapsto \sum_n e_n \otimes \beta_n x_n$. Then $T = \sum_n e_n \otimes x_n \in \mathcal{I}(\ell_1, X)$ and thus T admits an integral factorization $\kappa_X \circ T : \ell_1 \xrightarrow{i} C(K) \xrightarrow{j} X^{**}$, where K is a weak* compact norming subset of B_{ℓ_∞} and j is 1-summing. Proposition 1.3 of [7] furnishes an X^{**}-valued measure ν of bounded variation such that $\overline{j(B_{C(K)})} = \mathrm{rg}(\nu)$ and hence $\{x_n : n \in \mathbb{N}\} \subset \mathrm{rg}(\|i\|\nu)$, that is $(x_n) \in \mathcal{R}_{bv}(X^{**})$. □

Which Banach spaces X have the property that every sequence lying in the range of an X-valued measure actually lies in the range of a vector measure of bounded variation ? The theorem stated below handles this question.

Theorem 5. *The following statements are equivalent.*

(i) $\mathcal{R}(X) \subset \mathcal{R}_{iv}(X)$.
(ii) $\Pi_1(X, \ell_1) = \Pi_2(X, \ell_1)$.

Proof. (i)⇒(ii). Let us select any sequence $(x_n) \in \mathcal{R}_c(X)$. We use this sequence to define an operator $T : \ell_1 \to X$ via $T(\alpha_n) = \sum \alpha_n x_n$. We call on proposition 1.4 of [7] to derive that there exists an unconditionally convergent series $\sum_k y_k$ in X so that $x_n = \sum_k \delta_{k,n} y_k$, where $\|(\delta_{k,n})_k\|_\infty \le 1$. We exploit the fact that there exist a weakly summable sequence (z_k) in X and a sequence (λ_k) in B_{C_0} for which $y_k = \lambda_k z_k$ to see that $T \in \mathcal{N}_\infty(\ell_1, X)$. In fact $T = \sum_k \lambda_k \delta_k \otimes z_k$, where $\delta_k = (\delta_{k,n})_n$. The hypothesis (i) assures us that there exist an isometric $J : X \to X_0$ and a vector measure $\mu : \sum \to X_0$ with bounded variation such that $\{Jx_n : n \in \mathbb{N}\} \subset \mathrm{rg}\,\mu$. Choose $A_n \in \sum$ so that

$\mu(A_n) = Jx_n$ for each $n \in \mathbb{N}$. Define operators $u : L_\infty(|\mu|) \to X_0$ and $v : \ell_1 \to L_\infty(|\mu|)$ via $u(f) = \int f d\mu$ and $v(\alpha_n) = \sum \alpha_n \chi_{A_n}$, respectively. These operators validate the following :

$$uv(\alpha_n) = u(\sum_n \alpha_n \chi_{A_n}) = \int \sum_n \alpha_n \chi_{A_n} d\mu = \sum_n \alpha_n J x_n = JT\alpha_n.$$

Since u is 1-summing, the same is true of $uv = JT$. As J is an isometric, T is 1-summing. This yields that $\mathcal{N}_\infty(\ell_1, X) \subset \Pi_1(\ell_1, X)$, We make use of the fact that $\Pi_1(\ell_1, X) = \Pi_2(\ell_1, X)$ and $\nu_2(S) = \pi_2(S)$ for any finite rank operator S to obtain that $\mathcal{N}_\infty(\ell_1, X) \subset \mathcal{N}_2(\ell_1, X)$. Using the trace duality we draw that $\Pi_2(X, \ell_1) \subset \Pi_1(X, \ell_1)$. This gives the desired equality because $\Pi_1(X, \ell_1) \subset \Pi_2(X, \ell_1)$.

(ii)$\Rightarrow$(i). Take any sequence $(x_n) \in \mathcal{R}(X)$. Consider an operator $T = \sum_n e_n \otimes x_n \in \mathcal{B}(\ell_1, X)$. Let us assume that $S = \sum_n x_n^* \otimes e_n \in \mathcal{N}_2(X, \ell_1)$. Since S is 2-summing, it follows from the hypothesis (ii) that S is 1-summing. By virtue of theorem 3.4 of [8], we have that $S \circ T : \ell_1 \to \ell_1$ is nuclear. Since $\langle S \circ T(\alpha_n), e_i \rangle = \sum_n \langle x_n, x_i^* \rangle \alpha_n$, we apply proposition 6.3.6 of [6] to produce that $\sum_i \sup_n |\langle x_n, x_i^* \rangle| < \infty$ and so $\sum_i |\langle x_i, x_i^* \rangle| < \infty$. This permits us to define a continuous map $\Phi : \mathcal{N}_2(X, \ell_1) \to \ell_1$ by $\Phi(\sum_n x_n^* \otimes e_n) = (\langle x_n, x_n^* \rangle)_n$. From the proof of proposition 4 we know that $\Phi^* : \ell_\infty \to \Pi_2(\ell_1, X)$ is given by $\Phi^*(\beta_n) = \sum_n e_n \otimes \beta_n x_n$. Then $T = \sum_n e_n \otimes x_n \in \Pi_2(\ell_1, X)$ and so $T \in \Pi_1(\ell_1, X)$ because $\Pi_1(\ell_1, X) = \Pi_2(\ell_1, X)$. Let $i_X \circ T : \ell_1 \xrightarrow{i} C(K) \xrightarrow{j} \ell_\infty(B_{X^*})$ be a 1-summing factorization, where K is a weak* compact norming subset of B_{ℓ_∞} and j is 1-summing. Thanks to proposition 1.3 of [7] there exists an $\ell_\infty(B_{X^*})$-valued measure ν with bounded variation satisfying $\overline{j(B_{C(K)})} = \text{rg}(\nu)$ and hence $\{i_X(x_n) : n \in \mathbb{N}\} \subset \text{rg}(\|i\|\nu)$, that is $(x_n) \in \mathcal{R}_{iv}(X)$. $\square$

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