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Oscillation of Certain Second Order Damped Quasilinear Elliptic Equations via the Weighted Averages

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Abstract. By using the weighted averaging techniques, we establish oscillation criteria for the second order damped quasilinear elliptic differential equation

$$\sum_{i,j=1}^{N} D_i(a_{ij}(x)\|Dy\|^{p-2}D_jy) + \langle\, b(x), \|Dy\|^{p-2}Dy\,\rangle + c(x)f(y) = 0, \quad p > 1.$$

The obtained theorems include and improve some existing ones for the undamped half-linear partial differential equation and the semilinear elliptic equation.

1. Introduction

In this paper, we are concerned with the oscillation of the second order damped quasilinear elliptic differential equation

$$\sum_{i,\,j=1}^{N} D_i[\,a_{ij}(x)\|Dy\|^{p-2}D_jy\,] + \langle\, b(x), \|Dy\|^{p-2}Dy\,\rangle + c(x)f(y) = 0 \tag{1.1}$$

in the exterior domain $\Omega(r_0) := \{x \in \mathbb{R}^N : \|x\| \geq r_0\}$ for some $r_0 > 0$, where $x = (x_i)_{i=1}^N \in \Omega(r_0) \subset \mathbb{R}^N$, $N \geq 2$, $p > 1$, $D_iy = \partial y/\partial x_i$ for all i, $Dy = (D_iy)_{i=1}^N$, $\|\cdot\|$ and $\langle\,\cdot,\cdot\,\rangle$ denote the usual Euclidean norm and the usual scalar product in $\mathbb{R}^N$, respectively.

Throughout this paper, we tacitly assume that the following conditions holds.

(A1) $A = (a_{ij}(x))_{N\times N}$ is a real symmetric positive define matrix function with $a_{ij} \in C^{1+\nu}_{loc}(\Omega(r_0), \mathbb{R})$ for all i, j, $0 < \nu < 1$;

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(A2) $b(x) = (b_i(x))_{i=1}^N$ with $b_i \in \mathbf{C}_{loc}^{\nu}(\Omega(r_0), \mathbb{R})$ for all i, and $c \in \mathbf{C}_{loc}^{\nu}(\Omega(r_0), \mathbb{R})$;

(A3) $f \in \mathbf{C}(\mathbb{R}, \mathbb{R}) \cup \mathbf{C}^1(\mathbb{R} - \{0\}, \mathbb{R})$ such that $yf(y) > 0$ and

$$\frac{f'(y)}{|f(y)|^{(p-2)/(p-1)}} \geq \varepsilon > 0 \quad \text{for} \quad y \neq 0.$$

By a solution of Eq.(1.1) is meant a function $y \in \mathbf{C}^{2+\nu}(\Omega(r_0), \mathbb{R})$ with $a_{ij}(x)\|Dy\|^{p-2} D_j y \in \mathbf{C}^{1+\nu}(\Omega(r_0), R)$ which satisfies Eq.(1.1) for all $x \in \Omega(r_0)$. Regarding the question of existence of solutions of Eq.(1.1), we refer the reader to the monograph [2]. In what follows, our attention is restricted to those solutions which don't vanish identically in any neighborhood of ∞. The oscillation is considered in the usual sense, i.e., a solution y of Eq.(1.1) is said to be oscillatory if it has arbitrarily large zeros, i.e., the set $\{x \in \mathbb{R}^N : y(x) = 0\}$ is unbounded. Equation (1.1) is said to be oscillatory if every solution (if any exists) is oscillatory. Conversely, Equation (1.1) is nonoscillatory if there exists a solution which is not oscillatory.

The partial differential equation (PDE) with $p-$Laplacian have applications in various physical and biological problems – in the study of non-Newtonian fluids, in the glaciology and slow diffusion problems. For more detailed discussion about applications of PDE with $p-$Laplacian, see [2].

The special cases of equation (1.1) are the undamped half-linear PDE

$$\operatorname{div}(\|Dy\|^{p-2}Dy) + c(x)|y|^{p-2}y = 0, \tag{1.2}$$

and the damped half-linear PDE

$$\operatorname{div}(\|Dy\|^{p-2}Dy\,) + \langle\, b(x), \|Dy\|^{p-2}Dy\rangle + c(x)|y|^{p-2}y = 0. \tag{1.3}$$

In the qualitative theory of nonlinear PDE, one of important themes is to determine whether or not solutions of the equation under consider are oscillatory. In the last decade, the oscillation of Eq.(1.2) has received much attention and extensively studied by many authors, see, eg., [3], [5]-[8], [11], [13]-[16] and the references therein. Particularly, in [11], Usami used the Riccati technique due to Noussair and Swanson [10], and extended Fite-Wintner [4], [12] to Eq.(1.2). Very recently, by using Riccati inequality method, Mařík [9] has obtained some oscillation results for Eq.(1.3), which seems to be the first paper to study the oscillation of Eq.(1.3). However, those known results can not be applied to Eq.(1.1).

Motivated by the ideas of Mařík [9], Noussair and Swanson [10], Usami [11], in this paper, by using the weighted averaging techniques in Coles [1], we establish some oscillation criteria for Eq.(1.1). Particularly, when $p = 2$ in Eq.(1.1), we give a sharp oscillation criterion for it. The results obtained here include and improve the ones in [10], [11] and [15], and are essentially new ever for Eqs. (1.2) and (1.3).

2. Notations and lemmas

The following notations will be used throughout this paper. Let $\Phi(r, r_0)$ denote the class of all positive and locally integrable functions but not integrable functions which contains all bounded functions for all $r \geq r_0$. For arbitrary given functions $\phi \in \Phi(r, r_0)$ and $\theta \in \mathbf{C}([r_0, \infty), \mathbb{R})$, we define

$$\begin{aligned}
C_1(x) &= c(x) - \frac{1}{p}\Big(\frac{2}{\varepsilon q}\Big)^{p-1} \frac{\|A\|^p}{\lambda_{\min}^{p-1}(x)} \|b(x)A^{-1}\|^p, \\
C_2(x) &= c(x) - \frac{1}{2\varepsilon}\lambda_{\max}(x)\|b(x)A^{-1}\|^2, \\
C_{Mi}(r) &= \int_{S_r} C_i(x)d\sigma, \quad i = 1, 2, \quad \alpha(r, b) = \int_b^r \phi(s)ds, \\
Q(r, b; \theta) &= \frac{1}{\alpha(r, b)} \int_b^r \phi(s) \int_b^r \theta(u)\, du\, ds,
\end{aligned}$$

where $\lambda_{\max}(x)$ and $\lambda_{\min}(x)$ denote the largest and smallest eigenvalue of the matrix A, respectively, $\|A\| = [\sum_{i,j=1}^{N} a_{ij}^2(x)]^{1/2}$ denotes the norm of the matrix A, $S_r = \{x \in \mathbb{R}^N : \|x\| = r\}$, σ represents the measure on S_r, and q denotes the conjugate number to p, i.e., $1/p + 1/q = 1$.

Members of the set Φ will be called weighted functions [1].

Before we state our main theorems, we need the following technical lemmas. The first one is new, the second is a modified version of Lemma 1 in [10] for semiliner equations.

Lemma 2.1. *Suppose that* (1.1) *has a nonoscillatory solution $y = y(x) \neq 0$ for all $x \in \Omega(r_1)$, ($r_1 \geq r_0$). Then the N-dimensional vector Riccati function $W(x)$ is well defined on $\Omega(r_1)$ by*

$$W(x) = (W_i(x))_{i=1}^N, \quad W_i(x) = \frac{1}{f(y)}\Big(\sum_{j=1}^{N} a_{ij}(x)\|Dy\|^{p-2}D_j y\Big), \tag{2.1}$$

and satisfies the following partial Riccati-type inequality

$$\operatorname{div} W(x) \leq -C_1(x) - \frac{\varepsilon}{2}\frac{\lambda_{\min}(x)}{\|A\|^q}\|W(x)\|^q. \tag{2.2}$$

Proof. Without loss of generality let us consider that $y = y(x) > 0$ on $\Omega(r_1)$. Differentiation of $W_i(x)$ with respect to x_i gives

$$D_i W_i(x) = -\frac{f'(y)}{f^2(y)} D_i y\Big(\sum_{j=1}^{N} a_{ij}(x)\|Dy\|^{p-2}D_j y\Big) + \frac{1}{f(y)} D_i\Big(\sum_{j=1}^{N} a_{ij}(x)\|Dy\|^{p-2}D_j y\Big),$$

for $i = 1, \cdots, N$. Summation over i and use of (1.1) lead to

$$\mathrm{div} W(x) = -c(x) - \frac{f'(y)}{f^2(y)} \|Dy\|^{p-2} (Dy)^T A(Dy) - \Big\langle b(x), \frac{\|Dy\|^{p-2} Dy}{f(y)} \Big\rangle. \tag{2.3}$$

Note that

$$\|W(x)\| \le \frac{1}{f(y)} \|A\| \|Dy\|^{p-1}, \quad (Dy)^T A(Dy) \ge \lambda_{\min}(x) \|Dy\|^2.$$

This, together (2.3) with (A3), implies that

$$\begin{aligned} \mathrm{div} W(x) \quad &\le \quad -c(x) - \frac{f'(y)}{|f(y)|^{(p-2)/(p-1)}} \\ &\qquad \times \frac{\lambda_{\min}(x)}{\|A\|^q} \|W(x)\|^q - \langle\, b(x) A^{-1},\, W(x)\, \rangle \\ &\le \quad -c(x) - \frac{\varepsilon \lambda_{\min}(x)}{\|A\|^q} \|W(x)\|^q - \langle\, b(x) A^{-1}, W(x)\, \rangle. \end{aligned} \tag{2.4}$$

Application of Young's inequality yields

$$\begin{aligned} &\frac{\varepsilon \lambda_{\min}(x)}{\|A\|^q} \|W(x)\|^q + \langle\, b(x) A^{-1}, W(x)\, \rangle \\ = \;& \frac{\varepsilon q}{2} \frac{\lambda_{\min}(x)}{\|A\|^q} \Big[\frac{1}{q} \|W(x)\|^q + \frac{2}{\varepsilon q} \frac{\|A\|^q}{\lambda_{\min}(x)} \langle\, b(x) A^{-1}, W(x)\, \rangle + \frac{1}{q} \|W(x)\|^q \Big] \\ \ge \;& -\frac{1}{p} \Big(\frac{2}{\varepsilon q} \Big)^{p-1} \frac{\|A\|^p}{\lambda_{\min}^{p-1}(x)} \|b(x) A^{-1}\|^p + \frac{\varepsilon}{2} \frac{\lambda_{\min}(x)}{\|A\|^q} \|W(x)\|^q. \end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5), we get that (2.2) holds. □

Let $p = 2$ in Eq.(1.1). Eq.(1.1) reduces the following damped semilinear equation

$$\sum_{i,\, j=1}^{N} D_i[\, a_{ij}(x) D_j y\,] + \langle b(x), Dy\, \rangle + c(x) f(y) = 0. \tag{2.6}$$

Lemma 2.2. *Suppose that* (2.6) *has a nonoscillatory solution $y = y(x) \ne 0$ for all $x \in \Omega(r_1)$, $(r_1 \ge r_0)$. Then the N-dimensional vector Riccati function $\overline{W}(x)$ is well defined on $\Omega(r_1)$ by*

$$\overline{W}(x) = (\overline{W_i}(x))_{i=1}^N, \quad \overline{W_i}(x) = \frac{1}{f(y)} \Big(\sum_{j=1}^{N} a_{ij}(x) D_j y \Big), \tag{2.7}$$

and satisfies the following partial Riccati-type inequality

$$\mathrm{div}\, \overline{W}(x) \le -C_2(x) - \frac{\varepsilon}{2\lambda_{\max}(x)} \|\overline{W}(x)\|^2. \tag{2.8}$$

Proof. Similar to the proof of Lemma 2.1, we get

$$\operatorname{div}\overline{W}(x) = -c(x) - f'(y)(\overline{W}^T A^{-1}\overline{W})(x) - \Big\langle b(x), \frac{Dy}{f(y)}\Big\rangle. \tag{2.9}$$

Note that

$$(\overline{W}^T A^{-1}\overline{W})(x) \geq \frac{1}{\lambda_{\max}(x)}\|\overline{W}(x)\|^2,$$

this, together (2.9) with (A3), implies that

$$\operatorname{div}\overline{W}(x) \leq -c(x) - \frac{\varepsilon}{\lambda_{\max}(x)}\|\overline{W}(x)\|^2 - \langle b(x)A^{-1}, \overline{W}(x)\rangle. \tag{2.10}$$

By Young's inequality,

$$\begin{aligned} &\frac{\varepsilon}{\lambda_{\max}(x)}\|\overline{W}(x)\|^2 + \langle\, b(x)A^{-1}, \overline{W}(x)\,\rangle \qquad (2.11)\\ = \;& \frac{\varepsilon}{\lambda_{\max}(x)}\Big[\frac{1}{2}\|\overline{W}(x)\|^2 + \frac{\lambda_{\max}(x)}{\varepsilon}\langle\, b(x)A^{-1}, \overline{W}(x)\,\rangle + \frac{1}{2}\|\overline{W}(x)\|^2\Big]\\ \geq \;& \frac{\varepsilon}{2\lambda_{\max}(x)}\|\overline{W}(x)\|^2 - \frac{\lambda_{\max}(x)}{2\varepsilon}\|b(x)A^{-1}\|^2. \end{aligned}$$

Combining (2.10) and (2.11), we get that (2.8) holds. □

3. Main results

Theorem 3.1. *Suppose that there exist $\varphi \in \mathbf{C}^1([r_0,\infty),\mathbb{R}^+)$, $\phi \in \Phi(r, r_0)$, $\lambda \in \mathbf{C}([r_0,\infty),\ \mathbb{R}^+)$, and $l > 1$ such that*

$$\min_{\|x\|=r}\frac{\lambda_{\min}(x)}{\|A\|^q} \geq \lambda(r), \quad r \geq r_0, \tag{3.1}$$

$$\int_{r_0}^{\infty}\frac{\phi(s)\,\alpha^{\mu}(s,b)}{\beta_1(s,b)}ds = \infty, \quad 0 \leq \mu < q-1, \tag{3.2}$$

and

$$\lim_{r\to\infty} Q(r, b; \Theta_1) = \infty, \quad b \geq r_0, \tag{3.3}$$

where

$$h_1(r) = \frac{\varepsilon}{2}\lambda(r)[\omega_N r^{N-1}\varphi(r)]^{1/(1-p)}, \quad \beta_1(r,b) = \left(\int_b^r \phi^p(s)h_1^{1-p}(s)ds\right)^{1/(p-1)},$$

$$\Theta_1(r) = \varphi(r)C_{M1}(r) - \frac{1}{p}\Big(\frac{l}{q}\Big)^{p-1}\Big|\frac{\varphi'(r)}{\varphi(r)}\Big|^p h_1^{1-p}(r),$$

and $\omega_N = \int_{S_1} d\sigma = 2\pi^{N/2}/\Gamma(N/2)$ *denotes the surface measure of unit sphere. Then Eq.*(1.1) *is oscillatory.*

Proof. Let $y = y(x)$ be a nonoscillatory solution of Eq.(1.1). Without loss of generality we assume that $y(x) > 0$ for $x \in \Omega(r_1)$ for some sufficient large $r_1 \geq r_0$. Hence the $N-$dimensional vector Riccati function $W(x)$ is well defined on $\Omega(r_1)$ by (2.1). It follows from Lemma 2.1 that (2.2) holds. Put

$$Z(r) = \varphi(r) \int_{S_r} \langle W(x), \upsilon(x) \rangle d\sigma, \tag{3.4}$$

where $\upsilon(x) = x/\|x\|$, $x \neq 0$, denotes the outward unit normal. By the Green formula in (3.4), noting that (2.2) and (3.1), we have

$$\begin{aligned} Z'(r) &= \frac{\varphi'(r)}{\varphi(r)} Z(r) + \varphi(r) \int_{S_r} \mathrm{div} W(x) d\sigma \\ &\leq \frac{\varphi'(r)}{\varphi(r)} Z(r) - \varphi(r) \Big[\int_{S_r} C_1(x) d\sigma + \frac{\varepsilon}{2} \lambda(r) \int_{S_r} \|W(x)\|^q d\sigma \Big]. \end{aligned} \tag{3.5}$$

By Hölder's inequality,

$$\begin{aligned} |Z(r)| &\leq \varphi(r) \int_{S_r} \|W(x)\| \, \|\upsilon(x)\| d\sigma \\ &\leq \varphi(r) \Big(\int_{S_r} d\sigma \Big)^{1/p} \Big(\int_{S_r} \|W(x)\|^q d\sigma \Big)^{1/q} \\ &= \varphi(r) \Big(\omega_N r^{N-1} \Big)^{1/p} \Big(\int_{S_r} \|W(x)\|^q d\sigma \Big)^{1/q}, \end{aligned}$$

and equivalently,

$$\int_{S_r} \|W(x)\|^q d\sigma \geq \varphi^{-q}(r) (\omega_N r^{N-1})^{1/(1-p)} |Z(r)|^q,$$

which, together with (3.5), implies that

$$\begin{aligned} Z'(r) &\leq -\varphi(r) C_{M1}(r) + \frac{\varphi'(r)}{\varphi(r)} Z(r) \\ &\quad - \frac{\varepsilon}{2} \lambda(r) \Big[\omega_N r^{N-1} \varphi(r) \Big]^{1/(1-p)} |Z(r)|^q \\ &= -\varphi(r) C_{M1}(r) + \frac{\varphi'(r)}{\varphi(r)} Z(r) - h_1(r) |Z(r)|^q. \end{aligned} \tag{3.6}$$

By Young's inequality,

$$\begin{aligned} \frac{\varphi'(r)}{\varphi(r)} Z(r) - h_1(r)|Z(r)|^q &\leq \frac{q\, h_1(r)}{l} \Big[\frac{l}{q\, h_1(r)} \Big| \frac{\varphi'(r)}{\varphi(r)} \Big| |Z(r)| - \frac{1}{q} |Z(r)|^q - \frac{l}{l^* q} |Z(r)|^q \Big] \\ &\leq \frac{1}{p} \left(\frac{l}{q} \right)^{p-1} \Big| \frac{\varphi'(r)}{\varphi(r)} \Big|^p h_1^{1-p}(r) - \frac{1}{l^*} h_1(r) |Z(r)|^q, \end{aligned}$$

where l^* is the conjugate number to l, i.e., $1/l + 1/l^* = 1$, which, together with (3.6), follows that

$$
\begin{aligned}
(3.7)\quad Z'(r) &\leq -\varphi(r)C_{M1}(r) + \frac{1}{p}\Big(\frac{l}{q}\Big)^{p-1}\Big|\frac{\varphi'(r)}{\varphi(r)}\Big|^p h_1^{1-p}(r) - \frac{1}{l^*}h_1(r)|Z(r)|^q \\
&= -\Theta_1(r) - \frac{1}{l^*}h_1(r)|Z(r)|^q.
\end{aligned}
$$

Integrating (3.7) from b to r, $r \geq b \geq r_1$, we have

$$Z(r) + \frac{1}{l^*}\int_b^r h_1(s)|Z(s)|^q ds \leq Z(b) - \int_b^r \Theta_1(s)ds.$$

Multiplying the above by $\phi(r)$ and integrating it from b to r, we get

$$\int_b^r \phi(s)Z(s)ds + \frac{1}{l^*}\int_b^r \phi(s)\int_b^s h_1(u)|Z(u)|^q\, duds \leq \Big[Z(b) - Q(r,b;\Theta_1)\Big]\alpha(r,b).$$

Note that (3.3), there exists a $b_1 \geq b$ such that

$$Z(b) - Q(r,b;\Theta_1) < 0 \quad \text{for all} \quad r \geq b_1.$$

Thus, for every $r \geq b_1$,

$$H(r) := \frac{1}{l^*}\int_b^r \phi(s)\int_b^s h_1(u)|Z(u)|^q duds \leq -\int_b^r \phi(s)Z(s)ds.$$

Since H is nonnegative, then

$$H^q(r) \leq \Big(\int_b^r \phi(s)|Z(s)|ds\Big)^q, \quad r \geq b_1. \tag{3.8}$$

By the Hölder inequality,

$$
\begin{aligned}
(3.9)\qquad & \Big(\int_b^r \phi(s)|Z(s)|\, ds\Big)^q \\
\leq\ & \Big(\int_b^r \phi^p(s)h_1^{1-p}(s)\, ds\Big)^{1/(p-1)}\Big(\int_b^r h_1(s)|Z(s)|^q\, ds\Big) \\
=\ & l^*\Big(\int_b^r \phi^p(s)h_1^{1-p}(s)ds\Big)^{1/(p-1)}\frac{H'(r)}{\phi(r)}.
\end{aligned}
$$

Therefore, for all $r \geq b_1$,

$$
\begin{aligned}
H(r) &= \frac{1}{l^*}\int_b^r \phi(s)\int_b^s h_1(u)|Z(u)|^q\, du\, ds \\
&\geq \frac{1}{l^*}\Big(\int_b^{b_1} h_1(s)|Z(s)|^q\, ds\Big)\Big(\int_b^r \phi(s)\, ds\Big) \\
&= \frac{C}{l^*}\alpha(r,b).
\end{aligned}
$$

where $C = \int_b^{b_1} h_1(s)|Z(s)|^q\,ds$. Form (3.8) and (3.9), for all $r \geq b_1$ and some μ, $0 \leq u < q-1$, we get

$$C^\mu \phi(r)\Big(\int_b^r \phi^p(u) h_1^{1-p}(u)\,du\Big)^{1/(1-p)} \alpha^\mu(r,b) \leq (l^*)^{\mu+1} H^{\mu-q}(s) H'(r).$$

Integrating the above from b_1 to r, we obtain

$$C^\mu \int_{b_1}^r \frac{\phi(s)\alpha^\mu(s,b)}{\beta_1(s,b)} ds \leq \frac{(l^*)^{\mu+1}}{q-1-\mu} \frac{1}{H^{q-1-\mu}(b_1)} < \infty, \quad 0 \leq \mu < q-1,$$

which contradicts condition (3.2). □

Corollary 3.1. *Assume that there exists* $\varphi \in \mathbf{C}^1([r_0,\infty),\mathbb{R}^+)$ *such that*

$$\int_{r_0}^\infty h_1(s)\,ds = \int_{r_0}^\infty \Theta_1(s)\,ds = \infty. \tag{3.10}$$

Then Eq.(1.1) *is oscillatory.*

Proof. Let $\phi(r) = h_1(r)$. It is easy to show that (3.2) and (3.3) hold from (3.10). Hence, by Theorem 3.1, Eq.(1.1) is oscillatory . □

Remark 3.1. Corollary 3.1 improves Theorem 4 in [11] and Theorem 3.1 in [15].

Next, for Eq. (2.6), we should establish a sharp oscillation theorem which improve the main results in [10] for undamped semilinear equations.

Theorem 3.2. *Suppose that there exist* $\phi \in \Phi(r,a)$, $\eta \in \mathbf{C}^1([r_0,\infty),\mathbb{R}^+)$, $\lambda \in \mathbf{C}([r_0,\infty),\ \mathbb{R}^+)$ *such that*

$$\lambda(r) \geq \max_{\|x\|=r} \lambda_{\max}(x), \quad r \geq r_0, \tag{3.11}$$

$$\int_{r_0}^\infty \frac{\phi(s)\,\alpha^\mu(s,b)}{\beta_2(s,b)} ds = \infty, \quad 0 \leq \mu < 1, \tag{3.12}$$

and

$$\lim_{r\to\infty} Q(r,b;\Theta_2) = \infty, \quad b \geq r_0, \tag{3.13}$$

where

$$\begin{aligned} h_2(r) &= \frac{\varepsilon\, r^{1-N}}{2\omega_N \lambda(r)\psi(r)} \quad \beta_2(r,b) = \int_b^r \frac{\phi^2(s)}{h_2(s)} ds, \\ \psi(r) &= \exp\Big[-\frac{\varepsilon}{\omega_N}\int_{r_0}^r \frac{\eta(s)s^{1-N}}{\lambda(s)} ds\Big], \\ \Theta_2(r) &= \psi(r)\Big[C_{M2}(r) + \frac{\varepsilon\,\eta^2(r) r^{1-N}}{2\omega_N\,\lambda(r)} - \eta'(r)\Big]. \end{aligned}$$

Then Eq.(2.6) *is oscillatory.*

Proof. Let $y = y(x)$ be a nonoscillatory solution of Eq.(1.1). Without loss of generality we assume that $y(x) > 0$ for $x \in \Omega(r_1)$ for some sufficient large $r_1 \geq r_0$. Hence the $N-$dimensional vector Riccati function $\overline{W}(x)$ is well defined on $\Omega(r_1)$ by (2.7). It follows from lemma 2.2 that (2.8) holds. Define

$$Z(r) = \psi(r)\Big[\int_{S_r} \langle \overline{W}(x), \upsilon(x) \rangle d\sigma + \eta(r)\Big]. \tag{3.14}$$

Using Green's formula in (3.14), and in view of (2.8) and (3.11), we get

$$\begin{aligned} Z'(r) &= \frac{\psi'(r)}{\psi(r)} Z(r) + \psi(r)\Big\{\int_{S_r} \mathrm{div}\overline{W}(x)d\sigma + \eta'(r)\Big\} \\ &\leq \frac{\psi'(r)}{\psi(r)} Z(r) + \psi(r)\Big[-\int_{S_r} C_2(x)d\sigma - \frac{\varepsilon}{2\lambda(r)}\int_{S_r} \|W(x)\|^2\, d\sigma + \eta'(r)\Big] \\ &= -\psi(r)[C_{M2}(r) - \eta'(r)] + \frac{\psi'(r)}{\psi(r)} Z(r) - \frac{\varepsilon\, \psi(r)}{2\lambda(r)} \int_{S_r} |W(x)|^2\, d\sigma. \end{aligned} \tag{3.15}$$

The Schwartz inequality follows that

$$\int_{S_r} |\overline{W}(x)|^2 d\sigma \geq \frac{r^{1-N}}{\omega_N}\Big[\int_{S_r} \langle \overline{W}(x), \upsilon(x) \rangle\, d\sigma\Big]^2,$$

which, together with (3.15), implies that

$$\begin{aligned} Z'(r) &\leq -\psi(r)\Big[C_{M2}(r) - \eta'(r)\Big] + \frac{\psi'(r)}{\psi(r)} Z(r) - \frac{\varepsilon\, \psi(r) r^{1-N}}{2\,\omega_N \lambda(r)}\Big[\int_{S_r} \langle \overline{W}(x), \upsilon(x)\rangle d\sigma\Big]^2 \\ &= -\psi(r)\Big[C_{M2}(r) - \eta'(r)\Big] + \frac{\psi'(r)}{\psi(r)} Z(r) - \frac{\varepsilon\, \psi(r) r^{1-N}}{2\omega_N \lambda(r)}\Big[\frac{Z(r)}{\psi(r)} - \eta(r)\Big]^2 \\ &= -\Theta_2(r) - h_2(r) Z^2(r). \end{aligned}$$

The remainder of the proof is similar to that of Theorem 3.1 and omit the details.
□

Corollary 3.2. *Assume that there exists* $\varphi \in \mathbf{C}^1([r_0, \infty), \mathbb{R}^+)$ *such that*

$$\int_{r_0}^{\infty} h_2(s)\, ds = \int_{r_0}^{\infty} \Theta_2(s) ds = \infty. \tag{3.16}$$

Then Eq.(2.6) *is oscillatory.*

Proof. Let $\phi(r) = h_2(r)$. It is easy to show that (3.12) and (3.13) hold from (3.16), and therefore Eq.(2.6) is oscillatory from Theorem 3.2. □

Remark 3.2. Corollary 3.2 improves Theorem 4 in [10].

Corollary 3.3. *Assume that there exists* $\varphi \in \mathbf{C}^1([r_0,\infty),\mathbb{R}^+)$ *such that*

$$\lim_{r\to\infty}\frac{1}{r}\int_{r_0}^{r}\int_{r_0}^{s}\Theta_2(u)duds=\infty, \tag{3.17}$$

and

$$\lim_{r\to\infty}\frac{1}{r}\int_{r_0}^{r}\frac{ds}{h_2(s)}=\gamma>0. \tag{3.18}$$

Then Eq.(2.6) *is oscillatory.*

Proof. Let $\phi(r)=1$. It follows that Eq.(2.6) is oscillatory from Theorem 3.2. □

For illustration, we consider the following two examples.

Example 3.1. Consider Eq.(1.1) with

$$\begin{aligned} A(x) &= \operatorname{diag}\Big(\frac{1}{\|x\|},\frac{1}{\|x\|}\Big), b(x)=\Big(\frac{1}{\|x\|^2},\frac{1}{\|x\|^2}\Big),\\ c(x) &= \frac{2+\cos\|x\|-2\|x\|\sin\|x\|}{4\|x\|^{5/2}},\quad f(y)=|y|y, \end{aligned} \tag{3.19}$$

for $x\in\Omega(\pi/2)$, where $N=2$, $p=3$. It is easy to see that

$$\lambda_{\min}(x)=\frac{1}{\|x\|},\quad \|A\|=\frac{\sqrt{2}}{\|x\|},\quad \lambda(r)=2^{-3/4}\sqrt{r},\quad \varepsilon=2,$$

$$C_1(x)=\frac{2+\cos\|x\|-2\|x\|\sin\|x\|}{4\|x\|^{5/2}}-\frac{32}{27\|x\|^4},$$

and

$$C_{M1}(r)=\frac{\pi(2+\cos r-2r\sin r)}{2\,r^{3/2}}-\frac{64\pi}{27r^3}.$$

Let $\varphi(r)=r$, $l=3$. Thus, for $r\geq\pi/2$

$$h_1(r)=\frac{1}{2^{5/4}\sqrt{\pi\,r}},\quad \Theta_1(r)=\frac{\pi(2+\cos r-2r\sin r)}{2\sqrt{r}}-\frac{4\pi(4+9\sqrt{2})}{27r^2}.$$

Clearly,

$$\int_{\pi/2}^{\infty}h_1(s)ds=\infty,$$

and

$$\int_{\pi/2}^{r}\Theta_1(s)ds=\pi\Big[\sqrt{r}(2+\cos r)-\sqrt{2\pi}\Big]+\frac{4\pi(4+9\sqrt{2})}{27}\Big(\frac{1}{r}-\frac{2}{\pi}\Big)\to\infty,\quad \text{as}\quad r\to\infty.$$

Hence, all conditions of Corollary 3.1 are satisfied and hence Eq.(3.19) is oscillatory.

Example 3.2. Consider Eq.(2.6) with

$$
\begin{aligned}
A(x) &= \operatorname{diag}(1,1), \quad b(x) = \Big(\frac{1}{\|x\|^2}, \frac{1}{\|x\|^2}\Big), \\
c(x) &= \frac{1+\|x\|\sin\|x\|}{\|x\|}, \quad f(y) = y + y^3,
\end{aligned} \tag{3.20}
$$

for $x \in \Omega(1)$, where $N = 2$. It is easy to see that

$$\lambda_{\max}(\|x\|) = 1, \quad \|A\| = \sqrt{2}, \quad \lambda(r) = 1, \quad \varepsilon = 1.$$

and

$$C_2(x) = \frac{1+\|x\|\sin\|x\|}{\|x\|} - \frac{1}{\|x\|^4}, \quad C_{M2}(r) = 2\pi(1 + r\sin r) - \frac{2\pi}{r^3}.$$

Let $\eta(r) = 2\pi$. Thus, for $r \geq 1$,

$$\psi(r) = \frac{1}{r}, \quad h_2(r) = \frac{1}{4\pi}, \quad \Theta_2(r) = \frac{2\,\pi(1 + r\sin r)}{r} + \frac{\pi}{r^2} - \frac{2\pi}{r^4}.$$

Clearly,

$$\int_1^\infty h_2(s)ds = \int_1^\infty \Theta_2(s)ds = \infty.$$

Thus, all conditions of Corollary 3.2 are satisfied and hence Eq.(3.20) is oscillatory.

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