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INVERTIBLE INTERPOLATION ON $AX = Y$ IN ALG$\mathcal{L}$

JOO HO KANG

ABSTRACT. Given operators X and Y acting on a Hilbert space $\mathcal{H}$, an interpolating operator is a bounded operator A such that $AX = Y$. An interpolating operator for n-operators satisfies the equation $AX_i = Y_i$, for $i = 1, 2, \cdots, n$. In this article, we showed the following : Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$ and let X and Y be operators in $\mathcal{B}(\mathcal{H})$. Then the following are equivalent:

(1) $\sup\left\{\frac{\|E^{\perp}Yf\|}{\|E^{\perp}Xf\|} : f\in\mathcal{H}, E\in\mathcal{L}\right\} < \infty$, $\sup\left\{\frac{\|Xf\|}{\|Yf\|} : f\in\mathcal{H}\right\} < \infty$ and $\overline{range\ X} = \mathcal{H} = \overline{range\ Y}$.

(2) There exists an invertible operator A in Alg$\mathcal{L}$ such that $AX = Y$.

1. INTRODUCTION AND PRELIMINARIES

One form of interpolation problems in operator algebras is the following : Given operators X and Y on a Hilbert space $\mathcal{H}$ and an operator algebra $\mathcal{A}$ on $\mathcal{H}$, when does there exist an operator A in $\mathcal{A}$ such that $AX = Y$? Interpolation problems have been investigated in several operator algebras by many mathematicians. The author has studied interpolation problems in Alg$\mathcal{L}$ and tridiagonal algebras.

The n-vector interpolation problem was considered for a C^*-algebra $\mathcal{U}$ by Kadison [6]. In the case of nest algebra $\mathcal{U}$, the (one-vector) interpolation problem was solved by Lance [7]: his result was extended by Hopenwasser [2] to the case that $\mathcal{U}$ is a CSL-algebra. Munch[8] obtained conditions for interpolation in case A is required to lie in the ideal of Hilbert-Schmidt operators in a nest algebra. Hopenwasser [3] once again extended the interpolation condition to the ideal of Hilbert-Schmidt operators in a CSL-algebra.

Jo-Kang-Park obtained a necessary and sufficient condition for the existence of an interpolating operator that is in Alg$\mathcal{L}$ in [4]. In this paper we showed when there exists an invertible interpolating operator in Alg$\mathcal{L}$ from the previous results.

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A subspace lattice $\mathcal{L}$ is a strongly closed lattice of projections acting on a Hilbert space $\mathcal{H}$. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. The symbol Alg$\mathcal{L}$ is the algebra of all bounded linear operators on $\mathcal{H}$ that leave invariant all the projections in $\mathcal{L}$. Let M be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{M}$ means the closure of M and $\overline{M}^{\perp}$ means the orthogonal complement of $\overline{M}$. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers.

2. RESULTS

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded operators acting on $\mathcal{H}$. Let $\mathcal{L}$ be a subspace lattice (i.e. a complete lattice of orthogonal projections which contains 0 and I) on $\mathcal{H}$. Alg$\mathcal{L}$ is the algebra of all bounded linear operators on $\mathcal{H}$ which leave invariant each projection E in $\mathcal{L}$. Assume that X and Y are operators in $\mathcal{B}(\mathcal{H})$ and A is an operator in Alg$\mathcal{L}$ such that $AX = Y$. Then $\|E^{\perp}Yf\| = \|E^{\perp}AXf\| = \|E^{\perp}AE^{\perp}Xf\| \leq \|A\|\|E^{\perp}Xf\|$, for all $E \in \mathcal{L}$. If, for convenience, we adopt the convention that a fraction whose numerator and denominator are both zero is equal to zero, then the inequalities above may be stated in the form

$$\sup_{E\in\mathcal{L}} \frac{\|E^{\perp}Yf\|}{\|E^{\perp}Xf\|} \leq \|A\| < \infty.$$

In [4], we showed that the above fact is a necessary and sufficient condition for existence of an interpolating operator in Alg$\mathcal{L}$.

Theorem A ([4]). *Let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$ and let X and Y be operators in $\mathcal{B}(\mathcal{H})$. Let P be the projection onto $\overline{rangeX}$. If $PE = EP$ for each $E \in \mathcal{L}$, then the following statements are equivalent:*

(1) *There exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$.*

(2) $\sup\left\{\frac{\|E^{\perp}Yf\|}{\|E^{\perp}Xf\|} : f \in \mathcal{H}, E \in \mathcal{L}\right\} = K < \infty.$

Moreover, if condition (2) holds, then we may choose an operator A such that $\|A\| = K$.

Theorem B ([1]). *Let X and Y be bounded operators acting on a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

(1) *range*$Y^* \subseteq$ *range* X^*

(2) $Y^*Y \leq \lambda^2 X^*X$ *for some* $\lambda \geq 0$

(3) *there exists a bounded operator* A *on* $\mathcal{H}$ *so that* $AX = Y$.

Moreover, if (1), (2) *and* (3) *are valid, then there exists a unique operator* A *so that*

(a) $\|A\|^2 = \inf\{\mu : Y^*Y \leq \mu X^*X\}$

(b) *ker*Y^* = *ker*A^* *and*

(c) *range*$A^* \subseteq \overline{rangeX}$.

Now, we investigate invertible interpolation problems in Alg$\mathcal{L}$.

Lemma 2.1. *Let* A, X *and* Y *be operators in* $\mathcal{B}(\mathcal{H})$. *If* $Y = AX$ *and* $A|_{\overline{range\ X}^\perp} = 0$, *then Ker* A^* =*Ker* Y^*.

Proof. Let f be a vector in Ker A^*. Then $A^*f = 0$. So $X^*A^*f = 0$. Hence $Y^*f = 0$ and f is in Ker Y^*.

Conversely, if f is in Ker Y^*, then for any g in $\mathcal{H}$,

$$\begin{aligned} < f, Ag > &=< A^*f, g > \\ &=< A^*f, Xg_1 + g_2 > \text{ for some } g_1 \in \mathcal{H} \text{ and } g_2 \in \overline{range\ X}^\perp \\ &=< A^*f, Xg_1 > + < A^*f, g_2 > \\ &=< f, AXg_1 > + < f, Ag_2 > \\ &=< f, Yg_1 > + 0 = 0. \end{aligned}$$

Hence $f \in \overline{range\ A}^\perp$(=Ker A^*). □

Theorem 2.2. *Let* $\mathcal{L}$ *be a subspace lattice on a Hilbert space* $\mathcal{H}$ *and let* X *and* Y *be operators in* $\mathcal{B}(\mathcal{H})$. *Then the following assertions are equivalent:*

(1) $\sup\left\{\frac{\|E^\perp Yf\|}{\|E^\perp Xf\|} : f \in \mathcal{H}, E \in \mathcal{L}\right\} < \infty$, $\sup\left\{\frac{\|Xf\|}{\|Yf\|} : f \in \mathcal{H}\right\} < \infty$ *and* $\overline{range\ X} = \mathcal{H} = \overline{range\ Y}$.

(2) *There exists an invertible operator* A *in Alg*$\mathcal{L}$ *such that* $AX = Y$.

Proof. Assume that

$$\sup\left\{\frac{\|E^\perp Yf\|}{\|E^\perp Xf\|} : f \in \mathcal{H},\ E \in \mathcal{L}\right\} < \infty \text{ and } \sup\left\{\frac{\|Xf\|}{\|Yf\|} : f \in \mathcal{H}\right\} < \infty.$$

Let P be the projection onto $\overline{rangeX}$. Since $\overline{rangeX} = \mathcal{H}$, $P = I$. Hence $PE = EP$ for all E in $\mathcal{L}$. So there exist operators A in Alg$\mathcal{L}$ and B in $\mathcal{B}(\mathcal{H})$ such that

$AX = Y$ and $BY = X$ by Theorem A and Theorem B. So $X = BY = BAX$ and $Y = AX = ABY$. Since $\overline{range\ X} = \mathcal{H} = \overline{range\ Y}$, $AB = I = BA$. Hence A is invertible.

Conversely, if there exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$ and $AB = I = BA$ for some bounded operator B, then $X = BY$.

Hence

$$\sup\left\{\frac{\|E^{\perp}Yf\|}{\|E^{\perp}Xf\|} : f\in\mathcal{H},\ E\in\mathcal{L}\right\} < \infty$$

and

$$\sup\left\{\frac{\|Xf\|}{\|Yf\|} : f\in\mathcal{H}\right\} < \infty.$$

Since A and B are onto, KerA^* = KerY^* and KerB^* = KerX^*, $\overline{range\ Y}^{\perp} = 0 = \overline{range\ X}^{\perp}$ by Lemma 2.1. Hence X and Y have dense ranges in $\mathcal{H}$. □

Theorem 2.3. *Let $\mathcal{L}$ be a subspace lattice acting on a Hilbert space $\mathcal{H}$ and let X_i and Y_i be operators in $\mathcal{B}(\mathcal{H})$ for $i = 1, 2, \cdots, n$. Then the following assertions are equivalent:*

(1)

$$\sup\left\{\frac{\|E^{\perp}(\sum_{i=1}^{n} Y_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|} : f_i\in\mathcal{H},\ E\in\mathcal{L}\right\} < \infty,$$

$$\sup\left\{\frac{\|\sum_{i=1}^{n} X_i f_i\|}{\|\sum_{i=1}^{n} Y_i f_i\|} : f_i\in\mathcal{H}\right\} < \infty$$

and $\overline{range\ X_k} = \mathcal{H} = \overline{range\ Y_k}$ for some k in $\{1, 2, \cdots, n\}$.

(2) *There exists an invertible operator A in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots, n$.*

Proof. Assume that

$$\sup\left\{\frac{\|E^{\perp}(\sum_{i=1}^{n} Y_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|} : f_i\in\mathcal{H},\ E\in\mathcal{L}\right\} < \infty$$

and

$$\sup\left\{\frac{\|\sum_{i=1}^{n} X_i f_i\|}{\|\sum_{i=1}^{n} Y_i f_i\|} : f_i\in\mathcal{H}\right\} < \infty.$$

Let P_k be the projection onto $\overline{rangeX_k}$ for given k. Since $\overline{rangeX_k} = \mathcal{H}$, $P_k = I$. Hence $P_kE = EP_k$, for all E in $\mathcal{L}$. So there exist operators A in Alg$\mathcal{L}$ and B in $\mathcal{B}(\mathcal{H})$ such that $AX_i = Y_i$ and $BY_i = X_i$ for $i = 1, 2, \cdots, n$ by Theorem 2.2[4]. So $X_i = BY_i = BAX_i$ and $Y_i = AX_i = ABY_i$. Since $\overline{range\ X_k} = \mathcal{H} = \overline{range\ Y_k}$ for some $k = 1, 2, \cdots, n$, $AB = I = BA$. Hence A is invertible.

Conversely, if there exists an operator A in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots, n$ and $AB = I = BA$ for some bounded operator B, then $X_i = BY_i$ for all $i = 1, 2, \cdots, n$. Hence

$$\sup\left\{\frac{\|E^{\perp}(\sum_{i=1}^{n} Y_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|} : f_i \in \mathcal{H},\ E \in \mathcal{L}\right\} < \infty$$

and

$$\sup\left\{\frac{\|\sum_{i=1}^{n} X_i f_i\|}{\|\sum_{i=1}^{n} Y_i f_i\|} : f_i \in \mathcal{H}\right\} < \infty.$$

By Lemma 2.1, KerA^* = KerY_i^* and KerB^* = KerX_i^*. Since A and B are onto, $\overline{range\ Y_i}^{\perp} = 0 = \overline{range\ X_i}^{\perp}$. Hence X_i and Y_i have dense ranges in $\mathcal{H}$ for all $i = 1, 2, \cdots, n$. □

Theorem 2.4. *Let $\mathcal{L}$ be a subspace lattice acting on a Hilbert space $\mathcal{H}$ and let X_i and Y_i be operators in $\mathcal{B}(\mathcal{H})$ for $i = 1, 2, \cdots$. Then the following are equivalent:*

(1)

$$\sup\left\{\frac{\|E^{\perp}(\sum_{i=1}^{n} Y_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|} : f_i \in \mathcal{H},\ n \in \mathbb{N},\ E \in \mathcal{L}\right\} < \infty,$$

$$\sup\left\{\frac{\|\sum_{i=1}^{n} X_i f_i\|}{\|\sum_{i=1}^{n} Y_i f_i\|} : f_i \in \mathcal{H},\ n \in \mathbb{N}\right\} < \infty$$

and $\overline{range\ X_k} = \mathcal{H} = \overline{range\ Y_k}$ for some k in $\{1, 2, \cdots, n\}$.

(2) *There exists an invertible operator A in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots$.*

Proof. Assume that

$$\sup\left\{\frac{\|E^{\perp}(\sum_{i=1}^{n} Y_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|} : f_i \in \mathcal{H},\ n \in \mathbb{N},\ E \in \mathcal{L}\right\} < \infty$$

and

$$\sup\left\{\frac{\|\sum_{i=1}^{n} X_i f_i\|}{\|\sum_{i=1}^{n} Y_i f_i\|} : f_i \in \mathcal{H},\ n \in \mathbb{N}\right\} < \infty.$$

Let P_k be the projection onto $\overline{rangeX_k}$ for given k. Since $\overline{rangeX_k} = \mathcal{H}$, $P_k = I$. Hence $P_k E = EP_k$, for all E in $\mathcal{L}$. So there exist operators A in Alg$\mathcal{L}$ and B in $\mathcal{B}(\mathcal{H})$ such that $AX_i = Y_i$ and $BY_i = X_i$ for $i = 1, 2, \cdots$ (see [4, Theorem 2.3]). Hence $X_i = BY_i = BAX_i$ and $Y_i = AX_i = ABY_i$ for all $i = 1, 2, \cdots$. Since $\overline{range\ X_k} = \mathcal{H} = \overline{range\ Y_k}$, $AB = I = BA$. Hence A is invertible.

Conversely, if there exists an operator A in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots$ and $AB = I = BA$ for some bounded operator B, then $X_i = BY_i$ for all

$i = 1, 2, \cdots$. Hence

$$\sup\left\{\frac{\|E^{\perp}(\sum_{i=1}^{n} Y_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|} : f_i \in \mathcal{H},\, n \in \mathbb{N},\, E \in \mathcal{L}\right\} < \infty$$

and

$$\sup\left\{\frac{\|\sum_{i=1}^{n} X_i f_i\|}{\|\sum_{i=1}^{n} Y_i f_i\|} : f_i \in \mathcal{H},\, n \in \mathbb{N}\right\} < \infty.$$

By Lemma 2.1, $\text{Ker}A^* = \text{Ker}Y_i^*$ and $\text{Ker}B^* = \text{Ker}X_i^*$ for all $i = 1, 2, \cdots$. Since A and B are onto, $\overline{range\ Y_i}^{\perp} = 0 = \overline{range\ X_i}^{\perp}$ for all $i = 1, 2, \cdots$. Hence X_i and Y_i have dense ranges in $\mathcal{H}$ for all $i = 1, 2, \cdots$. □

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Department of Mathematics, Daegu University, Daegu, Korea
Email address: jhkang@daegu.ac.kr