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A FAST CONSTRUCTION OF GENERALIZED MANDELBROT SETS USING MAIN COMPONENTS WITH EPICYCLOIDAL BOUNDARIES

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Abstract. The main components in the generalized Mandelbrot sets are under a theoretical investigation for their parametric boundary equations. Using the boundary geometries, a fast construction algorithm is introduced for the generalized Mandelbrot set. This fast algorithm definitely reduces the construction CPU time in comparison with the naive algorithm. Its graphic implementation displays the mysterious and beautiful fractal sets

1. Introduction

Some geometric properties of the generalized Mandelbrot set $\boldsymbol{M}$ have been investigated by a number of researchers [1-5, 7]. By establishing an epicycloidal boundary equation of the main component of $\boldsymbol{M}$, we provide a theoretical background for an *escape-time*[1] algorithm rapidly constructing $\boldsymbol{M}$ shown in Figure 1. We denote $i = \sqrt{-1}$, the set of natural numbers by $\boldsymbol{N}$, the set of real numbers by $\boldsymbol{R}$ and the set of complex numbers by $\boldsymbol{C}$. Let ${P_c}^k$ denote the k-fold composite mapping $P_c(P_c(P_c(\cdots)))$. An attracting period-k component(bulb) $\boldsymbol{M}_k{}'$ in $\boldsymbol{M}$ is a component[6] of the set:

$$\left\{c \in \boldsymbol{C} : \text{there exists } \xi \in \boldsymbol{C} \text{ such that } {P_c}^k(\xi) = \xi, \left| \frac{d}{dz}{P_c}^k(z)|_{z=\xi} < 1\right.\right\}.$$

When $k = 1$ it is called the *main component(bulb)* which is shaded in blue. We first introduce a theoretical background regarding the properties of $\boldsymbol{M}$. Throughout the analysis we assume an integer $n \geq 2$.

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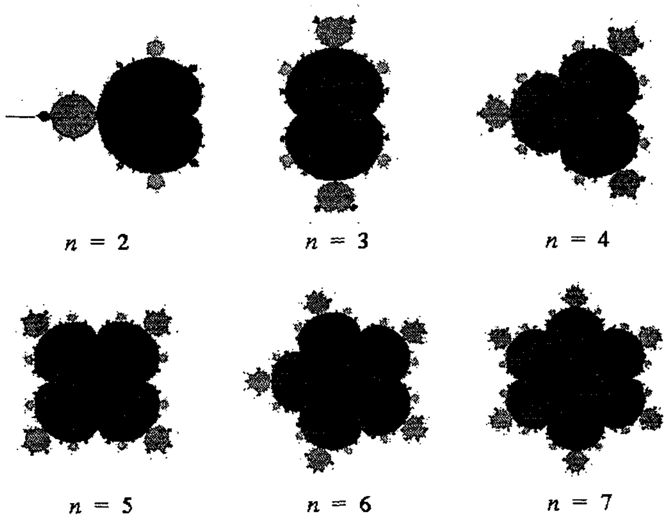

Figure 1. Typical examples of the degree-n bifurcation set

Definition 1.1. The set $S = \{c \in C : c = re^{i\theta}, r > 0, 0 < \theta \leq \pi/(n-1)\}$ is called the *principal sector*. The sets $P_m = \{c \in C : c = re^{i\phi_m}, r > 0, \phi_m = m\pi/(n-1)\}$ (for $m = 1, 2, \cdots, 2n-2$) are called the *rays of symmetry*.

Definition 1.2. Let $P_c(z) = z^n + c$ with $c,\ z \in C$. Then the *degree-n bifurcation set* or *the generalized Mandelbrot set* is defined as the set

$$M = \left\{c \in C : \lim_{k \to \infty} {P_c}^k(0) \neq \infty\right\}.$$

If $n = 2$, then M reduces to the *Mandelbrot set* [1-5,7].

Theorem 1.1. *The degree-n bifurcation set M is symmetric in the c-parameter plane about P_m for all $m \in \{1, 2, \cdots, 2n-2\}$.*

Proof. See [4, p. 224]. □

Theorem 1.2. *Let $B = 2^{\frac{1}{n-1}}$. Then*

$$M = \left\{c \in C : |{P_c}^k(0)| \leq B \text{ for all } k \in N\right\} \subset \left\{c \in C : |c| \leq B\right\}$$

Proof. See [3, p. 114-115]. □

2. Epicyloidal Boundaries of Main Components

Let $\partial \boldsymbol{M}_1'$ denote the boundary of $\boldsymbol{M}_1'$. Let $\boldsymbol{\Gamma} = \partial \boldsymbol{M}_1' \cap \bar{\boldsymbol{S}}$, where $\bar{\boldsymbol{S}}$ is the closure of the principal sector $\boldsymbol{S}$. The governing equation for $\partial \boldsymbol{M}_1'$ is characterized as a set of c-values with $c,\ z \in \boldsymbol{C}$ satisfying the following equations :

$$P_c(z) = z^n + c = z, \tag{1}$$

$$|P_c'(z)| = |nz^{n-1}| = 1. \tag{2}$$

Solving these two equations by means of a parameter ψ yields the following Theorem 2.1.

Theorem 2.1. *Let $c \in \partial \boldsymbol{M}_1'$ and $\rho = (1/n)^{n/(n-1)}$. As shown in Figure 2, let a fixed point* P *on a circle of radius ρ describe an epicycloid $\boldsymbol{E}$ as it rolls on the outside of a circle of radius $(n-1)\rho$. Let ψ denote a rolling angle traced by* P. *Then $c = \rho(n\cos\psi - \cos n\psi) + i\rho(n\sin\psi - \sin n\psi)$ with $0 \le \psi < 2\pi$ lies on the epicycloid $\boldsymbol{E}$, that is, $\partial \boldsymbol{M}_1' = \boldsymbol{E}$.*

Proof. We parametrize Eq.(2) so that $z^{n-1} = (1/n)e^{i\phi}$ with $0 \le \phi < 2\pi$ yields $(n-1)$ distinct solution branches $z_k = (1/n)^{1/(n-1)} e^{i\left(\frac{\phi}{n-1}\right) + i\left(\frac{2k\pi}{n-1}\right)}$ for $k = 0, 1, \cdots, n-2$. Let $\psi = \phi/(n-1)$. Then Eq.(1) shows that

$$c_0 = z_0 - {z_0}^n = \rho(n\cos\psi - \cos n\psi) + i\rho(n\sin\psi - \sin n\psi),\ \ 0 \le \psi < 2\pi/(n-1)$$

defines a principal branch in the c-parameter plane. All other branches $c_k = z_k - {z_k}^n$ are rotations of c_0 by an angle of $2k\pi/(n-1)$. This implies that $c(\psi) \in \partial \boldsymbol{M}_1'$ has a period of $2\ \pi/(n-1)$ in ψ. For $k = 0, 1, 2, \cdots, n-2$, let $\boldsymbol{H}_k = \{c_k(\psi) \in \partial \boldsymbol{M}_1' : 2k\pi/(n-1) \le \psi < 2(k+1)\pi/(n-1)\}$ denote an arc shown in Figure 2. It is obvious that $\partial \boldsymbol{M}_1' = \bigcup_{k=0}^{n-2} \boldsymbol{H}_k$. Hence a point $c \in \partial \boldsymbol{M}_1'$ is parametrized for $\psi \in [0,\ 2\pi)$ with $x,\ y \in \boldsymbol{R}$ such that

$$c = c(\psi) = x(\psi) + iy(\psi),\ x = \rho(n\cos\psi - \cos n\psi),\ y = \rho(n\sin\psi - \sin n\psi). \tag{3}$$

The above parametric equations represent the epicycloid $\boldsymbol{E}$, a special case of a general epicycloid whose point (x, y) is described by

$$x = (a+b)\cos\psi - b\cos\Big(\frac{a+b}{b}\Big)\psi,\ y = (a+b)\sin\psi - b\sin\Big(\frac{a+b}{b}\Big)\psi \tag{4}$$

with $a = (n-1)\rho$, $b = \rho$ and $\psi \in [0,\ 2\pi)$. □

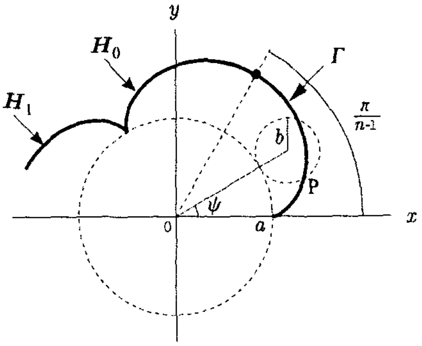

*Figure 2. An epicycloid **E** with an arc **Γ***

An explicit calculation for $c \in \boldsymbol{\Gamma}$ using Eq.(3) shows that

$$c = r(\cos\theta + i\sin\theta),\ r = \rho\sqrt{n^2+1-2n\cos(n-1)\psi}, \tag{5}$$

$$\theta = \begin{cases} \mathrm{Tan}^{-1}(n\cos\psi - \cos n\psi,\ n\sin\psi - \sin n\psi) & \text{for } \psi \in \boldsymbol{I}^* \\ \pi/2 & \text{for } \psi = \cos^{-1}((1-\sqrt{3})/2)\ if\ n=2 \end{cases},$$

where $\mathrm{Tan}^{-1}(x,y)$ gives a polar angle of the point (x,y) and $\boldsymbol{I}^* = [0, \pi/(n-1)]$ if $n \geq 3$ or $\boldsymbol{I}^* = [0,\pi] - \{\cos^{-1}((1-\sqrt{3})/2)\}$ if $n = 2$. The following Theorem 2.2 determines whether or not a given point $c = (x, y) \in \boldsymbol{C}$ belongs to $\boldsymbol{M}_1'$.

Theorem 2.2. *Let $\omega = \pi/(n-1)$, $\rho = (1/n)^{n/(n-1)}$, $r_{min} = (n-1)\rho$ and $r_{max} = (n+1)\rho$. For a given point (x, y), let $r = \sqrt{x^2+y^2}$, $0 \leq \theta = \mathrm{Tan}^{-1}(x,y) < 2\pi$ and $k = [\theta/\omega]$ be the integer part of θ/ω. Let $\theta^* = \theta - k\omega$ if k is even, and $\theta^* = (k+1)\omega - \theta$ if k is odd. Then there exist unique $\psi = \psi_b$ and $\theta = \theta_b$ satisfying Eq.(5). In addition, we have the following criteria:*

(a) *if $r < r_{min}$, then $(x,y) \in \boldsymbol{M}_1'$.*

(b) *if $r_{min} \leq r \leq r_{max}$, then*

$$(x,y) \in \boldsymbol{M}_1' \text{ for } \theta_b < \theta^* \text{ and } (x,y) \notin \boldsymbol{M}_1' \text{ for } \theta_b > \theta^*.$$

(c) *if $r > r_{max}$, then $(x,y) \notin \boldsymbol{M}_1'$.*

Proof. By Theorem 1.1, it suffices to consider $\boldsymbol{\Gamma}$ with $\theta = k\omega + \theta^*$ for even k. Besides the criteria, the existence as well as uniqueness for ψ_b and θ_b is intuitively clear in view of Figure 3. □

3. ALGORITHM AND COMPUTATIONAL RESULTS

Based on Theorem 2.2, we have developed a C++ subroutine INTM1(x, y, n)

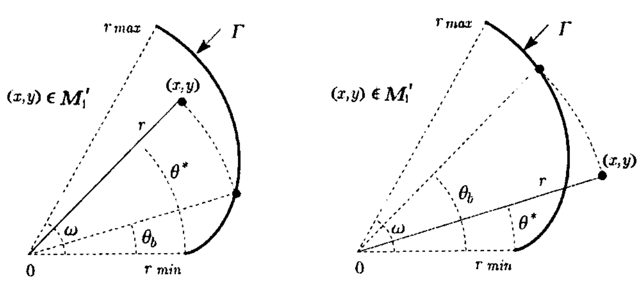

Figure 3. *How to determine whether or not* $(x, y) \in \boldsymbol{M}_1{}'$

checking whether or not $(x, y) \in \boldsymbol{M}_1'$. The following is an algorithm for a fast construction of $\boldsymbol{M}$.

Algorithm 3.1.

Step 1. *Let* n_c *be the number of colors, ITER be the maximum number of iterations, BGCO be the background color and* $B = 2^{\frac{1}{n-1}}$. *Select a rectangular region* $\boldsymbol{\Omega}$ *covering* $\{c \in \boldsymbol{C} : |c| \leq B\}$. *Divide* $\boldsymbol{\Omega}$ *into* $p \times q$ *subregions.*

Step 2. *For a point c in a subregion, compute* $P_c{}^j(0)$ *for all* $0 \leq j \leq ITER$ *and store the last* $(n_c - 1)$ *points.*

Step 3. *If* $|P_c{}^j(0)| > B$ *for some* j, *then stop iterating and paint c in BGCO.*

Step 4. *Use INTM1*(x, y, n) *to check whether or not* $(x, y) \in \boldsymbol{M}_1'$. *If* $c \in \boldsymbol{M}_1'$, *then paint c in a color number 1. If* $c \notin \boldsymbol{M}_1'$ *and* $|P_c{}^j(0)| \leq B$ *for all* $0 \leq j \leq ITER$, *then*

(1) *compute the period k of the orbit* [5] *from the* $(n_c - 1)$ *stored points.*

(2) (a) *If* $1 \leq k \leq n_c - 1$, *then paint c in a color number k,*

(b) *or else paint c in BGCO.*

Step 5. *Repeat Steps* 2 – 4 *for all the remaining points in* $\boldsymbol{\Omega}$.

An implementation of the above algorithm displays Figure 1 showing typical examples of the generalized Mandelbrot set $\boldsymbol{M}$ for various values of n. It is interesting to observe that $\partial \boldsymbol{M}_1'$ represents an epicycloid in $\boldsymbol{M}$ while it represents a cardioid, being a special case of an epicycloid, in the Mandelbrot set. An elementary theory of plane curves shows that the resulting epicycloid is a nephroid when $n = 3$ and a ranunculoid when $n = 6$.

Two *escape-time* algorithms constructing $\boldsymbol{M}$ are a naive algorithm that does not use INTM1(x, y, n) and a fast algorithm that uses INTM1(x, y, n). Both algorithms were implemented with C++ Builder 3.0 for the screen size of 300×300 pixels within

216 iterations using a Pentium IV 2.8 Ghz personal computer under Windows XP operating system. Table 1 lists CPU times in seconds fully constructing M for both algorithms as a function of n. This fast algorithm has significantly reduced construction CPU times.

Observe that the construction time for even n is longer than that for odd n. The reason is that for even n each construction algorithm uses only one symmetry axis, while for odd n it uses two symmetry axes. When $(n-1)$ is a multiple of 4, each construction algorithm uses 4 symmetry axes. The current analysis can be extended to the case when $k=2$, despite the expected algebraic complexity.

Table 1. Construction CPU times in Seconds

Algorithms & Properties	n									
	2	3	4	5	6	7	8	9	···	1000
Naive Algorithm	7.91	4.40	10.66	3.30	13.30	7.20	15.60	4.45	···	122.59
Fast Algorithm	2.01	1.16	2.53	0.87	2.79	1.46	2.91	0.94	···	5.27

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