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WEAK CONVERGENCE OF VARIOUS MODELS TO FRACTIONAL BROWNIAN MOTION

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Abstract. We consider arrival process and ON/OFF source model which allows for long packet trains and long inter-train distances. We prove the weak convergence of theses processes to Fractional Brownian motion. Finally, we figure out the coefficients of $B_H(t)$ and time t when ON/OFF periods have the Pareto distribution.

1. Introduction

Many researchers have studied long range dependent process and self-similar process because of burstiness of network traffic at any time scale, Though the various models proposed for capturing the long-range dependent nature of network traffic are all either exactly or asymptotically second order self-similar, their effect on network performance can be very different([6], [7], [8]).

Self-similarity, long range dependence and heavy tailed process have been observed in many time series, i.e. network traffic and finance([4]). In particular, fractional Brownian motion and FARIMA in modern packet network traffic has been the focus of much attention ([5]). Various methods for estimating the self-similar parameter and intensity of long range dependence in time series has been investigated ([7], [9]). And, there has been a recent flood of literature and discussion on the tail behavior of queue-length distribution, motivated by potential applications to the design and control by high-speed telecommunication networks([1], [2], [3]).

In this paper we consider arrival process based on autoregressive process and show that the suitably scaled distributions of those processes

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converge to fractional Brownian motion in the sense of finite dimensional distributions.

On the other hand, we consider idealized ON/OFF source model which allows for long packet trains and long inter-train distances. [9] proved that the aggregate cumulative packet process behaves like linear combination of fractional Brownian motion $B_H(t)$ and time t. When ON/OFF periods have the Pareto distribution, we figure out the coefficients of $B_H(t)$ and time t.

In section 2, we define the short range dependence, long range dependence, fractional Brownian motion and self-similarity. In section 3, we prove the weak convergence of arrival process and autoregressive process to fractional Brownian motion. In section 4, we figure out the coefficients of $B_H(t)$ and time t when ON/OFF periods have the Pareto distribution.

2. Definition and Preliminary

In this section we first define short range dependance and long range dependance. Let $\tau_X(k)$ be the covariance of stationary stochastic process $X(t)$.

DEFINITION 2.1. *A stationary stochastic process $X(t)$ exhibits short range dependence if*

$$\sum_{k=-\infty}^{\infty} |\tau_X(k)| < \infty$$

DEFINITION 2.2. *A stationary stochastic process $X(t)$ exhibits long range dependence if*

$$\sum_{k=-\infty}^{\infty} |\tau_X(k)| = \infty$$

A standard example of a long range dependent process is fractional Brownian motion, with Hurst parameter $H > \frac{1}{2}$.

DEFINITION 2.3. *A stochastic process $\{B_H(t)\}$ is said to be a Fractional Brownian motion(FBM) with Hurst parameter H if*

1. *$B_H(t)$ has stationary increments*
2. *for $t > 0$, $B_H(t)$ is normally distributed with mean 0*
3. *$B_H(0) = 0$ a.s.*

4. *The increments of $B_H(t)$, $Z(j) = B_H(j+1) - B_H(j)$ satisfy*

$$\rho_Z(k) = \frac{1}{2}\{|k+1|^{2H} + |k-1|^{2H} - 2k^{2H}\}$$

Fractional Brownian motion is important example of self-similar process defined below.

Definition 2.4. *A continuous process $X(t)$ is self-similar with self-similarity parameter $H \geq 0$ if it satisfies the condition:*

$$X(t) \stackrel{d}{=} c^{-H} X(ct), \quad \forall\, t \geq 0, \forall c > 0,$$

where the equality is in the sense of finite-dimensional distributions.

3. Weak Convergence to Fractional Brownian motion

Let $X^j(i)$ be the number of arrivals in the ith time unit of jth source. Let

$$X_M(i) = \sum_{j=1}^{M} (X^j(i) - E(X^j(i)),$$

and $\tau(k)$ denote the covariance of $X_1(i)$.

Lemma 3.1. ([4]) *The stationary sequence*

$$\frac{1}{M^{1/2}} X_M(i)$$

converges in the sense of finite dimensional distributions to $G_H(i)$, where $G_H(i)$ represents a stationary Gaussian process with covariance function of the same form as $\tau(k)$, as $M \to \infty$.

Theorem 3.1.

$$\lim_{T\to\infty} \lim_{M\to\infty} \frac{1}{T^H M^{1/2}} \sum_{i=0}^{[Tt]} X_M(i)$$

converges to $\{\sigma_0 B_H(t) | 0 \leq t \leq 1\}$ in the sense of finite dimensional distributions. Furthermore, as $M \to \infty$ and $T \to \infty$,
(a) (Long Range dependence) If

$$\tau(k) \sim ck^{2H-2}, \quad c > 0 \ \text{ and } \ 1/2 < H < 1,$$

then $\sigma_0^2 = \dfrac{c}{H(2H-1)}$.
(b) *If*

$$\sum_{k=1}^{\infty} |\tau(k)| < \infty \ \ \text{and} \ \ \sum_{k=1}^{\infty} \tau(k) = c > 0,$$

then $\sigma_0^2 = c$.
(c) (*Short Range dependence*)

$$\tau(k) \sim ck^{2H-2}, \ \ c < 0 \ \ \text{and} \ \ 0 < H < 1/2,$$

then $\sigma_0^2 = -\dfrac{c}{H(2H-1)}$.

Proof. Set $Z_i = 1/M^{1/2} X_M(i)$. By Lemma 3.1, Z_i converges in the sense of finite dimensional distributions to $G_H(i)$ as M goes to infinity. By Theorem 7.2.11 of [5], the finite dimensional distributions of $T^{-H} \sum_{i=0}^{[Tt]} Z_i$ converges to those of $\{\sigma_0 B_H(t), 0 \le t \le 1\}$. □

THEOREM 3.2. *Let X_t be the autoregressive process of order one, i.e. $X_t = \phi_1 X_{t-1} + a_t$, where $a_t \sim N(0,1)$ for each t. Then*

$$\lim_{T\to\infty} \lim_{M\to\infty} \sum_{i=0}^{[Tt]} X_M(i) = \sqrt{\frac{\phi_1}{1-\phi_1}} B(t),$$

where, $B(t)$ is a Brownian Motion.

Proof. We know that

$$(1-\phi_1 B) X_t = a_t,$$

i.e.

$$X_t = \sum_{i=0}^{\infty} \phi_1^i a_{t-i}.$$

And, we get

$$Cov_{X_t}(k) = \phi_1^k, \ k \ge 1, |\phi_1| < 1.$$

Therefore,

$$\tau(k) = \phi_1^k,$$

for large M. Since

$$\sum \tau(k) = \sum \phi_1^k = \frac{\phi_1}{1-\phi_1} < \infty,$$

we get, by Theorem 3.1 (b),

$$\lim_{T\to\infty}\lim_{M\to\infty}\sum_{i=0}^{[Tt]} X_M(i) = \frac{\phi_1}{1-\phi_1}B_{1/2}(t) = \frac{\phi_1}{1-\phi_1}B(t).$$

□

4. Convergence of ON/OFF Source Model

Let us consider the stationary time series $\{X(t), t \geq 0\}$. $X(t) = 1$ means that there is a packet at time t and $X(t) = 0$ means that there is no packet. Viewing $X(t)$ as the reward at time t, we have a reward of 1 throughout on ON-period, then a reward of 0 throughout the following OFF-periods, then 1 again, and so on. Suppose the lengths of the ON-periods are i.i.d., those of the OFF-periods are i.i.d. and the lengths of ON-periods and OFF-periods are independent. But the ON-periods and OFF-periods may have the different distributions.

Suppose that there are M i.i.d. sources. Since each source sends its own sequence of packet trains, it has its own reward sequence $\{X^{(m)}(t)\}$. Therefore, the cumulative packet count at time t is

$$\sum_{m=1}^{M} X^{(m)}(t).$$

Rescaling time by a factor T, we consider the aggregated cumulative packet counts

$$X_M(Tt) = \int_0^{Tt} (\sum_{m=1}^{M} X^{(m)}(u))du$$

in the interval $[0, Tt]$. To specify the distributions of ON-period O_1 and OFF-periods O_2, let

$$\mu_1 = EO_1, \mu_2 = EO_2$$

and as $x \to \infty$, tailing distributions of O_1 , O_2 are

$$l_1 x^{-\alpha_1} L_1(x) \quad \text{and} \quad l_2 x^{-\alpha_2} L_2(x)$$

with $1 < \alpha_j < 2$, where is a constant $l_j > 0$ and $L_j > 0$ is a slowly varying function at infinity.

Notation. When $1 < \alpha_j < 2$, set

$$a_j = l_j(\Gamma(2-\alpha_j))/(\alpha_j - 1),$$

$$b = \lim_{t\to\infty} t^{\alpha_2-\alpha_1}\frac{L_1(t)}{L_2(t)}.$$

If $0 < b < \infty$ then set

$$\sigma^2 = \frac{2(\mu_2^2 a_1 b + \mu_1^2 a_2)}{(\mu_1+\mu_2)^3\Gamma(4-\alpha_{min})},$$

if $b = 0$ or $b = \infty$ then set

$$\sigma^2 = \frac{2\mu_{max}^2 a_{min}}{(\mu_1+\mu_2)^3\Gamma(4-\alpha_{min})}.$$

LEMMA 4.1. *For large M and T, the aggregate packet process*

$$\{X_M(Tt), t \geq 0\}$$

behaves statistically like

$$TM\frac{\mu_1}{\mu_1+\mu_2}t + T^H\sqrt{L(t)M}\,\sigma B_H(t)$$

where $H = (3-\alpha_{min})/2$ and σ is as above.

Proof. Theorem 1 of [9] □

Suppose that ON/OFF periods O_j has the Pareto distribution

$$P(O_j > x) = K^{\alpha_j}x^{-\alpha_j} \quad \text{for} \quad x \geq K > 0.$$

When $1 < \alpha_j < 2$, each periods has infinite variance.

THEOREM 4.1. *Let O_j be ON/OFF-periods that has the Pareto distributions as above. Then, for large M and T , the aggregate packet process $\{X_M(Tt), t \geq 0\}$ behaves statistically like*

$$TM\frac{\alpha_1\alpha_2-\alpha_1}{2\alpha_1\alpha_2-\alpha_1-\alpha_2}t + T^H\sigma B_H(t)$$

where, $H = (3-\alpha_{min})/2$.

Case 1. Suppose that O_j have the same distributions, i.e., $\alpha_1 = \alpha_2 = \alpha$, then

$$H = \frac{3-\alpha}{2}$$

and

$$\sigma^2 = \frac{K^{\alpha-1}\Gamma(2-\alpha)}{2\alpha\Gamma(4-\alpha)}$$

Case 2. If $\alpha_1 < \alpha_2$, then

$$H = \frac{3-\alpha_1}{2}$$

and

$$\sigma^2 = \frac{2K^2\alpha_1^2(\alpha_1-1)(\alpha_2-1)^3 a_{min}}{(2\alpha_1\alpha_2 K - \alpha_1 K - \alpha_2 K)^3}$$

Case 3. If $\alpha_1 > \alpha_2$, then

$$H = \frac{3-\alpha_2}{2}$$

and

$$\sigma^2 = \frac{2K^2\alpha_2^2(\alpha_2-1)(\alpha_1-1)^3 a_{min}}{(2\alpha_1\alpha_2 K - \alpha_1 K - \alpha_2 K)^3}$$

Proof. Since the expectation of the Pareto distribution is

$$\frac{\alpha_j K}{\alpha_j - 1}$$

for $j = 1, 2, \cdots$. By Lemma 4.1, the coefficient of time t is

$$\frac{\alpha_1\alpha_2 - \alpha_1}{2\alpha_1\alpha_2 - \alpha_1 - \alpha_2}.$$

Case 1. Since O_j have the same distributions, we get

$$\lim_{t\to\infty} t^{\alpha_2-\alpha_1} = 1.$$

And we know

$$\alpha_1 = \alpha_2 = K^\alpha\Gamma(2-\alpha)/(\alpha-1).$$

Thus, we get

$$\sigma^2 = \frac{K^{\alpha-1}\Gamma(2-\alpha)}{2\alpha\Gamma(4-\alpha)}.$$

In the similar way, we can get Case 2 and Case 3. □

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