

AMLEs for Rayleigh Distribution Based on Progressive Type-II Censored Data

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Abstract

In this paper, we shall propose the AMLEs of the scale parameter and the location parameter in the two-parameter Rayleigh distribution based on progressive Type-II censored samples when one parameter is known.

We also propose the AMLEs of the two parameters in the Rayleigh distribution based on progressive Type-II censored samples when two parameters are unknown. We simulate the mean squared errors of the proposed estimators through Monte Carlo simulation for various censoring schemes.

Keywords: Approximate maximum likelihood estimator; location parameter; progressive Type-II censored sample; Rayleigh distribution; scale parameter.

1. Introduction

The probability density function (*pdf*) of the random variable X having the Rayleigh distribution is

$$f(x; \theta, \sigma) = \frac{(x - \theta)}{\sigma^2} \exp \left\{ -\frac{(x - \theta)^2}{2\sigma^2} \right\}, \quad x \geq \theta, \quad \sigma > 0 \quad (1.1)$$

and the cumulative distribution function (*cdf*) is

$$F(x; \theta, \sigma) = 1 - \exp \left\{ -\frac{(x - \theta)^2}{2\sigma^2} \right\}, \quad x \geq \theta, \quad \sigma > 0, \quad (1.2)$$

where θ and σ are the location and the scale parameters, respectively.

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The Rayleigh distribution is very useful in communication engineering and is a special case of a two-parameter Weibull distribution.

In most cases of censored and truncated samples, the explicit estimators might be not obtained by the maximum likelihood method. So we need another method for the purpose of providing the explicit estimators. The approximated maximum likelihood estimation method was first developed by Balakrishnan (1989) for the purpose of providing the explicit estimators of the scale parameter in the Rayleigh distribution. Kang (1996) obtained the AMLE for the scale parameter of the double exponential distribution based on Type-II censored samples. Kang and Lee (2005) and Kang and Park (2005) obtained the AMLEs for the exponential and exponentiated exponential distributions based on multiple Type-II censored samples. Han and Kang (2006) obtained the AMLEs of the scale parameter and the location parameter in the two-parameter Rayleigh distribution under multiply Type-II censoring.

The scheme of the progressive Type-II censoring has been suggested in the field of life-testing experiments. Its allowance for the removal of live units from the experiment at various stages in an attractive feature as it will potentially save a lot for the experimenter in terms of cost and time. Ng *et al.* (2002) discussed the estimation of parameters from progressively censored data using the EM algorithm. Balakrishnan *et al.* (2003) suggested point and interval estimation for Gaussian distribution based on progressive Type-II censored samples. Balakrishnan *et al.* (2004) studied inference for the extreme value distribution under progressive Type-II censoring. Lin *et al.* (2006) discussed the MLEs of the parameters of the log-gamma distribution based on progressively Type-II censored samples, and they derived the AMLEs of the parameters and used them as initial values in the determination of the MLEs through the Newton-Raphson method. Kim (2006) dealt with the problem of estimating parameters of Burr Type-XII distribution, on the basis of a general progressive Type-II censored samples using bayesian viewpoints.

In this paper, we shall propose the AMLEs of the scale parameter when the location parameter is known and the AMLE of the location parameter when the scale parameter is known in the two-parameter Rayleigh distribution based on progressive Type-II censored samples.

We also propose the AMLEs of the location and scale parameters in the two-parameter Rayleigh distribution based on progressive Type-II censored samples when two parameters are unknown. We simulate the mean squared errors of the proposed estimators through Monte Carlo simulation for various progressive

censoring schemes.

2. AMLEs of the Scale Parameter

Let us consider the following progressive Type-II censoring scheme; Suppose n randomly selected units with the Rayleigh distribution in (1.1) were placed on a life test. Further, suppose that the observation of failures begins at the time of the $(r + 1)^{th}$ failure (before which r units are known to have failed but their exact failure times are not known) and a progressive Type-II censored sample of size $m - r$ is observed from this life-testing experiment as follows: At the time of the $(r + 1)^{th}$ failure, R_{r+1} surviving units are randomly withdrawn from the test; at the time of the $(r + 2)^{th}$ failure, R_{r+2} surviving units are randomly withdrawn from the test, and so on; finally, at the time of the m^{th} failure, all the remaining $R_m = n - m - R_{r+1} - R_{r+2} - \dots - R_{m-1}$ are withdrawn from the test.

Then let

$$X_{r+1:n} \leq X_{r+2:n} \leq \dots \leq X_{m:n} \quad (2.1)$$

denote such a progressive Type-II censored sample with (R_{r+1}, \dots, R_m) being the progressive censoring scheme.

Note that the case $m = n$ and $r = 0$, in which case $R_{r+1} = \dots = R_m = 0$, corresponds to the complete sample situation, while the case $R_{r+1} = \dots = R_{m-1} = 0, R_m = n - m$ corresponds to the usual Type-II censored sample. Some historical remarks and good summary of the progressive censoring may be found in Balakrishnan and Aggarwala (2000).

The likelihood function based on the general progressive Type-II censored sample in (2.1) is given by

$$L = C[F(x_{r+1:n}; \theta, \sigma)]^r \prod_{i=r+1}^m f(x_{i:n}; \theta, \sigma)[1 - F(x_{i:n}; \theta, \sigma)]^{R_i}, \quad (2.2)$$

where

$$\begin{aligned} C &= \binom{n}{r} (n - r)(n - r - R_{r+1} - 1)(n - r - R_{r+1} - R_{r+2} - 2) \times \dots \\ &\quad \times (n - r - R_{r+1} - R_{r+2} - \dots - R_{m-1} - (m - r) + 1) \\ &= \binom{n}{r} (n - r) \prod_{j=r+2}^m \left(n - \sum_{i=r+1}^{j-1} R_i - j \right). \end{aligned}$$

By putting $Z_{i:n} = (X_{i:n} - \theta)/\sigma$, the likelihood function can be rewritten as

$$L = C\sigma^{-A}[F(z_{r+1:n})]^r \prod_{i=r+1}^m f(z_{i:n})[1 - F(z_{i:n})]^{R_i}, \quad (2.3)$$

where $A = m - r$ is the size of the censored sample (2.1), and $f(z) = ze^{-z^2/2}$, and $F(z) = 1 - e^{-z^2/2}$ are the pdf and the cdf of the standard Rayleigh distribution, respectively. The log-likelihood function is given by

$$\begin{aligned} \ln L &= \ln C - A \ln \sigma + r \ln[F(z_{r+1:n})] \\ &\quad + \sum_{i=r+1}^m \ln f(z_{i:n}) + \sum_{i=r+1}^m R_i \ln[1 - F(z_{i:n})] \end{aligned} \quad (2.4)$$

Now, we can derive the AMLEs of the scale parameter σ when the location parameter θ is known.

From the equation (2.4), on differentiating with respect to σ and the equation to zero, we obtain the estimating equation as

$$\begin{aligned} \frac{\partial \ln L}{\partial \sigma} &= -\frac{1}{\sigma} \left[2A + r \frac{f(z_{r+1:n})}{F(z_{r+1:n})} z_{r+1:n} - \sum_{i=r+1}^m z_{i:n}^2 - \sum_{i=r+1}^m R_i z_{i:n}^2 \right] \\ &= 0. \end{aligned} \quad (2.5)$$

Since the likelihood equations is very complicated, the equation (2.5) does not admit any explicit solution for σ , so we will expand the following functions in Taylor series around the point $a_{r+1} = F^{-1}(p_{r+1}) = (-2 \ln q_{r+1})^{1/2}$ by

$$\frac{f(z_{r+1:n})}{F(z_{r+1:n})} \simeq \alpha_{10} + \beta_{10} z_{r+1:n} \quad (2.6)$$

and

$$\frac{f(z_{r+1:n})}{F(z_{r+1:n})} z_{r+1} \simeq \alpha_{20} + \beta_{20} z_{r+1:n}, \quad (2.7)$$

where

$$\begin{aligned} p_i &= \frac{i}{n+1}, \quad q_i = 1 - p_i, \\ \alpha_{10} &= q_{r+1}(-2 \ln q_{r+1})^{3/2} / p_{r+1}^2, \\ \beta_{10} &= q_{r+1} \left(1 + \frac{2}{p_{r+1}} \ln q_{r+1} \right) / p_{r+1}, \end{aligned}$$

$$\alpha_{20} = \frac{(2q_{r+1} \ln q_{r+1})(p_{r+1} + 2 \ln q_{r+1})}{p_{r+1}^2},$$

$$\beta_{20} = \frac{2q_{r+1} \left(1 + \frac{\ln q_{r+1}}{p_{r+1}}\right) (-2 \ln q_{r+1})^{1/2}}{p_{r+1}}.$$

From the equations (2.6) and (2.7), the approximate likelihood equations for σ are given by

$$\begin{aligned} \frac{\partial \ln L}{\partial \sigma} &\simeq \frac{\partial \ln L^*}{\partial \sigma} \\ &= -\frac{1}{\sigma} \left[2A + r(\alpha_{10} + \beta_{10} z_{r+1:n}) z_{r+1:n} - \sum_{i=r+1}^m z_{i:n}^2 - \sum_{i=r+1}^m R_i z_{i:n}^2 \right] \\ &= 0 \end{aligned} \quad (2.8)$$

and

$$\begin{aligned} \frac{\partial \ln L}{\partial \sigma} &\simeq \frac{\partial \ln L^*}{\partial \sigma} \\ &= -\frac{1}{\sigma} \left[2A + r(\alpha_{20} + \beta_{20} z_{r+1:n}) - \sum_{i=r+1}^m z_{i:n}^2 - \sum_{i=r+1}^m R_i z_{i:n}^2 \right] \\ &= 0. \end{aligned} \quad (2.9)$$

From the equation (2.8), we can obtain a quadratic equation for σ

$$A_{10}\sigma^2 + B_{10}\sigma + C_{10} = 0, \quad (2.10)$$

where

$$\begin{aligned} A_{10} &= 2A, \\ B_{10} &= r\alpha_{10}X_{r+1:n} - r\alpha_{10}\theta, \\ C_{10} &= r\beta_{10}X_{r+1:n}^2 - 2r\beta_{10}X_{r+1:n}\theta + r\beta_{10}\theta^2 - \sum_{i=r+1}^m X_{i:n}^2 + 2\theta \sum_{i=r+1}^m X_{i:n} \\ &\quad - A\theta^2 - \sum_{i=r+1}^m R_i X_{i:n}^2 + 2\theta \sum_{i=r+1}^m R_i X_{i:n} - \theta^2 \sum_{i=r+1}^m R_i. \end{aligned}$$

Upon solving the equation (2.10) for σ , we derive an AMLE of σ as follows;

$$\hat{\sigma}_{10} = \frac{-B_{10} + \sqrt{B_{10}^2 - 4A_{10}C_{10}}}{2A_{10}}. \quad (2.11)$$

From the equation (2.9), we can obtain another quadratic equation for σ

$$A_{20}\sigma^2 + B_{20}\sigma + C_{20} = 0, \quad (2.12)$$

where

$$\begin{aligned} A_{20} &= 2A + r\alpha_{20}, \\ B_{20} &= r\beta_{20}X_{r+1:n} - r\beta_{20}\theta, \\ C_{20} &= -\sum_{i=r+1}^m X_{i:n}^2 + 2\theta \sum_{i=r+1}^m X_{i:n} - A\theta^2 - \sum_{i=r+1}^m R_i X_{i:n}^2 \\ &\quad + 2\theta \sum_{i=r+1}^m R_i X_{i:n} - \theta^2 \sum_{i=r+1}^m R_i. \end{aligned}$$

Upon solving the equation (2.12) for σ , we derive another AMLE of σ as follows;

$$\hat{\sigma}_{20} = \frac{-B_{20} + \sqrt{B_{20}^2 - 4A_{20}C_{20}}}{2A_{20}}. \quad (2.13)$$

3. AMLE of the Location Parameter

In this section, we derive an AMLE of the location parameter θ when the scale parameter σ is known.

From the equation (2.4), on differentiating with respect to θ and the equation to zero, we obtain the estimating equation as

$$\begin{aligned} \frac{\partial \ln L}{\partial \theta} &= -\frac{1}{\sigma} \left[r \frac{f(z_{r+1:n})}{F(z_{r+1:n})} + \sum_{i=r+1}^m \frac{1}{z_{i:n}} - \sum_{i=r+1}^m z_{i:n} - \sum_{i=r+1}^m R_i z_{i:n} \right] \\ &= 0. \end{aligned} \quad (3.1)$$

Since the likelihood equation is very complicated, the equation (3.1) does not admit any explicit solution for θ , so we will expand the following function in Taylor series around the point $a_i = F^{-1}(p_i) = (-2 \ln q_i)^{1/2}$ by

$$\frac{1}{z_{i:n}} \simeq \alpha_{1i} + \beta_{1i} z_{i:n}, \quad (3.2)$$

where

$$\begin{aligned}\alpha_{1i} &= \frac{2}{(-2 \ln q_i)^{1/2}}, \\ \beta_{1i} &= \frac{1}{2 \ln q_i}.\end{aligned}$$

From the equations (2.6) and (3.2), the approximate likelihood equation for θ is given by

$$\begin{aligned}\frac{\partial \ln L}{\partial \theta} &\simeq \frac{\partial \ln L^*}{\partial \theta} \\ &= -\frac{1}{\sigma} \left[r(\alpha_{10} + \beta_{10} z_{r+1:n}) + \sum_{i=r+1}^m (\alpha_{1i} + \beta_{1i} z_{i:n}) - \sum_{i=r+1}^m z_{i:n} - \sum_{i=r+1}^m R_i z_{i:n} \right] \\ &= 0.\end{aligned}\tag{3.3}$$

Upon solving the equation (3.3) for θ , we derive an AMLE of θ as follows;

$$\hat{\theta}_{01} = B_{01}\sigma + C_{01},\tag{3.4}$$

where

$$\begin{aligned}B_{01} &= \left(-r\alpha_{10} - \sum_{i=r+1}^m \alpha_{1i} \right) / D_1, \\ C_{01} &= \left(-r\beta_{10} X_{r+1:n} - \sum_{i=r+1}^m \beta_{1i} X_{r+1:n} + \sum_{i=r+1}^m X_{i:n} + \sum_{i=r+1}^m R_i X_{i:n} \right) / D_1, \\ D_1 &= A - r\beta_{10} + \sum_{i=r+1}^m R_i - \sum_{i=r+1}^m \beta_{1i}.\end{aligned}$$

4. AMLEs of the Two Parameters

Now, we consider the Rayleigh distribution with the density function (2.1) when two parameters are unknown.

Now making use of the approximate expressions in (2.6) and (3.2), we may the approximate likelihood equations of (3.1) and (2.5) as follows;

$$\frac{\partial \ln L}{\partial \theta} \simeq \frac{\partial \ln L^*}{\partial \theta}\tag{4.1}$$

$$\begin{aligned}
&= -\frac{1}{\sigma} \left[r(\alpha_{10} + \beta_{10} z_{r+1:n}) + \sum_{i=r+1}^m (\alpha_{1i} + \beta_{1i} z_{i:n}) - \sum_{i=r+1}^m z_{i:n} - \sum_{i=r+1}^m R_i z_{i:n} \right] \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \ln L}{\partial \sigma} &\simeq \frac{\partial \ln L^*}{\partial \sigma} \\
&= -\frac{1}{\sigma} \left[2A + r(\alpha_{10} + \beta_{10} z_{r+1:n}) z_{r+1:n} - \sum_{i=r+1}^m z_{i:n}^2 - \sum_{i=r+1}^m R_i z_{i:n}^2 \right] \quad (4.2) \\
&= 0.
\end{aligned}$$

Upon solving the equation (4.1) for θ , we derive an AMLE of θ as follows;

$$\hat{\theta}_{31} = B_{01} \hat{\sigma}_{31} + C_{01}. \quad (4.3)$$

From the equation (4.2), we can obtain a quadratic equation for σ as

$$A_{31}\sigma^2 + B_{31}\sigma + C_{31} = 0, \quad (4.4)$$

where

$$\begin{aligned}
A_{31} &= 2A - r\alpha_{10}\beta_{10} + r\beta_{10}B_{01}^2 + AB_{01}^2 + B_{01}^2 \sum_{i=r+1}^m R_i, \\
B_{31} &= r\alpha_{10}X_{r+1:n} - r\alpha_{10}C_{01} - 2r\beta_{10}X_{r+1:n}B_{01} + 2r\beta_{10}B_{01}C_{01} - 2B_{01} \sum_{i=r+1}^m X_{i:n} \\
&\quad + 2AB_{01}C_{01} - 2B_{01} \sum_{i=r+1}^m R_i X_{i:n} + 2B_{01}C_{01} \sum_{i=r+1}^m R_i, \\
C_{31} &= r\beta_{10}X_{r+1:n}^2 - 2r\beta_{10}X_{r+1:n}C_{01} + r\beta_{10}C_{01}^2 + \sum_{i=r+1}^m X_{i:n}^2 - 2C_{01} \sum_{i=r+1}^m X_{i:n} \\
&\quad + AC_{01}^2 + \sum_{i=r+1}^m R_i X_{i:n}^2 - 2C_{01} \sum_{i=r+1}^m R_i X_{i:n} + C_{01}^2 \sum_{i=r+1}^m R_i.
\end{aligned}$$

Upon solving the equation (4.4) for σ , we derive the AMLE of σ

$$\hat{\sigma}_{31} = \frac{-B_{31} + \sqrt{B_{31}^2 - 4A_{31}C_{31}}}{2A_{31}}. \quad (4.5)$$

Now making use of the approximate expressions in (2.6), (2.7), and (3.2), we may approximate the likelihood equations (3.1) and (2.5) as follows;

$$\begin{aligned} \frac{\partial \ln L}{\partial \theta} &\sim \frac{\partial \ln L^*}{\partial \theta} \\ &= -\frac{1}{\sigma} \left[r(\alpha_{10} + \beta_{10} z_{r+1:n}) + \sum_{i=r+1}^m (\alpha_{1i} + \beta_{1i} z_{i:n}) - \sum_{i=r+1}^m z_{i:n} - \sum_{i=r+1}^m R_i z_{i:n} \right] \\ &= 0. \end{aligned} \quad (4.6)$$

and

$$\begin{aligned} \frac{\partial \ln L}{\partial \sigma} &\sim \frac{\partial \ln L^*}{\partial \sigma} \\ &= -\frac{1}{\sigma} \left[2A + r(\alpha_{20} + \beta_{20} z_{r+1:n}) - \sum_{i=r+1}^m z_{i:n}^2 - \sum_{i=r+1}^m R_i z_{i:n}^2 \right] \\ &= 0. \end{aligned} \quad (4.7)$$

Upon solving the equation (4.6) for θ , we derive another AMLE of θ as follows;

$$\hat{\theta}_{32} = B_{01} \hat{\sigma}_{32} + C_{01}. \quad (4.8)$$

From the equation (4.7), we can obtain another quadratic equation for σ as follows;

$$A_{32}\sigma^2 + B_{32}\sigma + C_{32} = 0, \quad (4.9)$$

where

$$\begin{aligned} A_{32} &= 2A + r\alpha_{20} - r\beta_{20}B_{01} - AB_{01}^2 - B_{01}^2 \sum_{i=r+1}^m R_i, \\ B_{32} &= r\beta_{20}X_{r+1:n} - r\beta_{20}C_{01} + 2B_{01} \sum_{i=r+1}^m X_{i:n} - 2AB_{01}C_{01} \\ &\quad + 2B_{01} \sum_{i=r+1}^m R_i X_{i:n} - 2B_{01}C_{01} \sum_{i=r+1}^m R_i, \\ C_{32} &= - \sum_{i=r+1}^m X_{i:n}^2 + 2C_{01} \sum_{i=r+1}^m X_{i:n} - AC_{01}^2 - \sum_{i=r+1}^m R_i X_{i:n}^2 \\ &\quad + 2C_{01} \sum_{i=r+1}^m R_i X_{i:n} - C_{01}^2 \sum_{i=r+1}^m R_i. \end{aligned}$$

Upon solving the equation (4.9) for σ , we derive another AMLE of σ as follows;

$$\hat{\sigma}_{32} = \frac{-B_{32} + \sqrt{B_{32}^2 - 4A_{32}C_{32}}}{2A_{32}}. \quad (4.10)$$

5. The Simulated Result

In order to evaluate the performance of the proposed AMLEs, the mean squared errors of all proposed estimators are simulated by Monte Carlo method. The simulations are carried out for sample sizes $n = 10(10)50$, different choices of the number of the left censored data r , different choices of the effective sample size $m - r$, and different progressive censoring schemes with the complete sample in each case. These values are given in Table 5.1 and Table 5.2. For simplicity in notation, we will denote the schemes $(0, 0, \dots, 0, n-m)$ by $((m-r-1)*0, n-m)$, for example, $(10 * 0)$ denotes the progressive censoring scheme $(0, 0, \dots, 0)$ and $(4 * 0, 3, 2 * 0)$ denotes the progressive censoring scheme $(0, 0, 0, 0, 3, 0, 0)$.

From Table 5.1 and 5.2, we have the following results;

- (i) The estimator $\hat{\sigma}_{10}$ is more efficient than $\hat{\sigma}_{20}$ in the sense of MSE when the location parameter θ is known. But $\hat{\sigma}_{20}$ is a little more efficient than $\hat{\sigma}_{10}$ when R_i is large for small value of i ($r+1 \leq i \leq m$).
- (ii) The estimator $\hat{\theta}_{31}$ is generally more efficient than $\hat{\theta}_{32}$ in the sense of MSE when two parameters are unknown.
- (iii) When n is small ($n = 10$) and large ($n = 50$), the estimator $\hat{\sigma}_{32}$ is generally more efficient than $\hat{\sigma}_{31}$, but when n is moderate ($n = 20, 30, 40$), the estimator $\hat{\sigma}_{31}$ is generally more efficient than $\hat{\sigma}_{32}$.
- (iv) The MSEs of all estimators decrease as sample size n increases.
- (v) For fixed n and r , the MSEs increase as the effective sample size $m - r$ decreases.
- (vi) For fixed n and r , the MSEs increase as R_i is large for large value of i even when the effective sample size $m - r$ is same.
- (vii) For fixed n and r , the MSEs increase significantly as the effective sample proportion $(m-r)/n$ decreases.

Table 5.1: The relative mean squared errors of the proposed estimators when one parameter is known.

n	r	$m - r$	scheme	σ is known	θ is known	
				$\hat{\theta}_{01}$	$\hat{\sigma}_{10}$	$\hat{\sigma}_{20}$
10	0	10	(10*0)	0.040708	0.024885	0.024885
		7	(3*0,3,3*0)	0.054151	0.036066	0.036066
		7	(4*0,3,2*0)	0.054440	0.039894	0.039894
		7	(6*0,3)	0.067459	0.071699	0.071699
		6	(1*0,2,2*0,2,1*0)	0.064884	0.043019	0.043019
		6	(1*0,2,1*0,2,2*0)	0.066184	0.040681	0.040681
		5	(2*0,5,2*0)	0.086436	0.051438	0.051438
		5	(4*0,5)	0.121196	0.119260	0.119260
	1	7	(2*0,1,2*0,1,1*0)	0.046185	0.031986	0.032546
		6	(3*0,3,2*0)	0.051512	0.036287	0.036619
		6	(5*0,3)	0.063168	0.067774	0.071519
		5	(1*0,2,2*0,2)	0.058796	0.052288	0.055018
	2	6	(2*0,1,1*0,1,1*0)	0.049991	0.032316	0.033307
		5	(3*0,3,1*0)	0.058321	0.038359	0.038994
		5	(4*0,3)	0.068084	0.067382	0.073092
		4	(1*0,2,1*0,2)	0.062495	0.054534	0.059578
	3	5	(1*0,1,1*0,1,1*0)	0.057546	0.033222	0.034300
		4	(1*0,2,1*0,1)	0.062444	0.043433	0.046792
		4	(3*0,3)	0.079014	0.065503	0.073013
20	0	20	(20*0)	0.018373	0.012069	0.012069
		17	(4*0,3,12*0)	0.020827	0.014134	0.014134
		17	(3*0,3,13*0)	0.021042	0.014071	0.014071
		17	(16*0,3)	0.023253	0.034068	0.034068
		16	(1*0,2,2*0,2,11*0)	0.023261	0.014861	0.014861
		16	(1*0,2,1*0,2,12*0)	0.023708	0.014836	0.014836
		15	(2*0,5,12*0)	0.024468	0.015750	0.015750
		15	(14*0,5)	0.029469	0.060681	0.060681
	1	16	(3*0,3,12*0)	0.019533	0.014084	0.014040
		16	(15*0,3)	0.021305	0.032982	0.033987
		15	(1*0,2,2*0,2,10*0)	0.020788	0.015019	0.015029
		11	(3*0,5,1*0,3,5*0)	0.023275	0.022759	0.023397
	2	15	(3*0,3,11*0)	0.020303	0.013942	0.013881
		15	(14*0,3)	0.022244	0.032711	0.034088
		14	(1*0,2,1*0,2,10*0)	0.021503	0.014892	0.014900
		10	(5*0,5,3*0,3)	0.024675	0.056530	0.060546
	3	15	(1*0,1,1*0,1,11*0)	0.021116	0.013380	0.013355
		14	(3*0,3,10*0)	0.021566	0.014155	0.014094
		14	(13*0,3)	0.023479	0.032371	0.033992
		10	(6*0,5,1*0,2,1*0)	0.024078	0.043655	0.047057

(Continued)

n	r	$m - r$ 0	scheme	σ is known	θ is known	
				$\hat{\theta}_{01}$	$\hat{\sigma}_{10}$	$\hat{\sigma}_{20}$
30	0	30	(30*0)	0.011758	0.008188	0.008188
		27	(4*0,3,22*0)	0.012895	0.008916	0.008916
		27	(3*0,3,23*0)	0.013119	0.009009	0.009009
		27	(26*0,3)	0.013746	0.021258	0.021258
		26	(1*0,2,2*0,2,21*0)	0.014104	0.009500	0.009500
		26	(1*0,2,1*0,2,22*0)	0.014205	0.009424	0.009424
		25	(2*0,5,22*0)	0.014310	0.009623	0.009623
		25	(24*0,5)	0.015968	0.037570	0.037570
	1	26	(3*0,3,22*0)	0.012142	0.009055	0.009015
		26	(25*0,3)	0.012748	0.020757	0.021249
		25	(1*0,2,2*0,2,20*0)	0.012504	0.009223	0.009187
		16	(3*0,10,3*0,3,8*0)	0.015648	0.015835	0.016172
40	2	25	(3*0,3,21*0)	0.012428	0.009061	0.009007
		25	(24*0,3)	0.013023	0.020506	0.021157
		24	(1*0,2,1*0,2,20*0)	0.012835	0.009290	0.009248
		15	(5*0,10,3*0,3,5*0)	0.015114	0.024180	0.025344
	3	25	(1*0,1,1*0,1,21*0)	0.012905	0.008774	0.008739
		24	(3*0,3,20*0)	0.012995	0.009115	0.009053
		24	(23*0,3)	0.013629	0.020517	0.021253
		15	(10*0,10,1*0,2,2*0)	0.015001	0.050562	0.053954
	0	40	(40*0)	0.008437	0.006067	0.006067
		37	(36*0,3)	0.009511	0.015044	0.015044
		35	(20*0,5,14*0)	0.008983	0.008433	0.008433
		35	(34*0,5)	0.010609	0.026461	0.026461
		33	(1*0,2,12*0,5,18*0)	0.009902	0.007528	0.007528
		30	(10*0,10,19*0)	0.010020	0.008890	0.008890
		30	(3*0,10,26*0)	0.011215	0.008085	0.008085
		26	(1*0,2,1*0,12,22*0)	0.013549	0.009123	0.009123
40	1	36	(35*0,3)	0.008865	0.014736	0.015028
		29	(3*0,10,25*0)	0.009992	0.008131	0.008092
		26	(3*0,10,3*0,3,18*0)	0.010585	0.009047	0.009084
		22	(1*0,15,2*0,2,17*0)	0.013087	0.010494	0.010462
	2	35	(34*0,3)	0.008987	0.014617	0.015009
		28	(3*0,10,24*0)	0.009860	0.007979	0.007919
		21	(1*0,2,1*0,15,17*0)	0.012016	0.010637	0.010720
		18	(5*0,10,3*0,10,8*0)	0.011478	0.023307	0.024390
	3	34	(33*0,3)	0.009346	0.014614	0.015054
		27	(3*0,10,23*0)	0.010159	0.008035	0.007969
		26	(1*0,1,10*0,10,13*0)	0.009881	0.012015	0.012409
		15	(10*0,20,1*0,2,2*0)	0.013086	0.079492	0.086303

(Continued)

n	r	m - r	scheme	σ is known	θ is known	
				$\hat{\theta}_{01}$	$\hat{\sigma}_{10}$	$\hat{\sigma}_{20}$
50	0	50	(50*0)	0.006573	0.005008	0.005008
		47	(46*0,3)	0.007211	0.011611	0.011611
		45	(44*0,5)	0.007944	0.019979	0.019979
		40	(20*0,10,19*0)	0.007296	0.008675	0.008675
		40	(10*0,10,29*0)	0.007639	0.006318	0.006318
		36	(1*0,2,10*0,12,23*0)	0.008337	0.007379	0.007379
		35	(20*0,15,14*0)	0.007744	0.016292	0.016292
	1	35	(10*0,10,12*0,5,11*0)	0.008096	0.011441	0.011441
		46	(45*0,3)	0.006741	0.011404	0.011602
		29	(3*0,10,3*0,10,21*0)	0.008625	0.008082	0.008127
		29	(3*0,20,25*0)	0.008893	0.008131	0.008081
	2	29	(1*0,15,10*0,5,16*0)	0.009300	0.008010	0.008070
		45	(44*0,3)	0.006884	0.011336	0.011606
		28	(3*0,20,24*0)	0.008875	0.008272	0.008204
		23	(1*0,10,5*0,15,15*0)	0.009596	0.011460	0.011773
	3	18	(5*0,20,3*0,10,8*0)	0.009776	0.027317	0.028704
		44	(43*0,3)	0.007169	0.011318	0.011623
		27	(3*0,20,23*0)	0.008921	0.008220	0.008157
		22	(1*0,15,10*0,10,9*0)	0.009596	0.018102	0.018970
	10	25	(10*0,10,5*0,5,8*0)	0.009564	0.020599	0.021924

Table 5.2: The relative mean squared errors for the proposed estimators of the location parameter θ

n	r	m - r	scheme	$\hat{\theta}_{31}$	$\hat{\theta}_{32}$	$\hat{\sigma}_{31}$	$\hat{\sigma}_{32}$
10	0	10	(10*0)	0.105033	0.105033	0.065746	0.065746
		7	(4*0,3,2*0)	0.097573	0.097573	0.088423	0.088423
		7	(3*0,3,3*0)	0.107290	0.107290	0.081942	0.081942
		7	(6*0,3)	0.144005	0.144005	0.111724	0.111724
		6	(1*0,2,2*0,2,1*0)	0.123292	0.123292	0.090298	0.090298
		6	(1*0,2,1*0,2,2*0)	0.128139	0.128139	0.088931	0.088931
		5	(2*0,5,2*0)	0.121221	0.121221	0.121596	0.121596
		5	(4*0,5)	0.141424	0.141424	0.198673	0.198673
	1	7	(2*0,1,2*0,1,1*0)	0.115220	0.117775	0.082333	0.083145
		6	(3*0,3,2*0)	0.118747	0.120124	0.114166	0.112880
		6	(5*0,3)	0.137676	0.143509	0.139016	0.138893
		5	(1*0,2,2*0,2)	0.126605	0.129248	0.136653	0.133918
2	2	6	(2*0,1,1*0,1,1*0)	0.153620	0.155416	0.101960	0.101900
		5	(4*0,3)	0.196791	0.197751	0.208252	0.202139
		5	(3*0,3,1*0)	0.204868	0.203066	0.220803	0.216460
		4	(1*0,2,1*0,2)	0.188641	0.187207	0.198249	0.189707
3	5	5	(1*0,1,1*0,1,1*0)	0.219256	0.220548	0.129943	0.129352
		4	(1*0,2,1*0,1)	0.258604	0.256187	0.193283	0.188174
		4	(3*0,3)	0.746957	0.738515	0.795667	0.786330

(Continued)

n	r	$m - r$	scheme	$\hat{\theta}_{31}$	$\hat{\theta}_{32}$	$\hat{\sigma}_{31}$	$\hat{\sigma}_{32}$
20	0	20	(20*0)	0.042202	0.042202	0.028173	0.028173
		17	(4*0,3,12*0)	0.043201	0.043201	0.030306	0.030306
		17	(3*0,3,13*0)	0.043436	0.043436	0.030081	0.030081
		17	(16*0,3)	0.069228	0.069228	0.051024	0.051024
		16	(1*0,2,2*0,2,11*0)	0.047236	0.047236	0.031606	0.031606
		16	(1*0,2,1*0,2,12*0)	0.047994	0.047994	0.031429	0.031429
		15	(2*0,5,12*0)	0.045647	0.045647	0.030864	0.030864
		15	(14*0,5)	0.075597	0.075597	0.058962	0.058962
	1	16	(3*0,3,12*0)	0.040641	0.041425	0.030023	0.030443
		16	(15*0,3)	0.057755	0.060394	0.043965	0.046093
		15	(1*0,2,2*0,2,10*0)	0.042184	0.043121	0.031657	0.032134
		11	(3*0,5,1*0,3,5*0)	0.042321	0.043188	0.042534	0.042448
	2	15	(3*0,3,11*0)	0.047595	0.048208	0.033532	0.033788
		15	(14*0,3)	0.062363	0.064741	0.045510	0.047161
		14	(1*0,2,1*0,2,10*0)	0.049034	0.049779	0.035487	0.035722
		10	(5*0,5,3*0,3)	0.069802	0.069623	0.115066	0.114072
	3	15	(1*0,1,1*0,1,11*0)	0.056104	0.056658	0.035996	0.036233
		14	(3*0,3,10*0)	0.056326	0.056823	0.037926	0.038083
		14	(13*0,3)	0.070302	0.072350	0.049411	0.050592
		10	(6*0,5,1*0,2,1*0)	0.075916	0.075672	0.088951	0.088121
30	0	30	(30*0)	0.025517	0.025517	0.017983	0.017983
		27	(4*0,3,22*0)	0.025741	0.025741	0.018186	0.018186
		27	(3*0,3,23*0)	0.025796	0.025796	0.018079	0.018079
		27	(26*0,3)	0.043960	0.043960	0.034419	0.034419
		26	(1*0,2,2*0,2,21*0)	0.027899	0.027899	0.019247	0.019247
		26	(1*0,2,1*0,2,22*0)	0.028323	0.028323	0.019251	0.019251
		25	(2*0,5,22*0)	0.026350	0.026350	0.018277	0.018277
		25	(24*0,5)	0.051810	0.051810	0.041714	0.041714
	1	27	(3*0,3,22*0)	0.024082	0.024421	0.018149	0.018354
		27	(25*0,3)	0.037039	0.038341	0.029109	0.030353
		25	(1*0,2,2*0,2,20*0)	0.024816	0.025200	0.018603	0.018831
		16	(3*0,10,3*0,3,8*0)	0.024946	0.025311	0.028200	0.028017
	2	25	(3*0,3,21*0)	0.026508	0.026773	0.019505	0.019644
		25	(24*0,3)	0.038193	0.039434	0.028489	0.029581
		24	(1*0,2,1*0,2,20*0)	0.027284	0.027603	0.020075	0.020232
		15	(5*0,10,3*0,3,5*0)	0.028736	0.028767	0.040541	0.040043
	3	25	(1*0,1,1*0,1,21*0)	0.029714	0.029950	0.020298	0.020420
		24	(3*0,3,20*0)	0.029613	0.029822	0.020964	0.021048
		24	(23*0,3)	0.040750	0.041910	0.029404	0.030342
		15	(10*0,10,1*0,2,2*0)	0.046452	0.046303	0.074646	0.074208

(Continued)

n	r	$m - r$	scheme	$\hat{\theta}_{31}$	$\hat{\theta}_{32}$	$\hat{\sigma}_{31}$	$\hat{\sigma}_{32}$
40	0	40	(40*0)	0.017533	0.017533	0.012735	0.012735
		37	(36*0,3)	0.030644	0.030644	0.025053	0.025053
		35	(20*0,5,14*0)	0.015783	0.015783	0.012676	0.012676
		35	(34*0,5)	0.037457	0.037457	0.031852	0.031852
		33	(1*0,2,12*0,5,18*0)	0.018591	0.018591	0.013865	0.013865
		30	(10*0,10,19*0)	0.017439	0.017439	0.014634	0.014634
		30	(3*0,10,26*0)	0.018320	0.018320	0.013864	0.013864
		26	(1*0,2,1*0,12,22*0)	0.023752	0.023752	0.016788	0.016788
	1	36	(35*0,3)	0.025685	0.026463	0.021100	0.021925
		29	(3*0,10,25*0)	0.016596	0.016821	0.014325	0.014407
		26	(3*0,10,3*0,3,18*0)	0.017231	0.017485	0.016074	0.016140
		22	(1*0,15,2*0,2,17*0)	0.019645	0.020141	0.017903	0.017990
	2	35	(34*0,3)	0.026482	0.027232	0.020746	0.021492
		28	(3*0,10,24*0)	0.018134	0.018302	0.015691	0.015711
		21	(1*0,2,1*0,15,17*0)	0.020483	0.020835	0.020153	0.020135
		18	(5*0,10,3*0,10,8*0)	0.019341	0.019467	0.028080	0.027758
	3	34	(33*0,3)	0.027904	0.028623	0.021011	0.021675
		27	(3*0,10,23*0)	0.019921	0.020055	0.016885	0.016868
		26	(1*0,1,10*0,10,13*0)	0.020619	0.020777	0.018631	0.018660
		15	(10*0,20,1*0,2,2*0)	0.057516	0.057269	0.161405	0.160642
50	0	50	(50*0)	0.013167	0.013167	0.010092	0.010092
		47	(46*0,3)	0.023109	0.023109	0.019848	0.019848
		45	(44*0,5)	0.028600	0.028600	0.025543	0.025543
		40	(20*0,10,19*0)	0.011836	0.011836	0.010681	0.010681
		40	(10*0,10,29*0)	0.013169	0.013169	0.010965	0.010965
		36	(1*0,2,10*0,12,23*0)	0.013826	0.013826	0.011733	0.011733
		35	(20*0,15,14*0)	0.011045	0.011045	0.013131	0.013131
		35	(10*0,10,12*0,5,11*0)	0.011305	0.011305	0.012158	0.012158
	1	46	(45*0,3)	0.019372	0.019887	0.016928	0.017500
		29	(3*0,10,3*0,10,21*0)	0.012722	0.012897	0.013654	0.013633
		29	(3*0,20,25*0)	0.013016	0.013209	0.013829	0.013814
		29	(1*0,15,10*0,5,16*0)	0.013051	0.013250	0.014079	0.014017
	2	45	(44*0,3)	0.019871	0.020375	0.016647	0.017180
		28	(3*0,20,24*0)	0.013900	0.014044	0.015383	0.015291
		23	(1*0,10,5*0,15,15*0)	0.014591	0.014767	0.019428	0.019218
		18	(5*0,20,3*0,10,8*0)	0.014961	0.014988	0.034315	0.033725
	3	44	(43*0,3)	0.020864	0.021344	0.016705	0.017186
		27	(3*0,20,23*0)	0.015328	0.015433	0.016797	0.016678
		22	(1*0,15,10*0,10,9*0)	0.016396	0.016449	0.025234	0.024899
	10	25	(10*0,10,5*0,5,8*0)	0.030977	0.030922	0.026947	0.026840

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