KYUNGPOOK Math. J. 47(2007), 155-164

On a Generalization of Closed Sets

Dedicated to Professor Takashi Noiri on the occasion of his retirement and his 63th birthday

Miguel Caldas
Departamento de Matematica Aplicada, Universidade Federal Fluminense, Rua Mario Santos Braga, s/n 24020-140, Niteroi, RJ, Brasil
e-mail : gmamccs@vm.uff.br

Maximilian Ganster
Department of Mathematics, Graz University of Technology, Steyrergasse 30, A-8010 Graz, Austria
e-mail : ganster@weyl.math.tu-graz.ac.at

Dimitrios N. Georgiou
Department of Mathematics, University of Patras, 26500 Patras, Greece
e-mail : georgiou@math.upatras.gr

Saeid Jafari
Department of Mathematics and Physics, Roskilde University, Postbox 260, 4000 Roskilde, Denmark
e-mail : sjafari@ruc.dk

Valeriu Popa
Department of Mathematics, Bacau University, Bacau, Romania
e-mail : vpopa@ub.ro

Abstract. It is the objective of this paper to study further the notion of Λ_s-semi-θ-closed sets which is defined as the intersection of a θ-Λ_s-set and a semi-θ-closed set. Moreover, we introduce some low separation axioms using the above notions. Also we present and study the notions of Λ_s-continuous functions, Λ_s-compact spaces and Λ_s-connected spaces.

1. Introduction

We begin to recall some known notions which will be used in the sequel.

Let (X, τ) be a space and A be a subset of X. We denote the interior and the closure of a set A by $Int(A)$ and $Cl(A)$, respectively. The subset A of X is said to be *semi-open* (see [7]) if there exists an open set U of X such that $U \subset A \subset Cl(U)$. The complement of

Received September 30, 2004.
2000 Mathematics Subject Classification: 54B05, 54C08, 54D05.
Key words and phrases: semi-θ-open, semi-θ-closed, Λ_S-semi-θ-closed set.

a semi-open set is called *semi-closed* set (see [2]). The intersection of all semi-closed sets containing A is called the *semi-closure* of A (see [2]) and is denoted by $sCl(A)$. The semi-θ-closure (see [3]) denoted by $sCl_\theta(A)$, is the set of all $x \in X$ such that $sCl(O) \cap S \neq \emptyset$ for every semi-open set O of X containing x. A subset A is called *semi-θ-closed* (see [3]) if $A = sCl_\theta(A)$. The set $\{x \in X : sCl(O) \subseteq A$ for some semi-open set O containing $x\}$ is called the *semi-θ-interior* of A and is denoted by $sInt_\theta(A)$. A subset A is called *semi-θ-open* (see [5]) if $A = sInt_\theta(A)$. By [6], it is proved that, for a subset A, $sCl_\theta(A)$ is the intersection of all semi-θ-closed sets each containing A. We denote the collection of all semi-θ-open (resp. semi-θ-closed) sets by $S\theta O(X,\tau)$ (resp. $S\theta C(X,\tau)$). The notion of θ-Λ_s-set is introduced and investigated by Caldas et al. [1] by utilizing semi-θ-open sets. These sets suggested a new class of sets which they called Λ_s-semi-θ-closed sets. They offered some properties of these sets. Among others, they proved that a topological space (X,τ) is semi-θ-T_0 if and only if every singleton of X is Λ_s-semi-θ-closed. Recall that a topological space is semi-θ-T_0 [1] if to each pair of points $x, y \in X$ and $x \neq y$, there exists a semi-θ-open set which contains one of them but not the other.

In what follows (X,τ) and (Y,σ) (or X and Y) denote topological spaces.

2. Preliminaries

In this section we recall the definitions of $\Lambda_\theta^{\Lambda_s}$ [1] and $\Lambda_\theta^{\Lambda_s^*}$-sets.

Definition 1 ([1]). Let A be a subset of a topological space X. By $\Lambda_\theta^{\Lambda_s}(A)$ we denote the set $\cap\{O \in S\theta O(X,\tau) \mid A \subset O\}$. A subset A of a topological space (X,τ) is called a $\Lambda_\theta^{\Lambda_s}$-set if $A = \Lambda_\theta^{\Lambda_s}(A)$.

Lemma 2.1. *For subsets A and A_i $(i \in I)$ of a space (X,τ), the following hold*:

(1) $A \subset \Lambda_\theta^{\Lambda_s}(A)$.

(2) $\Lambda_\theta^{\Lambda_s}(\Lambda_\theta^{\Lambda_s}(A)) = \Lambda_\theta^{\Lambda_s}(A)$.

(3) *If* $A \subset B$, *then* $\Lambda_\theta^{\Lambda_s}(A) \subset \Lambda_\theta^{\Lambda_s}(B)$.

(4) $\Lambda_\theta^{\Lambda_s}(\cap\{A_i : i \in I\}) \subset \cap\{\Lambda_\theta^{\Lambda_s}(A_i) : i \in I\}$.

(5) $\Lambda_\theta^{\Lambda_s}(\cup\{A_i : i \in I\}) = \cup\{\Lambda_\theta^{\Lambda_s}(A_i) : i \in I\}$.

(6) $\Lambda_\theta^{\Lambda_s}(A)$ *is a* $\Lambda_\theta^{\Lambda_s}$*-set.*

(7) *If A is semi-θ-open, then A is a* $\Lambda_\theta^{\Lambda_s}$*-set.*

(8) *If A_i is* $\Lambda_\theta^{\Lambda_s}$*-set for each $i \in I$, then* $\cap\{A_i : i \in I\}$ *and* $\cup\{A_i : i \in I\}$ *are* $\Lambda_\theta^{\Lambda_s}$*-sets.*

Theorem 2.2. *Let X be a topological space. We set* $\tau^{\Lambda_\theta^{\Lambda_s}} = \{A : A \text{ is a } \Lambda_\theta^{\Lambda_s} - \text{set of } X\}$. *The pair* $(X, \tau^{\Lambda_\theta^{\Lambda_s}})$ *is an Alexandroff space.*

Proof. This is an immediate consequence of Lemma 2.1. □

Definition 2. Let A be a subset of a topological space (X,τ). By $\Lambda_\theta^{\Lambda_s^*}(A)$, we denote the set $\cup\{B \in S\theta C(X,\tau) \mid B \subset A\}$. A subset A of a topological space (X,τ) is called a $\Lambda_\theta^{\Lambda_s^*}$-set if $A = \Lambda_\theta^{\Lambda_s^*}(A)$.

We obtain the following lemma which is similar to Lemma 2.1.

Lemma 2.3. *For subsets A, B and A_i $(i \in I)$ of a topological space (X, τ) the following properties hold*:

(1) $\Lambda_\theta^{\Lambda_s^*}(A) \subseteq A$.

(2) *If* $A \subseteq B$, *then* $\Lambda_\theta^{\Lambda_s^*}(A) \subseteq \Lambda_\theta^{\Lambda_s^*}(B)$.

(3) *If A is semi-θ-closed, then* $\Lambda_\theta^{\Lambda_s^*}(A) = A$.

(4) $\Lambda_\theta^{\Lambda_s^*}(\cap\{A_i : i \in I\}) = \cap\{\Lambda_\theta^{\Lambda_s^*}(A_i) : i \in I\}$.

(5) $\cup\{\Lambda_\theta^{\Lambda_s^*}(A_i) : i \in I\} \subseteq \Lambda_\theta^{\Lambda_s^*}(\cup\{A_i : i \in I\})$.

(6) $\Lambda_\theta^{\Lambda_s}(X - A) = X - \Lambda_\theta^{\Lambda_s^*}(A)$ *and* $\Lambda_\theta^{\Lambda_s^*}(X - A) = X - \Lambda_\theta^{\Lambda_s}(A)$.

(7) $\Lambda_\theta^{\Lambda_s^*}(A)$ *is a* $\Lambda_\theta^{\Lambda_s^*}$*-set.*

(8) *If A is a semi-θ-closed, then A is a* $\Lambda_\theta^{\Lambda_s^*}$*-set.*

(9) *If A_i is a* $\Lambda_\theta^{\Lambda_s^*}$*-set for each $i \in I$, then* $\cup\{A_i \mid i \in I\}$ *and* $\cap\{A_i \mid i \in I\}$ *are* $\Lambda_\theta^{\Lambda_s^*}$*-sets.*

Observe that if X is a topological space and $\tau^{\Lambda_\theta^{\Lambda_s^*}} = \{A : A \text{ is a } \Lambda_\theta^{\Lambda_s^*} - \text{set of } X\}$, then $(X, \tau^{\Lambda_\theta^{\Lambda_s}})$ is an Alexandroff space.

3. Λ_s-semi-θ-closed sets

Definition 3. A subset A of a topological space (X, τ) is called Λ_s-semi-θ-closed [1], denoted by $(\Lambda, s\theta)$-closed, if $A = T \cap C$, where T is a $\Lambda_\theta^{\Lambda_s}$-set and C is a semi-$\theta$-closed set.

Lemma 3.1 ([1], Lemma 2.23). *Let A be a subset of a space (X, τ). Then the following conditions are equivalent:*

(1) *A is $(\Lambda, s\theta)$-closed,*

(2) $A = P \cap sCl_\theta(A)$, *where P is a* $\Lambda_\theta^{\Lambda_s}$*-set,*

(3) $f\, A = \Lambda_\theta^{\Lambda_s}(A) \cap sCl_\theta(A)$.

Example 3.2. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$. The semi-$\theta$-closed sets of (X, τ) are $\{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. The set $A = \{c\}$ is $(\Lambda, s\theta)$-closed since it is semi θ-closed but it is not closed.

Example 3.3. Let $X = \{a, b, c, \}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. The set $A = \{c\}$ is closed but it is not $(\Lambda, s\theta)$-closed.

The Example 3.2 and Example 3.3 shown that the sets $(\Lambda, s\theta)$-closed and closed are independent of each other.

Note that every semi θ-closed set is $(\Lambda, s\theta)$-closed, but the converse is not true in general.

Example 3.4. Let (X, τ) be as in the Example 3.2. Then $B = \{b, c\}$ is $(\Lambda, s\theta)$-closed since it is $\Lambda_\theta^{\Lambda_s}$-set, but it is not semi θ-closed.

Lemma 3.5. *If A_i is $(\Lambda, s\theta)$-closed for each $i \in I$, then* $\cap\{A_i : i \in I\}$ *is $(\Lambda, s\theta)$-closed.*

Proof. Suppose that A_i is $(\Lambda, s\theta)$-closed for each $i \in I$. Then, for each $i \in I$ there exist a $\Lambda_\theta^{\Lambda_s}$-set T_i and a semi-θ-closed set C_i such that $A_i = T_i \cap C_i$. Now

$$\begin{aligned}\bigcap\{A_i : i \in I\} &= \bigcap\{T_i \cap C_i : i \in I\} \\ &= \bigcap\{T_i : i \in I\} \cap \bigcap\{C_i : i \in I\}.\end{aligned}$$

By Lemma 2.1, $\bigcap\{T_i : i \in I\}$ is a $\Lambda_\theta^{\Lambda_s}$-set and $\bigcap\{C_i : i \in I\}$ is semi-θ-closed. This shows that $\bigcap\{A_i : i \in I\}$ is $(\Lambda, s\theta)$-closed. □

Definition 4. A subset A of a space (X, τ) is said to be $(s\theta, s\theta)$-generalized closed if $sCl_\theta(A) \subseteq G$ holds whenever $A \subseteq G$ and $G \in S\theta O(X, \tau)$.

Lemma 3.6. A subset A of a space (X, τ) is $(s\theta, s\theta)$-generalized closed if and only if $sCl_\theta(A) \subseteq \Lambda_\theta^{\Lambda_s}(A)$.

Proof. Necessity: Suppose that there is a point $x \epsilon X$ such that $x \notin \Lambda_\theta^{\Lambda_s}(A)$. Then, there exists a subset $O \epsilon S\theta O(X, \tau)$ such that $A \subseteq O$ and $x \notin O$. This implies that $sCl_\theta(A) \subseteq O$. Hence $x \notin sCl_\theta(A)$ since A is $(s\theta, s\theta)$-generalized closed.
Sufficiency: Obvious. □

Theorem 3.7. *A subset A of a space (X, τ) is semi-θ-closed if and only if A is $(s\theta, s\theta)$-generalized closed and $(\Lambda, s\theta)$-closed.*

Proof. Necessity: Every semi θ-closed set is both $(s\theta, s\theta)$-generalized closed and $(\Lambda, s\theta)$-closed.

Sufficiency: Since A is $(s\theta, s\theta)$-generalized closed, then by Lemma 3.3, $sCl_\theta(A) \subseteq \Lambda_\theta^{\Lambda_s}(A)$. By assumption and Lemma 3.1 $A = \Lambda_\theta^{\Lambda_s}(A) \bigcap sCl_\theta(A) = sCl_\theta(A)$. i.e., A is semi-θ-closed. □

Definition 5. A subset A of a topological space (X, τ) is called $(\Lambda, s\theta)$-open if $X \setminus A$ is $(\Lambda, s\theta)$-closed.

Theorem 3.8. *The union of any family of $(\Lambda, s\theta)$-open sets is a $(\Lambda, s\theta)$-open set.*

Proof. The proof of this theorem follows by the fact that the intersection of a family of $(\Lambda, s\theta)$-closed sets is $(\Lambda, s\theta)$-closed. □

Lemma 3.9. *The following statements are equivalent for a subset A of a topological space X:*

(1) *A is $(\Lambda, s\theta)$-open*

(2) *$A = T \cup C$, where T is a $\Lambda_\theta^{\Lambda_s^*}$-set and C is a semi-θ-open set.*

Proof. The proof of this lemma is clear. □

Lemma 3.10. *Every $\Lambda_\theta^{\Lambda_s^*}$-set is $(\Lambda, s\theta)$-open.*

Proof. Take $A = A \cup \emptyset$, where A is a $\Lambda_\theta^{\Lambda_s^*}$-set, X is semi-θ-closed and $\emptyset = X \setminus X$. □

Definition 6. A subset A of a topological space X is called a $\Lambda_\theta^{\Lambda_s}$-$D$ set if there are two $(\Lambda, s\theta)$-open sets U and V in X such that $U \neq X$ and $A = U - V$.

It is true that every $(\Lambda, s\theta)$-open set U different from X is a $\Lambda_\theta^{\Lambda_s}$-$D$ set if $A = U$ and $V = \emptyset$.

Example 3.11. Let (X,τ) be a space as in the Example 3.2. Then the sets $\{c\}$ and $\{a,c\}$ are $(\Lambda, s\theta)$-closed sets since they are $\Lambda_\theta^{\Lambda_s}$-sets. Thus the sets $\{a,b\}$ and $\{b\}$ are $(\Lambda, s\theta)$-open sets. So the set $A = \{a\} = \{a,b\} - \{b\}$ is $\Lambda_\theta^{\Lambda_s}$-$D$ set which is not open and $(\Lambda, s\theta)$-open set.

Definition 7. A topological space (X,τ) is called:

(i) $\Lambda_\theta^{\Lambda_s}$-$D_0$ if for any distinct pair of points x and y of X there exists a $\Lambda_\theta^{\Lambda_s}$-$D$ set of X containing x but not y or a $\Lambda_\theta^{\Lambda_s}$-$D$ set of X containing y but not x.

(ii) $\Lambda_\theta^{\Lambda_s}$-$D_1$ if for any distinct pair of points x and y of X there exist a $\Lambda_\theta^{\Lambda_s}$-$D$ set of X containing x but not y and a $\Lambda_\theta^{\Lambda_s}$-$D$ set of X containing y but not x.

(iii) $\Lambda_\theta^{\Lambda_s}$-$D_2$ if for any distinct pair of points x and y of X there exist disjoint $\Lambda_\theta^{\Lambda_s}$-$D$ sets G and E of X containing x and y, respectively.

A topological space (X,τ) satisfies the $(\Lambda, s\theta)$-property if for any distinct pair of points in X, there is a $(\Lambda, s\theta)$-open set containing one of the points but not the other.

Remark 3.12.

(i) If (X,τ) satisfies the $(\Lambda, s\theta)$-property, then it is $\Lambda_\theta^{\Lambda_s}$-$D_0$.

(ii) If (X,τ) is $\Lambda_\theta^{\Lambda_s}$-$D_i$, then it is $\Lambda_\theta^{\Lambda_s}$-$D_{i-1}$, where $i = 1, 2$.

Theorem 3.13. *For a topological space (X,τ), the following statements are true*:

(1) *(X,τ) is $\Lambda_\theta^{\Lambda_s}$-$D_0$ if and only if it satisfies the $(\Lambda, s\theta)$-property.*

(2) *(X,τ) is $\Lambda_\theta^{\Lambda_s}$-$D_1$ if and only if it is $\Lambda_\theta^{\Lambda_s}$-$D_2$.*

Proof. The sufficiency for (1) and (2) follows from the above Remark 3.5.
Necessity condition for (1). Let (X,τ) be $\Lambda_\theta^{\Lambda_s}$-$D_0$ so that for any distinct pair of points x and y of X at least one belongs to a $\Lambda_\theta^{\Lambda_s}$-$D$ set O. Therefore we choose $x \in O$ and $y \notin O$. Suppose $O = U - V$ for which $U \neq X$ and U and V are $(\Lambda, s\theta)$-open sets in X. This implies that $x \in U$. For the case that $y \notin O$ we have (i) $y \notin U$, (ii) $y \in U$ and $y \in V$. For (i), the space X satisfies the $(\Lambda, s\theta)$-property since $x \in U$ and $y \notin U$. For (ii), the space X also satisfies the $(\Lambda, s\theta)$-property since $y \in V$ but $x \notin V$.
Necessity condition for (2). Suppose that X is $\Lambda_\theta^{\Lambda_s}$-$D_1$. It follows from the definition that for any distinct points x and y in X there exist $\Lambda_\theta^{\Lambda_s}$-$D$ sets G and E such that G containing x but not y and E containing y but not x. Let $G = U - V$ and $E = W - D$, where U, V, W and D are $(\Lambda, s\theta)$-open sets in X. By the fact that $x \notin E$, we have two cases, i.e. either $x \notin W$ or both W and D contain x. If $x \notin W$, then from $y \notin G$ either (i) $y \notin U$ or (ii) $y \in U$ and $y \in V$. If (i) is the case, then it follows from $x \in U - V$ that $x \in U - (V \cup W)$, and also it follows from $y \in W - D$ that $y \in W - (U \cup D)$. Thus we have $U - (V \cup W)$ and $W - (U \cup D)$ which are disjoint. If (ii) is the case, it follows that $x \in U - V$, $y \in V$ and $(U - V) \cap V = \emptyset$. If $x \in W$ and $x \in D$, we have $y \in W - D$, $x \in D$ and $(W - D) \cap D = \emptyset$. This shows that X is $\Lambda_\theta^{\Lambda_s}$-$D_2$. □

Definition 8. Let (X,τ) be a topological space. A point $x \in X$ which has only X as the $(\Lambda, s\theta)$-neighborhood is called a $\Lambda_\theta^{\Lambda_s}$-neat point.

Theorem 3.14. *For a topological space (X,τ) that satisfies the $(\Lambda, s\theta)$-property the following are equivalent*:

(1) *(X,τ) is $\Lambda_\theta^{\Lambda_s}$-$D_1$;*

(2) (X,τ) *has no* $\Lambda_\theta^{\Lambda_s}$*-neat point.*

Proof. (1) → (2). Since (X,τ) is $\Lambda_\theta^{\Lambda_s}$-$D_1$, so each point x of X is contained in a $\Lambda_\theta^{\Lambda_s}$-$D$ set $O = U - V$ and thus in U. By definition $U \neq X$. This implies that x is not a $\Lambda_\theta^{\Lambda_s}$-neat point.
(2) → (1). Since X satisfies the $(\Lambda, s\theta)$-property, then for each distinct pair of points $x, y \in X$, at least one of them, choose x for example has a $(\Lambda, s\theta)$-neighborhood U containing x and not y. Thus U which is different from X is a $\Lambda_\theta^{\Lambda_s}$-$D$ set. If X has no $\Lambda_\theta^{\Lambda_s}$-neat point, then y is not a $\Lambda_\theta^{\Lambda_s}$-neat point. This means that there exists a $(\Lambda, s\theta)$-neighborhood V of y such that $V \neq X$. Thus $y \in (V - U)$ but not x and $V - U$ is a $\Lambda_\theta^{\Lambda_s}$-$D$ set. Hence X is $\Lambda_\theta^{\Lambda_s}$-$D_1$. □

Remark 3.15. It is clear that a topological space (X,τ) that satisfies the $(\Lambda, s\theta)$-property is not $\Lambda_\theta^{\Lambda_s}$-$D_1$ if and only if there is a unique $\Lambda_\theta^{\Lambda_s}$-neat point in X. It is unique because if x and y are both $\Lambda_\theta^{\Lambda_s}$-neat point in X, then at least one of them say x has a $(\Lambda, s\theta)$-neighborhood U containing x but not y. But this is a contradiction since $U \neq X$.

Definition 9. Let (X,τ) be a topological space and $A \subseteq X$. A point $x \in X$ is called $(\Lambda, s\theta)$-cluster point of A if for every $(\Lambda, s\theta)$-open set U of X containing x we have $A \cap U \neq \emptyset$. The set of all $(\Lambda, s\theta)$-cluster points is called the $(\Lambda, s\theta)$-closure of A and is denoted by $A^{(\Lambda,s\theta)}$.

Lemma 3.16. *Let A and B be subsets of a topological space (X,τ). For the $(\Lambda, s\theta)$-closure, the following properties hold.*

(1) $A \subset A^{(\Lambda,s\theta)}$.

(2) $A^{(\Lambda,s\theta)} = \cap\{F \mid A \subset F \text{ and } F \text{ is } (\Lambda, s\theta)-\text{closed}\}$.

(3) *If* $A \subset B$, *then* $A^{(\Lambda,s\theta)} \subset B^{(\Lambda,s\theta)}$.

(4) A *is* $(\Lambda, s\theta)$*-closed if and only if* $A = A^{(\Lambda,s\theta)}$.

(5) $A^{(\Lambda,s\theta)}$ *is* $(\Lambda, s\theta)$*-closed.*

Proof. Straightforward. □

Definition 10. A topological space (X,τ) is called a $(\Lambda, s\theta)$-symmetric if for x and y in X, $x \in \{y\}^{(\Lambda,s\theta)}$ implies $y \in \{x\}^{(\Lambda,s\theta)}$.

In what follows the set $\{x\}^{(\Lambda,s\theta)}$ is denoted by $x^{(\Lambda,s\theta)}$ for every $x \in X$.

Theorem 3.17. *A topological space (X,τ) is $(\Lambda, s\theta)$-symmetric if and only if for $x \in X$, $x^{(\Lambda,s\theta)} \subseteq E$ whenever $x \in E$ and E is $(\Lambda, s\theta)$-open in (X,τ).*

Proof. Assume that $x \in y^{(\Lambda,s\theta)}$ but $y \notin x^{(\Lambda,s\theta)}$. This means that the complement of $x^{(\Lambda,s\theta)}$ contains y. Therefore the set $\{y\}$ is a subset of the complement of $x^{(\Lambda,s\theta)}$. This implies that $y^{(\Lambda,s\theta)}$ is a subset of the complement of $x^{(\Lambda,s\theta)}$. Now the complement of $x^{(\Lambda,s\theta)}$ contains x which is a contradiction.

Conversely, suppose that $\{x\} \subset E$ and E is $(\Lambda, s\theta)$-open in (X,τ) but $x^{(\Lambda,s\theta)}$ is not a subset of E. This means that $x^{(\Lambda,s\theta)}$ and the complement of E are not disjoint. Let y belongs to their intersection. Now we have $x \in y^{(\Lambda,s\theta)}$ which is a subset of the complement of E and $x \notin E$. But this is a contradiction. □

Theorem 3.18. *For a $(\Lambda, s\theta)$-symmetric topological space (X,τ), the following are equivalent:*

(1) (X,τ) *satisfies the* $(\Lambda, s\theta)$*-property;*

(2) (X,τ) *is* $\Lambda_\theta^{\Lambda_s}$*-*$D_0$*;*

(3) (X,τ) *is* $\Lambda_\theta^{\Lambda_s}$*-*$D_1$*.*

Proof. (1) $\leftrightarrow$ (2) : Lemma 3.10.
(3) $\to$ (2) : Remark 3.12.
(1) $\to$ (3) : Let $x \neq y$ and by (1), we may assume that $x \in E \subset \{y\}^c$ for some E $(\Lambda, s\theta)$-open in (X,τ). Then $x \notin y^{(\Lambda,s\theta)}$ and hence $y \notin x^{(\Lambda,s\theta)}$. Hence there exists a $(\Lambda, s\theta)$-open set F such that $y \in F \subset \{x\}^c$. Since every $(\Lambda, s\theta)$-open set is a $\Lambda_\theta^{\Lambda_s}$-$D$ set, we have that (X,τ) is a $\Lambda_\theta^{\Lambda_s}$-$D_1$ space. □

4. $(\Lambda, s\theta)$-continuous functions

Definition 11. Let (X,τ) and (Y,σ) be two topological spaces. A function $f : (X,\tau) \to (Y,\sigma)$ is called $(\Lambda, s\theta)$-continuous at a point $x \in X$ if for every $(\Lambda, s\theta)$-open set V of Y such that $f(x) \in V$ there exists a $(\Lambda, s\theta)$-open set U of X such that $x \in U$ and $f(U) \subseteq V$.

The function f is called $(\Lambda, s\theta)$*-continuous* if f is $(\Lambda, s\theta)$-continuous at every point $x \in X$.

Definition 12. Let (X,τ) be a topological space, $x \in X$ and $\{x_s, s \in S\}$ be a net of X. We say that the net $\{x_s, s \in S\}$ $(\Lambda, s\theta)$-converges to x if for every $(\Lambda, s\theta)$-open set U containing x there exists an element $s_0 \in S$ such that $s \geq s_0$ implies $x_s \in U$.

Theorem 4.1. *Let (X,τ) be a topological space and $A \subseteq X$. A point $x \in A^{(\Lambda,s\theta)}$ if and only if there exists a net $\{x_s, s \in S\}$ of A which $(\Lambda, s\theta)$-converges to x.*

Proof. The existence of such a net clearly implies that $x \in A^{(\Lambda,s\theta)}$. Suppose $x \in A^{(\Lambda,s\theta)}$ and let us denote by $\mathcal{U}$ the set of all $(\Lambda, s\theta)$-open subsets U of X such that $x \in U$ directed by the relation $\subseteq$, i.e., let us define that $U_1 \leq U_2$ if $U_2 \subseteq U_1$. The net $\{x_U, U \in \mathcal{U}\}$, where x_U is an arbitrary point of $A \cap U$, $(\Lambda, s\theta)$-converges to x. □

Theorem 4.2. *For a function $f : (X,\tau) \to (Y,\sigma)$, the following are equivalent:*

(1) *f is $(\Lambda, s\theta)$-continuous;*

(2) *$f^{-1}(V)$ is $(\Lambda, s\theta)$-open in (X,τ) for every $(\Lambda, s\theta)$-open set V of (Y,σ);*

(3) *$f^{-1}(F)$ is $(\Lambda, s\theta)$-closed in (X,τ) for every $(\Lambda, s\theta)$-closed set F of (Y,σ);*

(4) *$f(A^{(\Lambda,s\theta)}) \subset [f(A)]^{(\Lambda,s\theta)}$ for each subset A of X;*

(5) *$[f^{-1}(B)]^{(\Lambda,s\theta)} \subset f^{-1}(B^{(\Lambda,s\theta)})$ for each subset B of Y;*

(6) *For every $x \in X$ and every net $\{x_s, s \in S\}$ of X which $(\Lambda, s\theta)$-converges to x in X, the net $\{f(x_s), s \in S\}$ $(\Lambda, s\theta)$-converges to $f(x)$ in Y.*

Proof. (1) $\to$ (2): Let V be any $(\Lambda, s\theta)$-open set of (Y,σ) and $x \in f^{-1}(V)$. Since f is $(\Lambda, s\theta)$-continuous, there exists a $(\Lambda, s\theta)$-open set U_x containing x such that $f(U_x) \subset V$. Therefore, we have $x \in U_x \subset f^{-1}(V)$ and hence $f^{-1}(V) = \cup\{U_x \mid x \in f^{-1}(V)\}$. By Theorem 3.8, $f^{-1}(V)$ is $(\Lambda, s\theta)$-open in (X,τ).
(2) $\to$ (1): This is obvious.
(2) $\leftrightarrow$ (3): This is obvious from Definition 5.
(3) $\to$ (4): Let A be any subset of X. Since $A \subset f^{-1}([f(A)]^{(\Lambda,s\theta)})$, by Lemma 3.15 we

have $A^{(\Lambda,s\theta)} \subset f^{-1}([f(A)]^{(\Lambda,s\theta)})$ and hence $f(A^{(\Lambda,s\theta)}) \subset [f(A)]^{(\Lambda,s\theta)}$.
(4) → (5): Let B be any subset of Y. By (4) we have $f([f^{-1}(B)]^{(\Lambda,s\theta)}) \subset [f(f^{-1}(B))]^{(\Lambda,s\theta)} \subset B^{(\Lambda,s\theta)}$ and hence $[f^{-1}(B)]^{(\Lambda,s\theta)} \subset f^{-1}(B^{(\Lambda,s\theta)})$.
(5) → (3): Let F be any $(\Lambda, s\theta)$-closed set in (Y,σ). By Lemma 3.15, $[f^{-1}(F)]^{(\Lambda,s\theta)} \subset f^{-1}(F^{(\Lambda,s\theta)}) = f^{-1}(F)$ and $[f^{-1}(F)]^{(\Lambda,s\theta)} \subset f^{-1}(F)$. Therefore, we obtain $[f^{-1}(F)]^{(\Lambda,s\theta)} = f^{-1}(F)$. This shows that $f^{-1}(F)$ is $(\Lambda, s\theta)$-closed in (X,τ).
(1) → (6): Let $x \in X$ and $\{x_s \mid s \in S\}$ be a net $(\Lambda, s\theta)$-converging to x. For any $(\Lambda, s\theta)$-open set of (Y,σ) containing $f(x)$, by (1) there exists a $(\Lambda, s\theta)$-open set U of X containing x such that $f(U) \subset V$. Since $\{x_s \mid s \in S\}$ converges to x, there exists $s_0 \in S$ such that $s \geq s_0$ implies $x_s \in U$. Therefore, $f(x_s) \in V$ for any $s \geq s_0$ and the net $\{f(x_s) \mid s \in S\}$ $(\Lambda, s\theta)$-converges to $f(x)$.
(6) → (1): Let us suppose that there exists a point $x \in X$ and a $(\Lambda, s\theta)$-open neighbourhood V of $f(x)$ such that for every $(\Lambda, s\theta)$-open set U of X containing x such that $f(U) \not\subseteq V$. Then for every $(\Lambda, s\theta)$-open set U of X such that $x \in U$, we choose an element $x_U \in U$ such that $f(x_U) \notin V$. Let $\mathcal{U}$ be the set of all $(\Lambda, s\theta)$-open sets U of X containing x and is directed by the relation $\subseteq$ i.e., let us define that $U_1 \leq U_2$ if $U_2 \subseteq U_1$. Easily, the net $\{x_U, U \in \mathcal{U}\}$ $(\Lambda, s\theta)$-converges to x but the net $\{f(x_U), U \in \mathcal{U}\}$ does not $(\Lambda, s\theta)$-converge to $f(x)$ which is a contradiction. Thus there exists a $(\Lambda, s\theta)$-open set U of X such that $x \in U$ and $f(U) \subseteq V$. □

Remark 4.3. We recall that a function $f : (X,\tau) \to (Y,\sigma)$ is said to be quasi irresolute [4] if $f^{-1}(V)$ is semi-θ-open in (X,τ) for each semi-θ-open set V of (Y,σ).

Clearly, if a function $f : (X,\tau) \to (Y,\sigma)$ is quasi irresolute, then $f : (X, \tau^{\Lambda_\theta^{\Lambda_s^*}}) \to (Y, \sigma^{\Lambda_\theta^{\Lambda_s^*}})$ is continuous.

Indeed let V be any $\Lambda_\theta^{\Lambda_s^*}$-set of (Y,σ). Then $V = \Lambda_\theta^{\Lambda_s^*}(V) = \cup\{W \mid V \supset W \in S\theta C(Y,\sigma)\}$. Since f is quasi irresolute, we have $f^{-1}(V) = \cup\{f^{-1}(W) \mid f^{-1}(V) \supset f^{-1}(W) \in S\theta C(X,\tau)\} \subset \cup\{U \mid f^{-1}(V) \supset U \in S\theta C(X,\tau)\} = \Lambda_\theta^{\Lambda_s^*}(f^{-1}(V))$. By Lemma 2.3, we have $f^{-1}(V) \supset \Lambda_\theta^{\Lambda_s^*}(f^{-1}(V))$ and hence $f^{-1}(V)$ is a $\Lambda_\theta^{\Lambda_s^*}$-set of (X,τ).

Observe that if a function $f : (X,\tau) \to (Y,\sigma)$ is quasi irresolute, then $f : (X, \tau^{\Lambda_\theta^{\Lambda_s}}) \to (Y, \sigma^{\Lambda_\theta^{\Lambda_s}})$ is continuous.

Theorem 4.4. *If $f : (X,\tau) \to (Y,\sigma)$ is a quasi irresolute function, then it is $(\Lambda, s\theta)$-continuous.*

Proof. Let F be a $(\Lambda, s\theta)$-closed set of (Y,σ). Then there exist a $\Lambda_\theta^{\Lambda_s}$-set T and a semi-θ-closed set C such that $F = T \cap C$. Since f is quasi irresolute $f^{-1}(T)$ is a $\Lambda_\theta^{\Lambda_s}$-set of (X,τ) and $f^{-1}(C)$ is semi-θ-closed. Therefore, $f^{-1}(F) = f^{-1}(T) \cap f^{-1}(C)$ is $(\Lambda, s\theta)$-closed in (X,τ). By Theorem 4.2, f is $(\Lambda, s\theta)$-continuous. □

Example 4.5. Let $X = \{a,b,c\}$ and $\tau = \{\emptyset, X, \{b\}, \{c\}, \{b,c\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b,c\}\}$. The semi-$\theta$-closed sets of (X,τ) are $\{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}\}$, the $(\Lambda, s\theta)$-closed sets of (X,τ) are $\{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$ and the semi-θ-closed sets of (X,σ) are $\{\emptyset, X, \{a\}, \{b,c\}\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be the identity function. Then f is $(\Lambda, s\theta)$-continuous but it is not quasi-irresolute since $f^{-1}(\{b,c\})$ is not semi θ-closed in (X,τ).

5. $(\Lambda, s\theta)$-compactness and $(\Lambda, s\theta)$-connectedness

Definition 13. A topological space (X, τ) is called $(\Lambda, s\theta)$-compact (resp. semi-θ-compact) if every cover of X by $(\Lambda, s\theta)$-open (resp. semi-θ-open) sets has a finite subcover.

Theorem 5.1. *A topological space (X, τ) is $(\Lambda, s\theta)$-compact (resp. semi-θ-compact) if and only if for every family $\{A_i : i \in I\}$ of $(\Lambda, s\theta)$-closed (resp. semi-θ-closed) sets in X satisfying $\cap\{A_i : i \in I\} = \emptyset$, there is a finite subfamily $A_{i_1}, \cdots, A_{i_n}$ with $\cap\{A_{i_k} : k = 1, \cdots, n\} = \emptyset$.*

Proof. Straightforward. □

Theorem 5.2. *For a topological space (X, τ), the following hold:*

(1) *If $(X, \tau^{\Lambda_\theta^{\Lambda_s}})$ is compact, then (X, τ) is semi-θ-compact.*

(2) *If (X, τ) is $(\Lambda, s\theta)$-compact, then (X, τ) is semi-θ-compact.*

(3) *If (X, τ) is $(\Lambda, s\theta)$-compact, then $(X, \tau^{\Lambda_\theta^{\Lambda_s^*}})$ is compact.*

Proof. (1) This follows from Lemma 2.1.
(2) This follows from Theorem 5.1 and of the fact that every semi-θ-closed set is $(\Lambda, s\theta)$-closed.
(3) This follows from Lemma 3.10. □

Theorem 5.3. *If $f : (X, \tau) \to (Y, \sigma)$ is a $(\Lambda, s\theta)$-continuous surjection and (X, τ) is a $(\Lambda, s\theta)$-compact space, then (Y, σ) is $(\Lambda, s\theta)$-compact.*

Proof. Let $\{V_i \mid i \in I\}$ be any cover of Y by $(\Lambda, s\theta)$-open sets of (Y, σ). Since f is $(\Lambda, s\theta)$-continuous, by Theorem 4.2 $\{f^{-1}(V_i \mid i \in I\}$ is a cover of X by $(\Lambda, s\theta)$-open sets of (X, τ). By $(\Lambda, s\theta)$-compactness of (X, τ), there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_i) \mid i \in I_0\}$. Since f is surjective, we obtain $Y = f(X) = \cup_{i \in I_0} V_i$. This shows that (Y, σ) is $(\Lambda, s\theta)$-compact. □

Corollary 5.4. *The $(\Lambda, s\theta)$-compactness is preserved by quasi irresolute surjections.*

Proof. This is an immediate consequence of Theorem 5.3 and Theorem 4.4. □

Definition 14. A topological space (X, τ) is called $(\Lambda, s\theta)$-connected if X cannot be written as a disjoint union of two non-empty $(\Lambda, s\theta)$-open sets.

Theorem 5.5. *For a topological space (X, τ), the following statements are equivalent:*

(1) *The space X is $(\Lambda, s\theta)$-connected;*

(2) *The only subsets of X, which are both $(\Lambda, s\theta)$-open and $(\Lambda, s\theta)$-closed are the empty set $\emptyset$ and X.*

Proof. Straightforward. □

Open problems.

(1) Does there exist a space (X, τ) which is semi-θ-compact but the space $(X, \tau^{\Lambda_\theta^{\Lambda_s}})$ is not compact?

(2) Does there exist a space (X, τ) which is semi-θ-compact but the space (X, τ) is not $(\Lambda, s\theta)$-compact?

(3) Does there exist a space (X, τ) such that the space $(X, \tau^{\Lambda_\theta^{\Lambda_s^*}})$ is compact but the space (X, τ) is not $(\Lambda, s\theta)$-compact?

Acknowledgment. Part of this research was carried out while M.Caldas was visiting the Institute of Mathematics of the Universidad Autónoma de México under the TWAS-UNESCO

References

[1] M. Caldas and S. Jafari, *On θ-semigeneralized closed sets in topology*, Kyungpook Math. J., **43**(2003), 135-148.

[2] S. G. Grossley and S. K. Hildebrand, *Semi-closure,* Texas J. Sci., **22**(1971), 99-112.

[3] G. Di Maio and T. Noiri, *On s-closed spaces,* Indian J. Pure Appl. Math., **18(3)**(1987), 226-233.

[4] G. Di Maio and T. Noiri, *Weak and strong forms of irresolute functions*, Rend. Circ. Mat. Palermo (2) Suppl., **18**(1988), 255-273.

[5] S. Ganguly and C. K. Basu, *Further characterizations of s-closed spaces*, Indian J. Pure Appl. Math., **23(9)**(1992), 635-641.

[6] M. N. Mukherjee and C. K. Basu, *On semi-θ-closed sets, semi-θ-connectedness and some associated mappings,* Bull. Cal. Math. Soc., **83**(1991), 227-238.

[7] N. Levine, *Generalized closed sets in topology*, Rend. Circ. Mat. Palermo, **19(2)**(1970), 89-96.