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ON $N(k)$-QUASI EINSTEIN MANIFOLDS

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ABSTRACT. $N(k)$-quasi Einstein manifolds are introduced and studied.

1. Introduction

A non-flat Riemannian manifold (M^n, g) is said to be a *quasi Einstein manifold* [2] if its Ricci tensor S satisfies

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \qquad X,Y \in TM \tag{1.1}$$

or equivalently, its Ricci operator Q satisfies

$$Q = aI + b\eta \otimes \xi \tag{1.2}$$

for some smooth functions a and $b \neq 0$, where η is a nonzero 1-form such that

$$g(X,\xi) = \eta(X), \qquad g(\xi,\xi) = \eta(\xi) = 1 \tag{1.3}$$

for the associated vector field ξ. The 1-form η is called the associated 1-form and the unit vector field ξ is called the generator of the manifold. In an n-dimensional quasi Einstein manifold the Ricci tensor has precisely two distinct eigenvalues a and $a+b$, where a is of multiplicity $(n-1)$ and $a+b$ is simple [2]. A proper η-Einstein contact metric manifold ([1], [5]) is a natural example of a quasi Einstein manifold.

In this paper, we introduce the concept of $N(k)$-quasi Einstein manifolds. In section 2, it is proved that conformally flat quasi Einstein manifolds are certain $N(k)$-quasi Einstein manifolds. Semi-symmetric $N(k)$-quasi Einstein manifolds are studied in section 3. A necessary and a sufficient condition for an $N(k)$-quasi Einstein manifold to satisfy $R(\xi,X)\cdot S = 0$ are obtained in section 4. In section 5, Ricci-recurrent quasi Einstein manifolds are studied. In the last section, it is proved that a quasi-umbilical hypersurface of an $N(\bar{k})$-quasi Einstein manifold, such that it is normal to the generator of the ambient manifold, is an $N(k)$-quasi Einstein manifold.

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2. $N(k)$-quasi Einstein manifolds

The *k-nullity distribution* $N(k)$ [7] of a Riemannian manifold M is defined by

$$N(k) : p \to N_p(k) = \{Z \in T_pM \mid R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y)\}$$

for all $X, Y \in TM$, where k is some smooth function.

Motivated by the above definition, we give the following definition.

Definition 2.1. Let (M^n, g) be a quasi Einstein manifold. If the generator ξ belongs to the k-nullity distribution $N(k)$ for some smooth function k, then we say that (M^n, g) is an $N(k)$-quasi Einstein manifold.

Let (M^n, g) be a quasi Einstein manifold. From (1.2) and (1.3), it follows that

$$S(X, \xi) = (a+b)\,\eta(X), \tag{2.1}$$

$$Q\xi = (a+b)\,\xi, \tag{2.2}$$

$$r = na + b, \tag{2.3}$$

where r is the scalar curvature of M^n.

In an n-dimensional Riemannian manifold (M^n, g), the conformal curvature tensor C is given by [8]

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}\{g(Y,Z)QX - g(X,Z)QY \\ &\qquad + S(Y,Z)X - S(X,Z)Y\} \\ &\quad + \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{2.4}$$

If (M^n, g) is a conformally flat quasi Einstein manifold, then in view of (1.2), (2.3) and (2.4) we have

$$\begin{aligned} R(X,Y)Z &= \frac{(n-2)a-b}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad + \frac{b}{n-2}\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &\qquad + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}. \end{aligned} \tag{2.5}$$

Putting $Z = \xi$, in the above equation, we obtain

$$R(X,Y)\xi = \frac{a+b}{n-1}\{\eta(Y)X - \eta(X)Y\}, \tag{2.6}$$

that is, in an n-dimensional conformally flat quasi Einstein manifold, the generator ξ belongs to the $\left(\frac{a+b}{n-1}\right)$*-nullity distribution* $N\left(\frac{a+b}{n-1}\right)$. We can state this fact as the following:

Theorem 2.2. *An n-dimensional conformally flat quasi Einstein manifold is an $N\left(\frac{a+b}{n-1}\right)$-quasi Einstein manifold.*

Thus, we see that n-dimensional conformally flat quasi Einstein manifolds are natural examples of $N(k)$-quasi Einstein manifolds. It is well-known that in a 3-dimensional Riemannian manifold (M^3, g), the conformal curvature tensor vanishes, therefore we have the following

Corollary 2.3. *Each 3-dimensional quasi Einstein manifold is an $N(\frac{a+b}{2})$-quasi Einstein manifold.*

Let (M^n, g) be an $N(k)$-quasi Einstein manifold. Then, we have

$$R(Y,Z)\xi = k(\eta(Z)Y - \eta(Y)Z). \tag{2.7}$$

The equation (2.7) is equivalent to

$$R(\xi,Y)Z = k(g(Y,Z)\xi - \eta(Z)Y). \tag{2.8}$$

In particular, the above equation implies that

$$R(\xi,Y)\xi = k(\eta(Y)\xi - Y). \tag{2.9}$$

From (2.7) and (2.8), we have

$$\eta(R(Y,Z)\xi) = 0, \tag{2.10}$$

$$\eta(R(\xi,Y)Z) = k(g(Y,Z) - \eta(Y)\eta(Z)). \tag{2.11}$$

3. Semi-symmetric $N(k)$-quasi Einstein manifolds

As a generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and in turn their generalizations. A Riemannian manifold M is said to be semi-symmetric if its curvature tensor R satisfies

$$R(X,Y)\cdot R = 0, \qquad X,Y \in TM,$$

where $R(X,Y)$ acts on R as a derivation. In this section, we study $N(k)$-quasi Einstein manifolds M satisfying $R(\xi,X)\cdot R = 0$, $X \in TM$.

First, we prove the following theorem.

Theorem 3.1. *An $N(k)$-quasi Einstein manifold (M^n, g) satisfies $R(\xi,X)\cdot R = 0$ if and only if $k = 0$.*

Proof. The condition $R(\xi,X)\cdot R = 0$ implies that

$$0 = [R(\xi,X), R(Y,Z)]\xi - R(R(\xi,X)Y,Z)\xi - R(Y,R(\xi,X)Z)\xi,$$

which in view of (2.8) and (2.9) gives

$$\begin{aligned} 0 \;=\; & k\{g(X,R(Y,Z)\xi)\xi - \eta(X)R(Y,Z)\xi + R(Y,Z)X \\ & - g(X,Y)R(\xi,Z)\xi + \eta(Y)R(X,Z)\xi \\ & - g(X,Z)R(Y,\xi)\xi + \eta(Z)R(Y,X)\xi\}. \end{aligned}$$

In view of (2.7), the above equation yields

$$0 = k\{R(Y,Z)X - k(g(Z,X)Y - g(Y,X)Z)\}.$$

Therefore, either $k = 0$ or

$$R(Y,Z)X = k(g(Z,X)Y - g(Y,X)Z).$$

In the second case, M^n becomes an Einstein space, which is not possible. Thus we have $k = 0$. Conversely, if $k = 0$, in view of (2.8) M^n satisfies $R(\xi,X) \cdot R = 0$. This completes the proof. □

As a Corollary, we have the following

Corollary 3.2. *If (M^n, g) is a semi-symmetric $N(k)$-quasi Einstein manifold, then $k = 0$.*

Now, we apply the above two results to conformally flat quasi Einstein manifolds and in result we may state the following

Theorem 3.3. *A conformally flat quasi Einstein manifold satisfies $R(\xi,X) \cdot R = 0$ if and only if $a + b = 0$. In particular, each conformally flat semi-symmetric quasi Einstein manifold satisfies $a + b = 0$.*

4. $N(k)$-quasi Einstein manifolds satisfying $R(\xi,X) \cdot S = 0$

First, we prove the following

Theorem 4.1. *An $N(k)$-quasi Einstein manifold (M^n, g) satisfies $R(\xi,X) \cdot S = 0$ if and only if $k = 0$.*

Proof. Let (M^n, g) be a $N(k)$-quasi Einstein manifold. The condition $R(\xi,X) \cdot S = 0$ gives

$$S(R(\xi,X)Y,\xi) + S(Y,R(\xi,X)\xi) = 0. \tag{4.1}$$

In view of (2.1) and (2.11), we get

$$S(R(\xi,X)Y,\xi) = (a+b)k(g(X,Y) - \eta(X)\eta(Y)). \tag{4.2}$$

In view of (2.9) and (2.1) we have

$$S(R(\xi,X)\xi,Y) = -kS(X,Y) + (a+b)k\eta(X)\eta(Y). \tag{4.3}$$

From (4.1), (4.2) and (4.3), we have

$$k\{S - (a+b)g\} = 0. \tag{4.4}$$

Therefore, either $k = 0$ or $S = (a+b)g$. In the second case, M^n becomes an Einstein space, which is not possible. Thus we have $k = 0$. Conversely, if $k = 0$ then in view of (2.8) M^n satisfies $R(\xi,X) \cdot S = 0$. □

As an application, we have the following

Theorem 4.2. *A conformally flat quasi Einstein manifold satisfies $R(\xi,X) \cdot S = 0$ if and only if $a + b = 0$.*

5. Ricci-recurrent quasi Einstein manifolds

A non-flat Riemannian manifold M is called a *Ricci-recurrent manifold* [6] if its Ricci tensor S satisfies the condition

$$(\nabla_X S)(Y,Z) = A(X)\,S(Y,Z), \tag{5.1}$$

where ∇ is Levi-Civita connection of the Riemannian metric g and A is a 1-form on M.

Now, we prove the following

Theorem 5.1. *If M is a Ricci-recurrent quasi Einstein manifold, then*

$$(a+b)\,A(X) = X(a+b), \qquad X \in TM. \tag{5.2}$$

Proof. Using (5.1) in

$$(\nabla_X S)(Y,Z) = XS(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z),$$

we get

$$A(X)\,S(Y,Z) = XS(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z).$$

Putting $Y = Z = \xi$, in the above equation we obtain

$$S(\xi,\xi)\,A(X) = XS(\xi,\xi) - 2S(\nabla_X \xi, \xi),$$

from which, in view of (2.1), we get (5.2). □

A Ricci-recurrent manifold is Ricci-symmetric if and only if the 1-form A is zero. Thus we have the following two corollaries:

Corollary 5.2. *If M is a Ricci-symmetric quasi Einstein manifold, then $a+b$ is constant.*

Corollary 5.3. *If M is a Ricci-recurrent quasi Einstein manifold and if $a+b$ is constant, then either $a+b=0$ or M reduces to a Ricci-symmetric quasi Einstein manifold.*

6. Quasi-umbilical hypersurfaces

Let M^n be a hypersurface of a Riemannian manifold $(\bar{M}^{n+1}, g)$. We now assume that M^n is orientable and choose a unit vector field ξ of $\bar{M}^{n+1}$ normal to M^n. Then Gauss and Weingarten formulae are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)\xi, \quad (X, Y \in TM^n), \tag{6.1}$$

$$\bar{\nabla}_X \xi = -HX, \tag{6.2}$$

where $\bar{\nabla}$ and ∇ are respectively the Riemannian and induced Riemannian connections in $\bar{M}^{n+1}$ and M^n and h is the second fundamental form related to H by

$$h(X,Y) = g(HX, Y). \tag{6.3}$$

M^n is called a *quasi-umbilical hypersurface* [3] if

$$h(X,Y) = \alpha\, g(X,Y) + \beta\, u(X)\, u(Y), \qquad X, Y \in TM^n, \tag{6.4}$$

where α and β are some smooth functions and u is a 1-form. A quasi-umbilical hypersurface becomes umbilical, geodesic or cylindrical according as $\beta = 0$, $\alpha = 0 = \beta$ or $\alpha = 0$.

If M^n is a quasi-umbilical hypersurface of a Riemannian manifold $(\bar{M}^{n+1}, g)$, then the Gauss equation becomes [4]

$$\begin{aligned} \bar{R}(X,Y,Z,W) &= R(X,Y,Z,W) + \alpha^2 \left(g(X,Z)\, g(Y,W) - g(X,W)\, g(Y,Z)\right) \\ &\quad + \alpha\beta \left(u(Y)\, u(W)\, g(X,Z) + u(X)\, u(Z)\, g(Y,W)\right. \\ &\quad \left. - u(Y)\, u(Z)\, g(X,W) - u(X)\, u(W)\, g(Y,Z)\right) \end{aligned} \tag{6.5}$$

for all $X, Y, Z, W \in TM^n$. From the above equation, we get

$$\begin{aligned} \bar{S}(X,Y) &= \bar{R}(\xi, X, Y, \xi) + S(X,Y) \\ &\quad - \left(n\alpha^2 + \alpha\beta\right) g(X,Y) - (n-1)\,\alpha\beta\, u(X)\, u(Y), \end{aligned} \tag{6.6}$$

where $\bar{S}$ and S are Ricci tensors of $\bar{M}^{n+1}$ and M^n respectively.

Now, we prove the following:

Theorem 6.1. *If M^n is a quasi-umbilical hypersurface of an $N(\bar{k})$-quasi Einstein manifold $(\bar{M}^{n+1}, g)$, such that M^n is normal to the generator ξ of $\bar{M}^{n+1}$, then M^n is an $N(k)$-quasi Einstein manifold.*

Proof. Using (2.8) in (6.6), we see that M^n is a quasi Einstein manifold. Moreover, using (2.7) in (6.5), we find that M^n is an $N(k)$-quasi Einstein manifold, where $k = \bar{k} + \alpha(\alpha + \beta)$. □

Remark 6.2. The above result is also true in the following cases **(a)** if M^n is umbilical, geodesic or cylindrical and/or **(b)** $\bar{M}^{n+1}$ is a conformally flat quasi Einstein manifold.

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