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EXTENDED FUZZY EQUIVALENCE RELATIONS

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Abstract. We define an extended fuzzy equivalence relation, discuss some basic properties of extended fuzzy equivalence relations, find the extended fuzzy equivalence relation generated by a fuzzy relation in a set, and give some lattice theoretic properties of extended fuzzy equivalence relations.

1. Introduction

The concept of a fuzzy relation was first proposed by Zadeh ([8]). Subsequently, Goguen ([2]) and Sanchez ([7]) studied fuzzy relations in various contexts. In [4] Nemitz discussed fuzzy equivalence relations, fuzzy functions as fuzzy relations, and fuzzy partitions. Murali ([4]) developed some properties of fuzzy equivalence relations and certain lattice theoretic properties of fuzzy equivalence relations. Gupta et al. ([3]) defined a fuzzy G-equivalence relation on a set and develop some properties of that relation. The standard definition of a reflexive fuzzy relation μ on a set X, which most mathematicians or computer scientists used in their papers, is $\mu(x,x) = 1$ for all $x \in X$. We extend this standard definition to $\mu(x,x) \geq \epsilon > 0$ for all $x \in X$ and $\inf_{t \in X} \mu(t,t) \geq \mu(y,z)$ for all $y \neq z \in X$, which is called e-reflexive fuzzy relation in this note. Chon ([1]) defined an extended fuzzy congruence based on the e-reflexive fuzzy relation and characterized those congruences on semigroups.

In section 2 we define an extended fuzzy equivalence relation based on the e-reflexive fuzzy relation and review some basic properties of fuzzy relations which will be used in next sections. In section 3 we

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discuss some basic properties of extended fuzzy equivalence relations and find the extended fuzzy equivalence relation generated by a fuzzy relation on a set. In section 4 we find sufficient conditions for the composition $\mu \circ \nu$ of two extended fuzzy equivalence relations μ and ν on a set to be an extended fuzzy equivalence relation generated by $\mu \cup \nu$, show that the collection $E(S)$ of all extended fuzzy equivalence relations on a set S is a complete lattice, and show that if S is a group, then $E_k(S) = \{\mu \in E(S) : \mu(c,c) = k \text{ for all } c \in S\}$ is a modular lattice for $0 < \epsilon \leq k \leq 1$.

2. Preliminaries

In this section we define an extended fuzzy equivalence relation and recall some basic properties of fuzzy relations which will be used in next sections.

DEFINITION 2.1. A function B from a set X to the closed unit interval [0, 1] in $\mathbb{R}$ is called a *fuzzy set* in X. For every $x \in B$, $B(x)$ is called a *membership grade* of x in B.

The standard definition of a fuzzy reflexive relation μ in a set X demands $\mu(x,x) = 1$ for all $x \in X$, which seems to be too strong. We redefine this standard definition as follows.

DEFINITION 2.2. A *fuzzy relation* μ in a set X is a fuzzy subset of $X \times X$. μ is *e-reflexive* in X if $\mu(x,x) \geq \epsilon > 0$ and $\inf_{t \in X} \mu(t,t) \geq \mu(x,y)$ for all $x, y \in X$ such that $x \neq y$. μ is *symmetric* in X if $\mu(x,y) = \mu(y,x)$ for all x, y in X. The composition $\lambda \circ \mu$ of two fuzzy relations λ, μ in X is the fuzzy subset of $X \times X$ defined by

$$(\lambda \circ \mu)(x,y) = \sup_{z \in X} \min(\lambda(x,z), \mu(z,y)).$$

A fuzzy relation μ in X is *transitive* in X if $\mu \circ \mu \subseteq \mu$. A fuzzy relation μ in X is called an *extended fuzzy equivalence relation* if μ is e-reflexive, symmetric, and transitive.

Let $\mathcal{F}_X$ be the set of all fuzzy relations in a set X. Then it is easy to see that $\mathcal{F}_X$ is a monoid under the operation of composition $\circ$

and an extended fuzzy equivalence relation is an idempotent element of $\mathcal{F}_X$.

DEFINITION 2.3. Let μ be a fuzzy relation in a set X. μ^{-1} is defined as a fuzzy relation in X by $\mu^{-1}(x,y) = \mu(y,x)$.

It is easy to see that $(\mu \circ \nu)^{-1} = \nu^{-1} \circ \mu^{-1}$ for fuzzy relations μ and ν.

PROPOSITION 2.4. *Let $\mathcal{F}_X$ be a monoid of all fuzzy relations in X and let $\phi : F_X \to F_X$ be a map defined by $\phi(\mu) = \mu^{-1}$. Then ϕ is an antiautomorphism and $\phi(\mu^{-1}) = (\phi(\mu))^{-1} = \mu$.*

Proof. Straightforward. □

PROPOSITION 2.5. *Let μ be a fuzzy relation on a set X. Then $\cup_{n=1}^{\infty} \mu^n$ is the smallest transitive fuzzy relation on X containing μ, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.*

Proof. See Proposition 2.3 of [6]. □

PROPOSITION 2.6. *Let μ be a fuzzy relation on a set X. If μ is symmetric, then so is $\cup_{n=1}^{\infty} \mu^n$, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.*

Proof. See Proposition 2.4 of [6]. □

PROPOSITION 2.7. *If μ is an e-reflexive fuzzy relation on a set X, then $\mu^{n+1}(x,y) \geq \mu^n(x,y)$ for all natural numbers n and all $x, y \in X$.*

Proof.

$$\begin{aligned}\mu^2(x,y) = (\mu \circ \mu)(x,y) &= \sup_{z \in X} \min[\mu(x,z), \mu(z,y)] \\ &\geq \min[\mu(x,x), \mu(x,y)] = \mu(x,y).\end{aligned}$$

Suppose $\mu^{k+1}(x,y) \geq \mu^k(x,y)$ for all $x, y \in X$. Then

$$\begin{aligned}\mu^{k+2}(x,y) = (\mu \circ \mu^{k+1})(x,y) &= \sup_{z \in S} \min[\mu(x,z),\ \mu^{k+1}(z,y)] \\ &\geq \sup_{z \in S} \min[\mu(x,z),\ \mu^k(z,y)] = (\mu \circ \mu^k)(x,y) = \mu^{k+1}(x,y).\end{aligned}$$

By the mathematical induction, $\mu^{n+1}(x,y) \geq \mu^n(x,y)$ for all natural numbers n. $\square$

PROPOSITION 2.8. *Let μ and each ν_i be fuzzy relations in a set X for all $i \in I$. Then $\mu \circ (\bigcup_{i\in I} \nu_i) = \bigcup_{i\in I} (\mu \circ \nu_i)$, $(\bigcup_{i\in I} \nu_i) \circ \mu = \bigcup_{i\in I} (\nu_i \circ \mu)$, $\mu \circ (\bigcap_{i\in I} \nu_i) \subseteq \bigcap_{i\in I} (\mu \circ \nu_i)$, and $(\bigcap_{i\in I} \nu_i) \circ \mu \subseteq \bigcap_{i\in I} (\nu_i \circ \mu)$.*

Proof. Straightforward. $\square$

3. Extended fuzzy equivalence relations

In this section we discuss some basic properties of extended fuzzy equivalence relations and find the extended fuzzy equivalence relation generated by a fuzzy relation.

DEFINITION 3.1. Let μ be an extended fuzzy equivalence relation in a set X. The set $\{y \in X : \mu(y,x) > 0\}$, which is denoted by $[x]$, is called an *equivalence class* of x. The collection of equivalence classes of X, denoted by X/μ, is called *quotient* of X by μ. That is, $X/\mu = \{[x] : x \in X\}$.

PROPOSITION 3.2. *Let μ be an extended fuzzy equivalence relation in a nonempty set X and let $[x]$ be the equivalence class of $x \in X$.*

(1) *For every $x \in X$, $[x]$ is a nonempty set.*
(2) *$\mu(x,y) > 0$ iff $[x] = [y]$.*
(3) *$[x] \cap [y] \neq \emptyset$ iff $\mu(x,y) > 0$.*

Proof. (1) Straightforward.
(2) Suppose $\mu(x,y) > 0$. Let $a \in [x]$. Then $\mu(a,x) > 0$.

$$\mu(a,y) \geq \sup_{k\in X} \min\, [\mu(a,k), \mu(k,y)] \geq \min\, [\mu(a,x), \mu(x,y)] > 0.$$

That is, $a \in [y]$. Thus $[x] \subseteq [y]$. Similarly we may show $[y] \subseteq [x]$. Hence $[x] = [y]$. Conversely suppose that $[x] = [y]$. Then $x \in [y]$, that is, $\mu(x,y) > 0$.

(3) Suppose that $[x] \cap [y] \neq \emptyset$. Then there exists $c \in X$ such that $\mu(c,x) > 0$ and $\mu(c,y) > 0$. $\mu(x,y) \geq \sup_{k \in X} \min[\mu(x,k), \mu(k,y)] \geq \min\,[\mu(x,c), \mu(c,y)] > 0$. Conversely suppose that $\mu(x,y) > 0$. Then $x \in [y]$. Since $x \in [x]$, $[x] \cap [y] \neq \emptyset$. □

Proposition 3.3. *Let μ be an extended fuzzy equivalence relation in a nonempty set X. Then the quotient set X/μ is a partition of X.*

Proof. Straightforward from Proposition 3.2. □

Theorem 3.4. *Let μ be a fuzzy relation on a set S. If μ is e-reflexive, then so is $\cup_{n=1}^{\infty} \mu^n$, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.*

Proof. Clearly $\mu^1 = \mu$ is e-reflexive. Suppose μ^k is e-reflexive. Then

$$\begin{aligned}\mu^{k+1}(x,x) = (\mu^k \circ \mu)(x,x) &= \sup_{z \in S} \min[\mu^k(x,z), \mu(z,x)] \\ &\geq \min[\mu^k(x,x), \mu(x,x)] \geq \epsilon > 0\end{aligned}$$

for all $x \in S$. Let $x, y \in S$ with $x \neq y$. Then

$$\begin{aligned}\inf_{t \in S} \mu^{k+1}(t,t) = \inf_{t \in S} (\mu^k \circ \mu)(t,t) &= \inf_{t \in S} \sup_{z \in S} \min[\mu^k(t,z), \mu(z,t)] \\ &\geq \inf_{t \in S} \min[\mu^k(t,t), \mu(t,t)] \geq \min\,[\inf_{t \in S} \mu^k(t,t), \inf_{t \in S} \mu(t,t)] \\ &\geq \min[\mu^k(x,z), \mu(z,y)]\end{aligned}$$

for all $z \in S$ such that $z \neq x$ and $z \neq y$. That is, $\inf_{t \in S} \mu^{k+1}(t,t) \geq \sup_{z \in S - \{x,y\}} \min[\mu^k(x,z), \mu(z,y)]$. Clearly $\inf_{t \in S} \mu(t,t) \geq \min\,[\mu^k(x,x), \mu(x,y)]$ and $\inf_{t \in S} \mu^k(t,t) \geq \min\,[\mu^k(x,y), \mu(y,y)]$. Since $\mu^{k+1}(t,t) \geq \mu^k(t,t) \geq \mu(t,t)$ by Proposition 2.7,

$$\begin{aligned}&\inf_{t \in S} \mu^{k+1}(t,t) \geq \min\,[\mu^k(x,x), \mu(x,y)] \text{ and} \\ &\inf_{t \in S} \mu^{k+1}(t,t) \geq \min\,[\mu^k(x,y), \mu(y,y)].\end{aligned}$$

Thus

$$\begin{aligned}\inf_{t\in S}\ \mu^{k+1}(t,t) &\geq \max\,[\sup_{z\in S-\{x,y\}} \min(\mu^k(x,z),\mu(z,y)),\\ &\qquad \min\,(\mu^k(x,x),\mu(x,y)), \min\,(\mu^k(x,y),\mu(y,y))]\\ &= \sup_{z\in S}\min[\mu^k(x,z),\mu(z,y)] = (\mu^k\circ\mu)(x,y)\\ &= \mu^{k+1}(x,y).\end{aligned}$$

That is, μ^{k+1} is e-reflexive. By the mathematical induction, μ^n is e-reflexive for $n = 1, 2, \ldots$. Thus $\inf_{t\in S}\ [\cup_{n=1}^{\infty}\ \mu^n](t,t) = \inf_{t\in S}\sup[\mu(t,t),(\mu\circ\mu)(t,t),\ldots] \geq \sup\ [\inf_{t\in S}\ \mu(t,t), \inf_{t\in S}\ (\mu\circ\mu)(t,t),\ldots] \geq \sup[\mu(x,y),(\mu\circ\mu)(x,y),\ldots] = [\cup_{n=1}^{\infty}\mu^n](x,y)$. Clearly $[\cup_{n=1}^{\infty}\ \mu^n](x,x) \geq \epsilon > 0$. Hence $\cup_{n=1}^{\infty}\ \mu^n$ is e-reflexive. □

Proposition 3.5. *Let μ and ν be extended fuzzy equivalence relations in a set X. Then $\mu\cap\nu$ is an extended fuzzy equivalence relation.*

Proof. It is easy to see that $\mu\cap\nu$ is e-reflexive and symmetric. By Proposition 2.8, $[(\mu\cap\nu)\circ(\mu\cap\nu)] \subseteq [\mu\circ(\mu\cap\nu)]\cap[\nu\circ(\mu\cap\nu)] \subseteq [(\mu\circ\mu)\cap(\mu\circ\nu)]\cap[(\nu\circ\mu)\cap(\nu\circ\nu)] \subseteq [\mu\cap(\mu\circ\nu)]\cap[(\nu\circ\mu)\cap\nu] \subseteq \mu\cap\nu$. □

It is easy to see that even though μ and ν are extended fuzzy equivalence relations, $\mu\cup\nu$ is not necessarily an extended fuzzy equivalence relation. We find the extended fuzzy equivalence relation generated by $\mu\cup\nu$ in the following proposition.

Proposition 3.6. *Let μ and ν be extended fuzzy equivalence relations in a set X. Then the extended fuzzy equivalence relation generated by $\mu\cup\nu$ is $\cup_{n=1}^{\infty}(\mu\cup\nu)^n = (\mu\cup\nu)\cup[(\mu\cup\nu)\circ(\mu\cup\nu)]\cup\ldots$.*

Proof. Clearly $(\mu\cup\nu)(x,x) \geq \epsilon > 0$.

$$\begin{aligned}\inf_{t\in X}\ (\mu\cup\nu)(t,t) &= \inf_{t\in X}\ \max(\mu(t,t),\nu(t,t))\\ &\geq \max\,(\inf_{t\in X}\mu(t,t), \inf_{t\in X}\ \nu(t,t))\\ &\geq \max\,(\mu(x,y),\nu(x,y)) = (\mu\cup\nu)(x,y)\end{aligned}$$

for all $x \neq y$ in X. That is, $\mu \cup \nu$ is e-reflexive. By Theorem 3.4, $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ is e-reflexive. Since $\mu \cup \nu$ is symmetric, $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ is symmetric by Proposition 2.6. By Proposition 2.5, $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ is transitive. Hence $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ is an extended fuzzy equivalence relation. Let λ be an extended fuzzy equivalence relation in a set X containing $\mu \cup \nu$. Then $\cup_{n=1}^{\infty}(\mu \cup \nu)^n \subseteq \cup_{n=1}^{\infty}\lambda^n = \lambda \cup (\lambda \circ \lambda) \cup (\lambda \circ \lambda \circ \lambda) \cup \cdots \subseteq \lambda \cup \lambda \cup \cdots \subseteq \lambda$. Thus $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ is the extended fuzzy equivalence relation generated by $\mu \cup \nu$. $\square$

We now turn to the characterization of the extended fuzzy equivalence relation generated by a fuzzy relation.

THEOREM 3.7. *Let μ be a fuzzy relation in a set X. Then the extended fuzzy equivalence relation in X generated by μ is*

$$\begin{aligned} &\cup_{n=1}^{\infty} (\mu \cup \mu^{-1} \cup \theta)^n \\ &= (\mu \cup \mu^{-1} \cup \theta) \cup ((\mu \cup \mu^{-1} \cup \theta) \circ (\mu \cup \mu^{-1} \cup \theta)) \cup \ldots, \end{aligned}$$

where θ is a fuzzy relation in X such that $\theta(x,x) \geq \epsilon > 0$, $\theta = \theta^{-1}$, $\theta(x,y) \leq \mu(x,y)$, and $\inf_{t\in X} \theta(t,t) \geq \max\,[\mu(x,y), \mu^{-1}(x,y), \theta(x,y)]$ for all $x \neq y$ in X.

Proof. Let $\mu_1 = \mu \cup \mu^{-1} \cup \theta$. Then

$$\mu_1(x,x) = \max\,[\mu(x,x), \mu^{-1}(x,x), \theta(x,x)] \geq \epsilon > 0.$$

$$\begin{aligned} \inf_{t\in X} \mu_1(t,t) &= \inf_{t\in X} \max[\mu(t,t),\ \mu^{-1}(t,t),\ \theta(t,t)] \geq \inf_{t\in X} \theta(t,t) \\ &\geq \max[\mu(x,y),\ \mu^{-1}(x,y), \theta(x,y)] = \mu_1(x,y). \end{aligned}$$

Thus μ_1 is e-reflexive. By Theorem 3.4, $\cup_{n=1}^{\infty}\mu_1^n$ is e-reflexive. Since $\theta = \theta^{-1}$, $\mu_1(x,y) = (\mu \cup \mu^{-1} \cup \theta)(x,y) = \max[\mu(x,y), \mu^{-1}(x,y), \theta^{-1}(x,y)] = \max[\mu^{-1}(y,x), \mu(y,x), \theta(y,x)] = (\mu \cup \mu^{-1} \cup \theta)(y,x) = \mu_1(y,x)$. Thus μ_1 is a symmetric. By Proposition 2.6, $\cup_{n=1}^{\infty}\mu_1^n$ is symmetric. By Proposition 2.5, $\cup_{n=1}^{\infty}\mu_1^n$ is transitive. Hence $\cup_{n=1}^{\infty} \mu_1^n$ is an extended fuzzy equivalence relation containing μ. Let ν be an extended fuzzy equivalence relation containing μ. Then $\mu(x,y) \leq \nu(x,y)$, $\mu^{-1}(x,y) =$

$\mu(y,x) \le \nu(y,x) = \nu(x,y)$, and $\theta(x,y) \le \mu(x,y) \le \nu(x,y)$. Thus $\mu_1 \subseteq \nu$. Suppose that $\mu_1^k \subseteq \nu$. Then $\mu_1^{k+1}(x,y) = (\mu_1 \circ \mu_1^k)(x,y) = \sup_{z\in X} \min[\mu_1(x,z), \mu_1^k(z,y)] \le \sup_{z\in X} \min[\nu(x,z), \nu(z,y)] = (\nu \circ \nu)(x,y)$. Since ν is transitive, $\mu_1^{k+1} \subseteq \nu \circ \nu \subseteq \nu$. By the mathematical induction, $\mu_1^n \subseteq \nu$ for all natural numbers n. Thus $\cup_{n=1}^{\infty} \mu_1^n = \mu_1 \cup (\mu_1 \circ \mu_1) \cup (\mu_1 \circ \mu_1 \circ \mu_1) \cdots \subseteq \nu$. □

4. Lattice of extended fuzzy equivalence relations

In this section we discuss lattice theoretic properties of extended fuzzy equivalence relations. Let $E(S)$ be the collection of all extended fuzzy equivalence relations on a set S.

THEOREM 4.1. *$(E(S), \le)$ is a complete lattice, where $\le$ is a relation on the set of all extended fuzzy equivalence relations on S defined by $\mu \le \nu$ iff $\mu(x,y) \le \nu(x,y)$ for all $x, y \in S$.*

Proof. Clearly $\le$ is a partial order relation. It is easy to check that the relation σ defined by $\sigma(x,y) = 1$ for all $x, y \in S$ is in $E(S)$ and the relation λ defined by $\lambda(x,y) = \epsilon$ for $x = y$ and $\lambda(x,y) = 0$ for $x \ne y$ is in $E(S)$. Also σ is the greatest element and λ is the least element of $E(S)$ with respect to the ordering $\le$. Let $\{\mu_j\}_{j\in J}$ be a non-empty collection of extended fuzzy equivalence relations in $E(S)$. Let $\mu(x,y) = \inf_{j\in J} \mu_j(x,y)$ for all $x, y \in S$. It is easy to see that $\mu(x,x) \ge \epsilon$ for all $x \in S$, $\inf_{t\in X} \mu(t,t) \ge \mu(y,z)$ for all $y \ne z \in X$, and $\mu = \mu^{-1}$. $\mu \circ \mu(x,y) = \sup_{z\in X} \min[\inf_{j\in J} \mu_j(x,z), \inf_{j\in J} \mu_j(z,y)] = \sup_{z\in X} \inf_{j\in J} \inf_{i\in J} \min[\mu_j(x,z), \mu_i(z,y)] \le \inf_{j\in J} \sup_{z\in X} \min[\mu_j(x,z), \mu_j(z,y)] = \inf_{j\in J} \mu_j \circ \mu_j(x,y) \le \inf_{j\in J} \mu_j(x,y) = \mu(x,y)$. That is, $\mu \in E(S)$. Since μ is the greatest lower bound of $\{\mu_j\}_{j\in J}$, $(E(S), \le)$ is a complete lattice.□

Let $E_k(S) = \{\mu \in E(S) : \mu(c,c) = k \text{ for all } c \in S\}$. It is easy to see that $E_k(S)$ is a sublattice of $E(S)$ for $0 < \epsilon \le k \le 1$. We define addition and multiplication on $E_k(S)$ by $\mu + \nu = <\mu \cup \nu>$ and

$\mu \cdot \nu = \mu \cap \nu$, where $< \mu \cup \nu >$ is the extended fuzzy equivalence relation generated by $\mu \cup \nu$.

Definition 4.2. A lattice $(L, +, \cdot)$ is called *modular* if $(x+y)\cdot z \leq x + (y \cdot z)$ for all $x, y, z \in L$ with $x \leq z$.

Lemma 4.3. *Let μ and ν be extended fuzzy equivalence relations in a set X such that $\mu(t,t) = \nu(t,t)$ for all $t \in X$. If $\mu \circ \nu = \nu \circ \mu$, then $\mu \circ \nu$ is the extended fuzzy equivalence relation in X generated by $\mu \cup \nu$.*

Proof.

$$\begin{aligned} (\mu \circ \nu)(x,x) &= \sup_{z \in X} \min[\mu(x,z), \nu(z,x)] \\ &\geq \min(\mu(x,x), \nu(x,x)) \geq \epsilon > 0, \\ \inf_{t \in X} (\mu \circ \nu)(t,t) &= \inf_{t \in X} \sup_{z \in X} \min[\mu(t,z), \nu(z,t)] \\ &\geq \inf_{t \in X} \min[\mu(t,t), \nu(t,t)] \geq \min[\mu(x,z), \nu(z,y)] \end{aligned}$$

for all $z \in X$. Thus

$$\inf_{t \in X} (\mu \circ \nu)(t,t) \geq \sup_{z \in X} \min[\mu(x,z), \nu(z,y)] = (\mu \circ \nu)(x,y).$$

That is, $\mu \circ \nu$ is e-reflexive. Since μ and ν are symmetric, $(\mu \circ \nu)^{-1} = \nu^{-1} \circ \mu^{-1} = \nu \circ \mu = \mu \circ \nu$. Thus $\mu \circ \nu$ is symmetric. Since μ and ν are transitive and the operation $\circ$ is associative, $(\mu \circ \nu) \circ (\mu \circ \nu) = \mu \circ (\nu \circ \mu) \circ \nu = \mu \circ (\mu \circ \nu) \circ \nu = (\mu \circ \mu) \circ (\nu \circ \nu) \subseteq \mu \circ \nu$. Hence $\mu \circ \nu$ is an extended fuzzy equivalence relation. Let $x, y \in X$ with $x \neq y$. Since $\nu(r,r) \geq \mu(s,r)$, $(\mu \circ \nu)(x,y) = \sup_{z \in X} \min[\mu(x,z), \nu(z,y)] \geq \min(\mu(x,y), \nu(y,y)) = \mu(x,y)$. Since $\mu(p,p) \geq \nu(p,q)$, $(\mu \circ \nu)(x,y) = \sup_{z \in X} \min[\mu(x,z), \nu(z,y)] \geq \min(\mu(x,x), \nu(x,y)) = \nu(x,y)$. Thus $(\mu \circ \nu)(x,y) \geq \max(\mu(x,y), \nu(x,y)) = (\mu \cup \nu)(x,y)$ for $x \neq y$. $(\mu \circ \nu)(t,t) = \sup_{z \in X} \min[\mu(t,z), \nu(z,t)] \geq \min(\mu(t,t), \nu(t,t)) = (\mu \cup \nu)(t,t)$. That is, $\mu \cup \nu \subseteq \mu \circ \nu$. Let λ be an extended fuzzy equivalence relation in X containing $\mu \cup \nu$. Since λ is transitive, $\mu \circ \nu \subseteq (\mu \cup \nu) \circ (\mu \cup \nu) \subseteq \lambda \circ \lambda \subseteq \lambda$.

Thus $\mu \circ \nu$ is the extended fuzzy equivalence relation generated by $\mu \cup \nu$. $\square$

It is well known that if μ and ν are equivalence relations on a set S and $\mu \circ \nu = \nu \circ \mu$, then $\mu \circ \nu$ is the equivalence relation on S generated by $\mu \cup \nu$. Lemma 4.3 may be considered as a generalization of this in extended fuzzy equivalent relation.

THEOREM 4.4. *Let $0 < \epsilon \leq k \leq 1$, let S be a set, and let H be a sublattice of $(E_k(S), +, \cdot)$ such that $\mu \circ \nu = \nu \circ \mu$ for all $\mu, \nu \in H$. Then H is a modular lattice.*

Proof. Let $\mu, \nu, \rho \in H$ with $\mu \leq \rho$. Let $x, y \in S$.

$$\begin{aligned} \min\,[(\mu \circ \nu)(x,y), \rho(x,y)] &= \sup_{z \in S} \min\,[\mu(x,z), \nu(z,y), \rho(x,y)] \\ &\leq \sup_{z \in S} \min[\mu(x,z), \rho(x,z), \nu(z,y), \rho(x,y)] \\ &\leq \sup_{z \in S}\min[\mu(x,z), \nu(z,y), \rho(z,y)] \\ &= [\mu \circ \min(\nu, \rho)](x,y). \end{aligned}$$

Thus $(\mu \circ \nu) \cdot \rho \leq \mu \circ (\nu \cdot \rho)$. Since $\mu, \nu \in E_k(S)$, $\mu(c,c) = \nu(c,c) = k$ for all $c \in S$. By Lemma 4.3, $\mu \circ \nu$ is the extended fuzzy equivalence relation generated by $\mu \cup \nu$. That is, $\mu + \nu = \mu \circ \nu$. Similarly we may show $\mu + (\nu \cdot \rho) = \mu \circ (\nu \cdot \rho)$. Thus $(\mu + \nu) \cdot \rho \leq \mu + (\nu \cdot \rho)$. Hence H is modular. $\square$

PROPOSITION 4.5. *If S is a group, then $\mu \circ \nu = \nu \circ \mu$ for all $\mu, \nu \in E_k(S)$.*

Proof. Straightforward. $\square$

COROLLARY 4.6. *If S is a group and $0 < \epsilon \leq k \leq 1$, then $(E_k(S), +, \cdot)$ is modular.*

Proof. By Theorem 4.4 and Proposition 4.5, $(E_k(S), +, \cdot)$ is modular. $\square$

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