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EXISTENCE OF A POSITIVE SOLUTION FOR THE SYSTEM OF THE NONLINEAR BIHARMONIC EQUATIONS

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Abstract. We prove the existence of a positive solution for the system of the following nonlinear biharmonic equations with Dirichlet boundary condition

$$\begin{cases} \Delta^2 u + c\Delta u + av^+ = s_1\phi_1 + \epsilon_1 h_1(x) & \text{in } \Omega, \\ \Delta^2 v + c\Delta v + bu^+ = s_2\phi_1 + \epsilon_2 h_2(x) & \text{in } \Omega, \end{cases}$$

where $u^+ = \max\{u, 0\}$, $c \in R$, $s \in R$, Δ^2 denotes the biharmonic operator and ϕ_1 is the positive eigenfunction of the eigenvalue problem $-\Delta$ with Dirichlet boundary condition. Here ϵ_1, ϵ_2 are small numbers and $h_1(x)$, $h_2(x)$ are bounded.

1. Introduction and statement of main result

In this paper we investigate the existence of a positive solution of the system of the nonlinear biharmonic equations with Dirichlet boundary conditions

$$\begin{cases} \Delta^2 u + c\Delta u + av^+ = s_1\phi_1 + \epsilon_1 h_1(x) & \text{in } \Omega, \\ \Delta^2 v + c\Delta v + bu^+ = s_2\phi_1 + \epsilon_2 h_2(x) & \text{in } \Omega, \\ u = 0, \quad \Delta u = 0 & \text{on } \partial\Omega, \\ v = 0, \quad \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $u^+ = \max\{u, 0\}$, $c \in R$, $s \in R$, Δ^2 denotes the biharmonic operator and $\phi_1$is the positive eigenfunction of the eigenvalue problem

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$\Delta u + \lambda u = 0 \quad$ in Ω, $u = 0 \quad$ on $\partial\Omega$. Here ϵ_1, ϵ_2 are small numbers and $h_1(x)$, $h_2(x)$ are bounded and $\|h_1\| = \|h_2\| = 1$.

System (1.1) of the nonlinear biharmonic equations with Dirichlet boundary condition is considered as a model of the cross of the two nonlinear oscillations in differential equation.

For the case of the single biharmonic equation Tarantello([12]), Lazer and McKenna([7]), Choi and Jung ([4]) etc., investigate the multiplicity of the solutions via the degree theory or the critical point theory or the variational reduction method. In this paper we improve the multiplicity results of the single biharmonic equation to the case of the system of the nonlinear biharmonic equations. The system (1.1) can be rewritten by

$$\begin{cases} \Delta^2 U + c\Delta U + AU^+ = \binom{s_1\phi_1}{s_2\phi_1} + \binom{\epsilon_1 h_1}{\epsilon_2 h_2} & \text{in } \Omega, \\ U = \binom{0}{0}, \qquad \Delta U = \binom{0}{0} & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $U = \binom{u}{v}$, $U^+ = \binom{u^+}{v^+}$, $\Delta^2 U + c\Delta U = \binom{\Delta^2 u + c\Delta u}{\Delta^2 v + c\Delta v}$, $A = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \in M_{2\times 2}(R)$.

Let Ω be a bounded set in R^n with smooth boundary $\partial\Omega$. Let λ_k, $k = 1, 2, \ldots$, denote the eigenvalues and ϕ_k, $k = 1, 2, \ldots$, the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem

$$\Delta u + \lambda u = 0 \qquad \text{in } \Omega,$$

$$u = 0 \qquad \text{on } \partial\Omega,$$

where each eigenvalue λ_k is repeated as often as its multiplicity. We recall that $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \ldots \to +\infty$, and that $\phi_1(x) > 0$ for $x \in \Omega$. The eigenvalue problem

$$\Delta^2 u + c\Delta u = \nu u \qquad \text{in } \Omega,$$

$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega$$

has infinitely many eigenvalues

$$\nu_k = \lambda_k(\lambda_k - c), \qquad k = 1, 2, \ldots,$$

and corresponding eigenfunctions $\phi_k(x)$. The set of functions $\{\phi_k\}$ is an orthonormal base for $L^2(\Omega)$. Let us denote an element u, in $L^2(\Omega)$, as

$$u = \sum h_k \phi_k, \qquad \sum h_k^2 < \infty.$$

We define a subspace H of $L^2(\Omega)$ as follows

$$H = \{u \in L^2(\Omega) | \sum |\nu_k| h_k^2 < \infty\}.$$

Then this is a complete normed space with a norm

$$\|u\| = [\sum |\nu_k| h_k^2]^{\frac{1}{2}}.$$

Since $\lambda_k \to +\infty$ and c is fixed, we have
(1) $\Delta^2 u + c\Delta u \in H$ implies $u \in H$.
(2) $\|u\| \geq C\|u\|_{L^2(\Omega)}$ for some $C > 0$.
(3) $\|u\|_{L^2(\Omega)} = 0$ if and only if $|\|u\|| = 0$.

The space H is a Banach space with norm

$$\|u\|^2 = [\sum |\nu_j| h_j^2]^{\frac{1}{2}}.$$

Let us set $E = H \times H$. We endow the Hilbert E with the norm

$$\|(u,v)\|_E^2 = \|u\|^2 + \|v\|^2 \qquad \forall (u,v) \in E.$$

We are looking for the weak solutions of (1.1) in E, that is, (u,v) satisfying the equation

$$\int_\Omega (\Delta^2 u + c\Delta u) z + \int_\Omega (\Delta^2 v + c\Delta v) w + \int_\Omega (AU^+, (z,w)) - \int_\Omega (s_1\phi_1 + \epsilon_1 h_1) z$$
$$- \int_\Omega (s_2\phi_1 + \epsilon_2 h_1) w = 0, \qquad \forall (z,w) \in E,$$

where $u = \sum c_j\phi_j$, $v = \sum d_j\phi_j$ with $\Delta^2 u + c\Delta u = \sum \nu_j c_j \phi_j \in H$, $\Delta^2 v + c\Delta v = \sum \nu_j d_j \phi_j \in H$ i.e., with $\sum c_j^2 |\nu_j| < \infty$, $\sum d_j^2 |\nu_j| < \infty$, which implies u, $v \in H$. Now we state the main result :

THEOREM 1.1. *(Existence of a positive solution) Let $c < \lambda_1$ and $s_1, s_2 > 0$. Assume that*

$$\nu_j^2 - ab \neq 0, \qquad \text{for all } \ j, \tag{1.3}$$

$$a < 0, \qquad b < 0 \qquad \text{and} \qquad \nu_1^2 - ab > 0. \tag{1.4}$$

Then, for each $h_1(x)$, $h_2(x) \in H$ with $\|h_1(x)\| = 1$, $\|h_2(x)\| = 1$, there exist small numbers ϵ_1^* and ϵ_2^* such that if $\epsilon_1 < \epsilon_1^*$ and $\epsilon_2 < \epsilon_2^*$ then system (1.1) has a positive solution.

2. Proof of Theorem 1.1

We have some properties. Since $|\nu_j| > 0$ for all j, we have the following lemma.

LEMMA 2.1. *Let $Lu = \Delta^2 u + c\Delta u$. Then we have the followings.*
(i) $\|u\| \geq \|u\|_{L^2(Q)}$, where $\|u\|_{L^2(Q)}$ denotes the L^2 norm of u.
(ii) $\|u\| = 0$ if and only if $\|u\|_{L^2(Q)} = 0$.
(iii) $Lu \in H$ implies $u \in H$.
(iv) Suppose that c is not an eigenvalue of L. Let $f \in H$. Then we have $(L - c)^{-1} f \in H$.

For the proof we refer [4].

LEMMA 2.2. *Let $c < \lambda_1$ and $s_1, s_2 > 0$. Assume that the conditions (1.3) and (1.4) hold. Then the system*

$$\begin{cases} \Delta^2 u + c\Delta u + av = s_1\phi_1 & \text{in } \Omega, \\ \Delta^2 v + c\Delta v + bu = s_2\phi_1 & \text{in } \Omega, \\ u = 0, \qquad \Delta u = 0 & \text{on } \partial\Omega, \\ v = 0, \qquad \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

has a unique solution $(u_*, v_*) \in E$, which is of the form

$$u_* = [\frac{\nu_1 s_2 - s_2 a}{\nu_1^2 - ab}]\phi_1, v_* = [\frac{\nu_1 s_2 - s_1 b}{\nu_1^2 - ab}]\phi_1.$$

Proof. *If we put $u_* = \alpha\phi_1$, $v_* = \beta\phi_1$ and insert it into (2.1), then we get the solution (u_*, v_*) of system (2.1) of the above form.* □

The solution $(u_*, v_*) \in E$ satisfies that $u_* > 0$, $v_* > 0$. Thus we have the following lemma.

LEMMA 2.3. *Let $c < \lambda_1$ and $s_1, s_2 > 0$. Assume that the conditions (1.3) and (1.4) hold. Then the system*

$$\begin{cases} \Delta^2 u + c\Delta u + av^+ = s_1\phi_1 & \text{in } \Omega, \\ \Delta^2 v + c\Delta v + bu^+ = s_2\phi_1 & \text{in } \Omega, \\ u = 0, \qquad \Delta u = 0 & \text{on } \partial\Omega, \\ v = 0, \qquad \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

has a positive unique solution $(u_*, v_*) \in E$.

LEMMA 2.4. *Assume that the conditions (1.3) and (1.4) hold. Then the system*

$$\begin{cases} \Delta^2 U + c\Delta U + AU = \binom{0}{0} & \text{in } \Omega, \\ U = \binom{0}{0}, \qquad \Delta U = \binom{0}{0} & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where $U = \binom{u}{v}$, $\Delta^2 U + c\Delta U = \binom{\Delta^2 u + c\Delta u}{\Delta^2 v + c\Delta v}$, *has only the trivial solution* $U = \binom{0}{0}$.

Proof. We assume that there exists a nontrivial solution $U = (u, v) \in E$ of (2.2) of the form $u = \phi_j$ and $v = \phi_k$. Let $LU = \Delta^2 U + c\Delta U$. The equation

$$L\begin{pmatrix} \phi_j \\ \phi_k \end{pmatrix} + A\begin{pmatrix} \phi_j \\ \phi_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is equivalent to the equation

$$\begin{pmatrix} \nu_j \phi_j \\ \nu_k \phi_k \end{pmatrix} + \begin{pmatrix} a\phi_k \\ b\phi_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus when $j \neq k$, we have a contradiction since ϕ_j and ϕ_k are linearly independent. When $j = k$, we have $\nu_j + a = 0$ and $\nu_j + b = 0$, which means that $\nu_j^2 - ab = 0$. These contradicts to the assumption (1.3). $\square$

Lemma 2.5. *Assume that the conditions (1.3)and (1.4) hold. Suppose that* $h_1, h_2 \in H$ *are bounded functions with* $\int_\Omega h_1\phi_1 = \int_\Omega h_2\phi_1 = 0$. *Then the system*

$$\begin{cases} \Delta^2 U + c\Delta U + AU = \binom{h_1}{h_2} & \text{in } \Omega, \\ U = \binom{0}{0}, \qquad \Delta U = \binom{0}{0} & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

has a unique solution $(\check{u}, \check{v}) \in E$.

Proof. Let $\delta > 0$ and $\delta > \max\{a, b\}$. Let us consider the modified system

$$\begin{cases} \Delta^2 u + c\Delta u + av + \nu_1 u + \delta u = h_1(x) & \text{in } \Omega, \\ \Delta^2 v + c\Delta v + bu + \nu_1 v + \delta v = h_2(x) & \text{in } \Omega, \\ u = 0, \qquad \Delta u = 0 & \text{on } \partial\Omega, \\ v = 0, \qquad \Delta v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Let us set

$$L_\delta U = \Delta^2 U + c\Delta U + AU + \nu_1 U + \delta U, \qquad U = \begin{pmatrix} u \\ v \end{pmatrix}.$$

System (2.3) is invertible. Thus there exists an inverse operator $L_\delta^{-1} : H \times H \to E$ *which is a linear and compact operator such that* $(u, v) = L_\delta^{-1}(h_1, h_2)$. *Thus we have that if* (u, v) *is a solution of (2.2) if and only if*

$$(u, v) = L_\delta^{-1}(h_1, h_2) + \nu_1(u, v) + \delta(u, v)). \tag{2.4}$$

Thus we have

$$(I - (\nu_1 + \delta)L_\delta^{-1})((h_1, h_2) + \nu_1(u, v) + \delta(u, v))) = (h_1, h_2).$$

By the conditions (1.3) and (1.4), $\frac{1}{\nu_1+\delta} \notin \sigma(L_\delta^{-1})$. *Since* L_δ^{-1} *is a compact operator, system (2.4) has a unique solution, thus system (2.2) has a unique solution.* □

Proof of Theorem 1.1 *By Lemma 2.3 and Lemma 2.5,* $(u_* + \epsilon_1 \check{u}, v_* + \epsilon_2 \check{v})$ *is a solution of the system*

$$\begin{cases} \Delta^2 u + c\Delta u + av = s_1\phi_1 + \epsilon_1 h_1(x) & \text{in } \Omega, \\ \Delta^2 v + c\Delta v + bu = s_2\phi_1 + \epsilon_2 h_2(x) & \text{in } \Omega, \\ u = 0, \quad \Delta u = 0 & \text{on } \partial\Omega, \\ v = 0, \quad \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

where $u_* = [\frac{\nu_1 s_2 - s_2 a}{\nu_1^2 - ab}]\phi_1, v_* = [\frac{\nu_1 s_2 - s_1 b}{\nu_1^2 - ab}]\phi_1$. *Therefore if* $\nu_1^2 - ab < 0$, *then there exist small numbers* ϵ_1^*, ϵ_2^* *with* $\epsilon_1^* > 0$ *and* $\epsilon_2^* > 0$ *such that* $u_* + \epsilon_1\check{u} > 0$ *and* $v_* + \epsilon_2\check{v} > 0$ *for each* $\epsilon_1 < \epsilon_1^*$ *and* $\epsilon_2 < \epsilon_2^*$.

Therefore there exist small numbers ϵ_1^* *and* ϵ_2^* *such that if* $\epsilon_1 < \epsilon_1^*$ *and* $\epsilon_2 < \epsilon_2^*$ *then system (1.1) has a positive solution* $(u_* + \epsilon_1\check{u}, v_* + \epsilon_2\check{v})$. □

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