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CHARACTERIZATION OF SOME CONTINUOUS DISTRIBUTIONS BY PROPERTIES OF PARTIAL MOMENTS

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ABSTRACT

In this paper we present characterizations of the Pareto, Lomax, exponential and beta models by some properties of their r^{th} partial moment defined as $\alpha_r(t) = E[(X-t)^+]^r$, where $(X-t)^+ = \max(X-t, 0)$. Given the partial moments at a few truncation points, these results enable us to calculate the moments at many other points.

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1. INTRODUCTION

Consider a positive random variable X with absolutely continuous distribution function $F(x)$ and finite moment of order r. Then the r^{th} partial moment of X about a point t is defined as

$$\alpha_r(t) = E[\{(X-t)^+\}^r], \quad r = 0, 1, 2, \ldots, \tag{1.1}$$

where $(X-t)^+ = \max(X-t, 0)$. The quantity $(X-t)^+$ is interpreted as residual life in the context of life length studies (Lin, 2003) and the moments (1.1) are used in actuarial sciences in the analysis of risks (Denuit, 2002). In the assessment of income tax, t can be taken as the tax exemption level so that $(X-t)^+$ becomes the taxable income. Gupta and Gupta (1983) have shown that $\alpha_r(t)$ determine the

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underlying distribution uniquely for any positive real r and therefore information about partial moments enables model identification.

Like other types of moments, properties of $\alpha_r(t)$ can be used to characterize probability distributions. Chong (1977) established that

$$\alpha_1(t+s)\alpha_1(0) = \alpha_1(t)\alpha_1(s) \tag{1.2}$$

is a characteristic property of the exponential law. This result was later extended by Nair (1987) to the bivariate case. Discrete distributions like geometric, negative hypergeometric and Waring were characterized by Nair *et al.* (2000). The results of Chong (1977) were further strengthened and several new unique properties of the exponential law were found by Lin (2003). In the present paper we attempt to supplement the results of Lin (2003) by providing similar characterizations for the Pareto, Lomax, exponential and beta models. Unlike the survival functions of these distributions which can be deduced from one another through monotone transformations, the partial moments do not permit such deductions and therefore separate treatments are required for each distribution.

2. CHARACTERIZATIONS

In this section we present characterizations of several continuous distributions and deduce many known results as special cases. For the random variable defined in Section 1

$$\alpha_r(t) = \int_t^\infty (x-t)^r \, dF \tag{2.1}$$

is assumed to be continuous in t, through out the support of X in this paper.

2.1. Pareto distribution

THEOREM 2.1. (a) *X follows Pareto distribution with $F(x) = 1-(k/x)^a$, $x \ge k > 0$, where k, a are constants and $a > r$ for some positive integer r, then*

$$\alpha_r(t)\alpha_r(s) = \alpha_r(1)\alpha_r(ts) \quad \textit{for all } t,s > 1. \tag{2.2}$$

(b) *Conversely, if X satisfies* (2.2) *for some positive integer r, then X follows a Pareto distribution of the form given in* (a).

PROOF. For the Pareto distribution mentioned previously

$$\alpha_r(t) = \frac{r!\, k^a t^{r-a}}{(a-1)^{(r)}},$$

where $y^{(r)} = y(y-1)\cdots(y-r+1)$ is the descending factorial. Thus (2.2) is satisfied. Conversely, when (2.2) holds we have

$$G(r,t)G(r,s) = G(r,ts) \tag{2.3}$$

with $G(a,b) = \alpha_a(b)/\alpha_a(1)$. The only non-trivial solution of (2.3) is (Aczel, 1966, p. 39) of the form

$$G(r,t) = t^{\beta(r)} \text{ or } \alpha_r(t) = \alpha_r(1)t^{\beta(r)} \tag{2.4}$$

for some $\beta(r)$. Since at all continuity points of $F(x)$,

$$\alpha_r'(t) = -r\alpha_{r-1}(t)$$

with prime denoting differentiation, we have from (2.4)

$$\alpha_r(1)\beta(r)t^{\beta(r)-1} = -r\alpha_{r-1}(1)t^{\beta(r-1)}, \quad r \geq 1. \tag{2.5}$$

In order that (2.5) holds for all $t > 0$ the exponent of t via

$$\beta(r) - 1 - \beta(r-1), \quad r \geq 1$$

must be zero, otherwise the equation holds only for a fixed number of t-values obtained as the roots of (2.5) which is contradictory to the assumptions of the theorem. Hence

$$\beta(r) - 1 = \beta(r-1),$$

giving the solution by successive reduction as

$$\beta(r) = r - a, \quad a = -\beta_0.$$

Further

$$\alpha_r(t) = \alpha_r(1)t^{r-a}. \tag{2.6}$$

Now, from equation (2.1),

$$\begin{aligned} 1 - F(x) &= \frac{(-1)^r}{r!}\frac{d^r\alpha_r(x)}{dx^r} \\ &= \frac{(a-1)^{(r)}}{r!}\alpha_r(1)x^{-a}. \end{aligned}$$

Writing $k = \sup[x|F(x) \geq 0]$, the constant $\alpha_r(1)$ is evaluated at $F(k) = 0$ as

$$\alpha_r(1) = \frac{r!}{(a-1)^{(r)}}k^a$$

and thus

$$F(x) = 1 - \left(\frac{x}{k}\right)^{-a}, \quad a > 0.$$

This completes the proof. □

REMARK 2.1.

1. Pareto model is characterized by the property of expectations

$$E[(X-t)^+]E[(X-s)^+] = E[(X-1)^+]E[(X-ts)^+]. \tag{2.7}$$

2. The property (2.2) implies that for all $t_i \geq 1, i = 1, 2, \ldots, n$,

$$\alpha_r(t_1)\alpha_r(t_2)\cdots\alpha_r(t_n) = [\alpha_r(1)]^{n-1}\alpha_r(t_1 t_2 \cdots t_n). \tag{2.8}$$

Formulae (2.2), (2.7) and (2.8) gives a method of generating an infinite number of partial moments given the moments at more than one point of truncation. However, when the moment at one truncation point alone is known, we can use

$$[\alpha_r(t)]^n = [\alpha_r(1)]^{n-1}\alpha_r(t^n) \tag{2.9}$$

obtained by setting all $t_i = t$ in (2.8).

We will now show that (2.9) is also a characteristic property of the Pareto law.

THEOREM 2.2. *The relationship*

$$[\alpha_r(t)]^n = C\alpha_r(t^n) \tag{2.10}$$

holds for all $t \geq 1$ and all positive integers n, where C is independent of t if and only if X has Pareto distribution.

PROOF. When (2.10) holds for all $t \geq 1$, setting $t = 1$ gives $C = (\alpha_r(1))^{n-1}$, which shows that C is independent of t. Hence (2.10) is equivalent to

$$A_r^n(t) = A_r(t^n), \tag{2.11}$$

where $A_r(t) = \alpha_r(t)/\alpha_r(1)$. Setting $t = e^u$, $A_r(e^u) = g_r(u)$ will lead to the functional equation

$$g_r^n(u) = g_r(un) \text{ for all } u \text{ and positive integers } n.$$

From Galambos and Kotz (1978), the solution of the last equation is $g_r(u) = e^{bu}$, and hence

$$A_r(t) = t^{b_r} \quad \text{for all real } t \geq 1. \tag{2.12}$$

That (2.12) leads to the Pareto distribution is shown in Theorem 2.1. The converse follows from the expression for $\alpha_r(t)$ and the proof is complete. □

COROLLARY 2.1. *The partial mean satisfies the condition*

$$\alpha_1^n(t) = C\alpha_1(t^n)$$

for all $t \geq 1$ *and positive integers* n, *if and only if* X *follows Pareto distribution.*

2.2. Lomax distribution

The Lomax distribution is specified by the distribution function

$$F(x) = 1 - \sigma^c(x+\sigma)^{-c}, \quad \sigma, c > 0, \tag{2.13}$$

for $x > 0$.
The distribution is important in reliability analysis as one with linearly increasing (decreasing) failure rate (mean residual life). Its partial moments are given by

$$\alpha_r(t) = \frac{r!\sigma^r\left(1+\dfrac{t}{\sigma}\right)^{r-c}}{(c-1)^{(r)}}, \quad c > r. \tag{2.14}$$

We now prove the following theorems.

THEOREM 2.3. *Let* X *be a continuous random variable in* $(0, \infty)$ *with finite moments of order* r *and the partial moments* $\alpha_r(t)$ *are continuous in the support of* X. *Then for any* $r = 1, 2, \ldots,$

$$\alpha_r(t)\alpha_r(s) = C_r\, \alpha_r\left(t + s + \sigma^{-1}ts\right), \tag{2.15}$$

for all t, $s > 0$ *and* $\sigma > 0$ *holds if and only if* X *has Lomax distribution.*

PROOF. Assume that (2.15) holds. Taking $A_r(t) = C_r^{-1}\alpha_r(t)$ and $u = \sigma^{-1}t$, $v = \sigma^{-1}s$,

$$A_r(\sigma u)A_r(\sigma v) = A_r(\sigma u + \sigma v + \sigma uv). \tag{2.16}$$

Equation (2.16) can be written as

$$B_r(x-1)B_r(y-1) = B_r(xy-1)$$

by writing $x = (1+u)$, $y = (1+v)$ and $B_r(u) = A_r(\sigma u)$. This is however

$$G(r, x)G(r, y) = G(r, xy),$$

where $G_r(r,x) = B_r(x-1)$. From Theorem 2.1, $G(r,x) = x^{r-c}$ and we can work backwards to $A_r(t)$. Since $C_r = \alpha_r(0+)$, proceeding as in Theorem 2.1, we can reach at (2.13). The converse part readily follows from (2.14) applied in (2.15). □

THEOREM 2.4. *The random variable X in Theorem 2.3 follows Lomax distribution* (2.13) *if and only if*

$$\alpha_r^n(t) = C_r\,\alpha_r\left[\sigma\left\{\left(1+\frac{t}{\sigma}\right)^n - 1\right\}\right]. \tag{2.17}$$

PROOF. From (2.17), $C_r = \alpha_r(0+)^{n-1}$ and hence (2.17) becomes

$$A_r^n(\sigma\{(1+u)-1\}) = A_r(\sigma\{(1+u^n)-1\}) \tag{2.18}$$

with $A_r(t) = \alpha_r(t)/\alpha_r(0+)$ and $t = u\sigma$. Equation (2.18) now reduces to

$$C_r^n(v) = C_r(v^n) \tag{2.19}$$

by changing $\sigma\{(1+u)-1\}$ to v, by choosing some $w = 1+u$ and $v = \sigma(w-1)$. Equation (2.19) has the solution v^{b_r} and working back to $A_r(\cdot)$ we get the Lomax law. The sufficiency part follows from the expression for $\alpha_r(t)$. □

2.3. Exponential distribution

We present the following characterizations of the exponential distribution specified by

$$F(x) = 1 - e^{-\lambda x}, \quad x > 0,\ \lambda > 0 \tag{2.20}$$

whose partial moments are

$$\alpha_r(t) = \frac{r!e^{-\lambda t}}{\lambda^r},\ r = 1,2,\ldots. \tag{2.21}$$

THEOREM 2.5. *For a random variable X defined over $(0,\infty)$ with absolutely continuous distribution function with finite moments of order r and continuous partial moments $\alpha_r(t)$, the relationship, for any $r = 1,2,\ldots$,*

$$\alpha_r(t)\alpha_r(s) = C_r\alpha_r(t+s), \tag{2.22}$$

is satisfied for all $t,s > 0$ and a constant C_r independent of t and s if and only if X follows the exponential law.

PROOF. When X has exponential distribution, substituting (2.21) in (2.22) we verify that the conditions hold. Conversely when (2.22) is true, the functional equation

$$B_r(t) \cdot B_r(s) = B_r(t+s)$$

with $B_r(t) = \alpha_r(t)/C_r$ holds, which has solution $B_r(t) = e^{-b_r t}$.
Further at all continuity points of $F(t)$,

$$\frac{d}{dt}\alpha_r(t) = -r\alpha_{r-1}(t)$$

giving

$$b_r C_r e^{b_r t} = r C_{r-1} e^{b_{r-1} t} \tag{2.23}$$

for (2.17) to be true for all $t > 0$, $b_r = b_{r-1} = \lambda$, a constant. Then

$$C_r = r\lambda^{-1} C_{r-1} = r!\lambda^{-r} C_0.$$

Using the expression for $\alpha_r(t)$ as in Theorem (2.21), we get

$$F(t) = 1 - e^{-\lambda t}.$$

□

REMARK 2.2.

1. By induction on (2.22),

$$\alpha_r(t_1)\alpha_r(t_2)\cdots\alpha_r(t_n) = \alpha_r^{n-1}(0)\alpha_r(t_1 + t_2 + \cdots + t_n).$$

   When $t_n = t$ for $i = 1, 2, \ldots, n$,

$$\alpha_r^n(t) = c_n \alpha_1(nt),$$

   which is the result in Theorem 4 of Lin (2003). Specialising to $r = 1$,

$$\alpha_1^n(t) = c_n \alpha_1(nt)$$

   is also obtained by Lin (2003) in his Theorem 2.

2. Two other useful characteristic properties are

   (a) $\alpha_r(t-s)\alpha_r(t-s) = \alpha_r^2(t)$,

   (b) $\alpha_r\left(\dfrac{t+s}{2}\right) = [\alpha_r(t)\alpha_r(s)]^{\frac{1}{2}}$.

The last property means that the partial moments at the arithmetic mean of any two truncation points will be the geometric mean of the moments at those points.

2.4. Beta distribution

Consider the translated beta distribution with survival function

$$1 - F(x) = \left(1 - \frac{x}{R}\right)^c, \ 0 < x < R, \ R > 0. \tag{2.24}$$

This is used as a model of life lengths that possess linearly decreasing (increasing) failure rate (mean residual life). The r^{th} order partial moments are

$$\alpha_r(t) = \frac{r}{(c+1)^{(r)}} R^r \left(1 - \frac{t}{R}\right)^{c+r}. \tag{2.25}$$

The distribution function and the expression for $\alpha_r(t)$ are similar to those of the Lomax distribution and therefore the same type of arguments hold in proving the following theorems.

THEOREM 2.6. *Let X be a continuous random variable defined on $(0, R)$ with absolutely continuous distribution function, finite moments of order r and continuous $\alpha_r(t)$. Then, for any $r = 1, 2, \ldots$,*

$$\alpha_r(t)\alpha_r(s) = C_r \alpha_r(t + s - R^{-1}ts),$$

for some $R > 0$ and all $t, s > 0$ holds if and only if X has beta the distribution specified in equation (2.24).

THEOREM 2.7. *Under the condition of Theorem 2.6, the relationship*

$$\alpha_r^n(t) = C_r \alpha_r \left\{ R \left(1 - \frac{t}{R}\right)^n - 1 \right\}, \ r = 0, 1, 2, \ldots$$

holds for all $0 < t < R$ if and only if X has beta the distribution in (2.24).

It may be noted that the uniform distribution over $(0, R)$ is a special case of (2.24) when $c = 1$ and accordingly we get the corresponding results from Theorems 2.6 and 2.7 by specializing to $C_r = 1$.

3. CONCLUDING REMARKS

We have established characterization theorems that extend the result of Lin (2003) to other continuous probability distributions. The main applications of the results proved above consist in (a) to derive partial moments at many other truncation points when the corresponding values at a few points are known and (b) to calculate the complete moments $E(X^r)$ from the partial moments. In fact, the C's in the various theorems are the usual moments and they can be computed using the partial moments at any two points. Thus an approximate conclusion about complete data in terms of truncated observations can be drawn, which can be of some advantage in situations when the complete data is unavailable.

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