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MINIMUM PERMANENT ON A FACE OF Ω_6 CONTAINING TWO SQUARE ZERO-SUBMATRICES

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ABSTRACT. In this paper, when $n = 6$, we will determine the minimum permanent and minimizing matrices on the face of Ω_n which contains exactly two square zero-submatrices.

1. INTRODUCTION

Let Ω_n be the convex polytope of all n-square doubly stochastic matrices, that is, real nonnegative n-square matrices whose row and column sums are all equal to 1. For an $n \times n$ matrix $A = [a_{ij}]$, the *permanent* of A, written per$A$, is defined by

$$\text{per}A = \sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)}, \tag{1.1}$$

where S_n stands for the symmetric group on the set $\{1, 2, \cdots, n\}$. For an $n \times n$ $(0,1)$-matrix $C = [c_{ij}]$, let

$$\mathcal{F}(C) = \{X = [x_{ij}] \in \Omega_n | \ x_{ij} = 0 \ \text{ whenever } c_{ij} = 0\} \tag{1.2}$$

Then $\mathcal{F}(C)$ forms a face of Ω_n, and since it is compact, $\mathcal{F}(C)$ contains a minimizing matrix A such that per$A \leq$ per$X$ for all $X \in \mathcal{F}(C)$.

Egorycev [2] and Falikman [3] proved the van der Waerden permanent conjecture: If $A \in \Omega_n$, then there exists the unique minimizing matrix J_n on Ω_n such that

$$\text{per}A \geq \text{per}J_n,$$

where J_n is an n-square matrix all of whose entries are $\frac{1}{n}$. After these results, many mathematicians have solved problems determining minimum permanents in various faces of Ω_n. Knopp and Sinkhorn [7] determined the minimum permanent in a face of Ω_n with one prescribed zero. This result was extended by Friedland

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[4] to faces in which prescribed zeros form a submatrix. More generally, Hwang [5] determined the minimum permanents in faces of Ω_n that are determined by a staircase matrix. Similarly, Lee [8, 9, 10] investigated the minimum permanent and minimizing matrices on the face of Ω_n which contains exactly two square zero-submatrices. In particular, he proposed a conjecture: A minimizing matrix on the face of Ω_n which contains exactly two square zero-submatrices has all diagonal block zero-submatrices, and he showed that the conjecture is true for $n = 5$. But Choi [1] showed that the conjecture is not true for $n = 6$.

In this paper, when $n = 6$, we will determine the minimum permanent and minimizing matrices on the face of Ω_n which contains exactly two square zero-submatrices.

2. RESULTS

To prove the main result, we need some lemmas:

Lemma 2.1 ([11]). *If $A = [a_{ij}]$ is a minimizing matrix, then $a_{nk} > 0$ implies* $\mathrm{per}A(h|k) = \mathrm{per}A$, *where* $\mathrm{per}A(h|k)$ *is the submatrix obtained from A by deleting row h and column k.*

Let $\mathcal{K}$ be a set of n-square real matrices and let f be a real valued function on $\mathcal{K}$. A matrix $A \in \mathcal{K}$ is called an f-minimizing matrix on $\mathcal{K}$ if $f(A) \leq f(X)$ for all $X \in \mathcal{K}$.

Lemma 2.2 ([6, Row-column averaging lemma]). *Let $\mathcal{K}$ be a compact convex set of $n \times n$ real nonnegative matrices, and f be real valued function on $\mathcal{K}$. Let $A = [a_1, \cdots, a_n]$ be an f-minimizing matrix on $\mathcal{K}$. Suppose, for some k $(2 \leq k \leq n)$, that*

(i) *$a_1, \cdots, a_k$ have the same $(0,1)$ pattern,*

(ii) *for any $k \times k$ permutation matrix P,*

$$B = A(P \oplus I_{n-k}) \in K, \ f(A) = f(B),$$

and $f(tA + (1-t)B)$ is a polynomial in t of degree ≤ 3.

Then $A(J_k \oplus I_{n-k})$ is also a f-minimizing matrix on $\mathcal{K}$.

Let $D = [d_{ij}]$ be an n-square matrix such that

$$d_{ij} = \begin{cases} 0, & 1 \le i,j \le p \\ 0, & p+1 \le i,j \le p+q \\ 1, & \text{elsewhere} \end{cases} \tag{2.1}$$

where $n < 2(p+q)$, $2p < n$, $2q < n$.

Theorem 2.3. *Let $g(\beta) = -36\beta^5 + 42\beta^4 - 17\beta^3 + \frac{5}{2}\beta^2$. If D has the form as in (2.1) and $n = 6$, then the minimum permanent on $\mathcal{F}(D)$ is $g(\beta)$ where β, $\frac{1}{4} \le \beta < \frac{1}{2}$, is a real root of the equation $-180a^3 + 168a^2 - 51a + 5 = 0$.*

Proof. Since $6 < 2(p+q)$, $2p < 6$, and $2q < 6$, we have $p = q = 2$. By Lemma 2.2, minimizing matrix $X = [x_{ij}]$ will have the form of the following,

$$X = \begin{bmatrix} 0 & 0 & b & b & \frac{1-2b}{2} & \frac{1-2b}{2} \\ 0 & 0 & b & b & \frac{1-2b}{2} & \frac{1-2b}{2} \\ a & a & 0 & 0 & \frac{1-2a}{2} & \frac{1-2a}{2} \\ a & a & 0 & 0 & \frac{1-2a}{2} & \frac{1-2a}{2} \\ \frac{1-2a}{2} & \frac{1-2a}{2} & \frac{1-2b}{2} & \frac{1-2b}{2} & \frac{2a+2b-1}{2} & \frac{2a+2b-1}{2} \\ \frac{1-2a}{2} & \frac{1-2a}{2} & \frac{1-2b}{2} & \frac{1-2b}{2} & \frac{2a+2b-1}{2} & \frac{2a+2b-1}{2} \end{bmatrix}.$$

(Case 1) $a, b \ne \frac{1}{2}$. If $a = 0$, then $b = \frac{1}{2}$ from column 6 of X, which is a contradiction to the assumption. Thus $a > 0$ and $b > 0$ from similar method. Therefore

$$0 < a < \frac{1}{2} \text{ and } 0 < b < \frac{1}{2}\,. \tag{2.2}$$

We claim that $a = b$. Suppose that $a \ne b$. Since $x_{46}, x_{64} \ne 0$, by Lemma 2.1, perX(6|4) = perX(4|6) = perX. Thus

$$\frac{b^2(1-2a)^3}{2} + 2ab^2(1-2a)(2a+2b-1) + 2ab(1-2a)(1-2b)^2$$
$$= \frac{a^2(1-2b)^3}{2} + 2a^2b(1-2b)(2a+2b-1) + 2ab(1-2b)(1-2a)^2$$

Hence

$$(a-b)(24a^2b^2 - (a+b) - 24ab(a+b) + 18ab) = 0 \tag{2.3}$$

So

$$(a+b)(24ab+1) = 24a^2b^2 + 18ab. \tag{2.4}$$

Since $a, b > 0$ from (2.2), $(a+b) \ge 2\sqrt{ab}$. Thus we have

$$\sqrt{ab}(24ab+1) \le 12a^2b^2 + 9ab. \tag{2.5}$$

By taking $c = \sqrt{ab}$,

$$12c^3 - 24c^2 + 9c - 1 \geq 0. \tag{2.6}$$

Since $0 < c < \frac{1}{2}$ from (2.2), consider

$$f(c) = 12c^3 - 24c^2 + 9c - 1, \text{ where } 0 < c < \frac{1}{2}.$$

To find $\sup f(c)$, solving the equation

$$\frac{df}{dc} = 36c^2 - 48c + 9 = 0, \tag{2.7}$$

we have

$$c = \frac{4 \pm 7}{6}, \ \sup f(c) = f\left(\frac{4-\sqrt{7}}{6}\right),$$

which is the only maximum of $f(c)$ between $0 < c < \frac{1}{2}$. Then since we have $\sup f(c) < 0$ this contradicts to (2.6). Thus we have $a = b$.

On the other hand,

$$\text{per}X = \text{per}X(6|4) = -36a^5 + 42a^4 - 17a^3 + \frac{5}{2}a^2,$$

and from the facts that $\frac{1-2a}{2} > 0$ and $\frac{4a-1}{2} \geq 0$, we have $a \in [\frac{1}{4}, \frac{1}{2})$. Now, to find the minimum of per$X$, let

$$g(a) := -36a^5 + 42a^4 - 17a^3 + \frac{5}{2}a^2, \text{where } a \in \left[\frac{1}{4}, \frac{1}{2}\right).$$

Then, solving the equation $\frac{dg}{da} = 0$, i.e., $-180a^3 + 168a^2 - 51a + 5 = 0$, we have

$$\inf g(a) = g(\beta),$$

where β is a real root of the equation $-180a^3 + 168a^2 - 51a + 5 = 0$.

(Case 2) $a = \frac{1}{2}$ or $b = \frac{1}{2}$. Suppose, without loss of generality, $b = \frac{1}{2}$. Then the minimizing matrix X have the following form:

$$X = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ a & a & 0 & 0 & \frac{1-2a}{2} & \frac{1-2a}{2} \\ a & a & 0 & 0 & \frac{1-2a}{2} & \frac{1-2a}{2} \\ \frac{1-2a}{2} & \frac{1-2a}{2} & 0 & 0 & a & a \\ \frac{1-2a}{2} & \frac{1-2a}{2} & 0 & 0 & a & a \end{bmatrix}.$$

So

$$\text{per}X = 12a^4 - 12a^3 + 5a^2 - a + \frac{1}{8}.$$

Let $h(a) := \text{per}X = 12a^4 - 12a^3 + 5a^2 - a + \frac{1}{8}$, where $a \in [0, \frac{1}{2})$. Then we have

$$\inf h(a) = \min\left\{h(0), h\left(\frac{1}{4}\right), h\left(\frac{1}{2}\right)\right\} = \frac{3}{63} > g(\beta).$$

Hence, from (Case 1) and (Case 2), the minimum permanent of $\mathcal{F}(D)$ is $g(\beta)$. □

Note that since $\beta = 0.333\ldots$, we have $g(\beta) = 0.185185\ldots$. Thus, the minimum permanent on $\mathcal{F}(D)$ is $0.0185185\ldots$.

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