KYUNGPOOK Math. J. 47(2007), 227-237

Weak Distributive n-Semilattices and n-Lattices

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Abstract. We define weak distributive n-semilattices and n-lattices, using variants of the absorption law and those of the distributive law. From a weak distributive n-semilattice, we construct direct system of subalgebras which are weak distributive n-lattices and show that its direct limit is a reflection of the category **wD*n*-SLatt** of the weak distributive n-semilattices.

1. Introduction

A *semilattice* is an algebra, $S = (S, \vee)$, with one binary operation $\vee$ that is idempotent, commutative and associative, that is, the following identities hold in S:

$$\begin{aligned} x \vee x &= x && \text{(idempotence)}, \\ x \vee y &= y \vee x && \text{(commutativity)}, \\ (x \vee y) \vee z &= x \vee (y \vee z) && \text{(associativity)}. \end{aligned}$$

An algebra $(B, \vee, \wedge)$ with two binary operations $\vee$ and $\wedge$ is called a *bisemilattice* if both of its reducts $(B, \vee)$ and $(B, \wedge)$ are semilattices. This notion was introduced by J. Plonka in [8] under the name *quasilattice*. However, it is called *bisemilattice* by other author ([3], [6], [7]). In particular, a bisemilattice is *distributive* if it satisfies the following two distributivity:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Plonka has generalized distributive bisemilattice to distributive n-semilattice and distributive n-lattice([9]). *A distributive n-semilattice* (S, F) which is an algebra with a family $F = \{\circ_i \mid i \in [n]\}$ of n binary semilattice operations on a common set S in which each pair of semilattice operations satisfy both distributive laws. A distributive n-semilattice (S, F) is called a *distributive n-lattice* if it satisfies

Received January 16, 2006.
2000 Mathematics Subject Classification: 06A12, 06B99, 18A30, 18A40.
Key words and phrases: n-semilattice, n-lattice, weak distributive, generalized absorption law, reflection, direct limit.

moreover the following *generalized absorption law* for the sequence $I = (1, 2, \cdots, n)$ of indices of $F = \{\circ_i \mid i \in [n]\}$

$$a \circ_1 (a \circ_2 (\cdots (a \circ_{n-1} (a \circ_n b)) \cdots)) = a.$$

In 1971, R. Padmanbhan define a *weak distributive bisemilattice*, which is a bisemilattice satisfying the weak distributivity (it was studied under the name quasilattice in [7]):

$$((a \wedge b) \vee c) \wedge (b \vee c) = (a \wedge b) \vee c \text{ and } ((a \vee b) \wedge c) \vee (b \wedge c) = (a \vee b) \wedge c.$$

In this paper, we are concerned with categorical properties of certain algebras which we call *weak distributive n-semilattices*. These algebras generalize weak distributive bisemilattices. A *weak distributive n-semilattice* is an algebra with a family of n binary semilattice operations on a common underlying set which are mutually weak distributive. A weak distributive n-semilattice will be called a *weak distributive n-lattice*, if it satisfies the generalized absorption law, which generalizes the absorption law for lattices. Furthermore, weak distributive n-semilattices (or weak distributive n-lattices) generalize distributive n-semilattices (or distributive n-lattices, respectively). We show that every weak distributive n-semilattices has a partition consisting of weak distributive n-lattices and then the family of week distributive n-lattices in the partition forms a direct system in the category **wDn-Latt** of weak distributive n-lattices and homomorphisms. Furthermore, we prove that its direct limit gives rise to the reflection. For the terminology not introduced in the paper, we refer to [1] for the category theory, [2] for the ordered sets and [4], [5] for the abstract algebra.

2. Weak distributive n-semilattices

Let us start with a definition of weak distributive n-semilattice which is a generalization of both weak distributive bisemilattice and distributive n-semilattice.

Definition 2.1. An algebra $W = (W, F)$ is called a *weak distributive n-semilattice* if it has a family $F = \{\circ_i \mid i \in [n]\}$ consisting of n binary operations which satisfy the following equations for any $i, j \in I$:

$$\begin{aligned} a \circ_i a &= a && (\textit{idempotence}), \\ a \circ_i b &= b \circ_i a && (\textit{commutativity}), \\ (a \circ_i b) \circ_i c &= a \circ_i (b \circ_i c) && (\textit{associativity}), \\ ((a \circ_i b) \circ_j c) \circ_i (b \circ_j c) &= (a \circ_i b) \circ_j c && (\textit{weak distributivity}). \end{aligned}$$

A weak distributive n-semilattice is called a *weak distributive n-lattice* if it satisfies the generalized absorption law:

$$(*) \qquad a \circ_{\sigma(1)} \left(a \circ_{\sigma(2)} \left(\cdots \left(a \circ_{\sigma(n-1)} \left(a \circ_{\sigma(n)} b\right)\right) \cdots\right)\right) = a$$

for any permutation $\sigma \in Sym(n)$.

In the case $n = 2$, it is clear that a weak distributive n-semilattice is a weak distributive bisemilattice and a weak distributive n-lattice is a lattice. In a weak distributive n-lattice, the condition $(*)$ can be reduced to the condition

$$a \circ_1 (a \circ_2 (\cdots (a \circ_{n-1} (a \circ_n b)) \cdots)) = a,$$

because it can be easily shown by the weak distributivity. A distributive n-semilattice (or n-lattice) is a weak distributive n-semilattice (or n-lattice, respectively). But a weak distributive n-semilattice (or n-lattice) need not be a distributive n-semilattice (or n-lattices, respectively).

From now on, an n-semilattice $W = (W, F)$ with a family $F = \{\circ_i \mid i \in [n]\}$ of n semilattice operations will be denoted by $W = (W, F)$ or W, simply.

Remark 2.2.

(1) It is easy to see that an n-semilattice $W = (W, F)$ is weak distributive if and only if $a \circ_i b = b$ implies $(a \circ_j c) \circ_i (b \circ_j c) = b \circ_j c$ for any $j \in [n]$ and any $c \in W$.

(2) Let (W, F) be a weak distributive n-semilattice. If $a \circ_i b = a$ and $c \circ_i d = c$, then for any $j \in [n]$, we have by (1),

$$(a \circ_j c) \circ_i (b \circ_j d) = a \circ_j c.$$

Now we obtain some properties of weak distributive n-semilattices and n-lattices, which will be needed in the formation of the direct system in the category **wDn-Latt** of weak distributive n-lattices and homomorphisms.

Lemma 2.3. *Let $W = (W, F)$ be a weak distributive n-semilattice. Then for any $i, j \in [n]$, the following equations hold.*

$$\begin{aligned}
&(1) & a \circ_i (b \circ_j a) &= (a \circ_i b) \circ_j a, \\
&(2) & a \circ_i (a \circ_j b) \circ_i (c \circ_j b) &= a \circ_i (c \circ_j b), \\
&(3) & a \circ_i (a \circ_j b) \circ_i (a \circ_j b \circ_j c) &= a \circ_i (a \circ_j b \circ_j c), \\
&(4) & a \circ_i b \circ_i (a \circ_j c) &= a \circ_i b \circ_i (b \circ_j c), \\
&(5) & a \circ_i (a \circ_j (b \circ_i c)) &= a \circ_i (a \circ_j b) \circ_i (a \circ_j c), \\
&(6) & a \circ_i (a \circ_j (b \circ_i (b \circ_j c))) &= a \circ_i (a \circ_j (c \circ_i (c \circ_j b))), \\
&(7) & a \circ_i (a \circ_j (b \circ_i (b \circ_j c))) &= a \circ_i (a \circ_j b) \circ_i ((a \circ_i (a \circ_j b)) \circ_j c), \\
&(8) & a \circ_i b \circ_i ((a \circ_i b) \circ_j c) &= a \circ_i (a \circ_j c) \circ_i b = a \circ_i b \circ_i (b \circ_j c), \\
&(9) & a \circ_i (a \circ_j (b \circ_i (b \circ_j (a \circ_i b)))) &= a \circ_i (a \circ_j b).
\end{aligned}$$

Proof. (1) It follows from the definition of weak distributive n-semilattice.
(2) From the associativity, weak distributivity and (2) of Remark 2.2, we have

$$\begin{aligned} a \circ_i (a \circ_j b) \circ_i (c \circ_j b) &= ((c \circ_j b) \circ_i a) \circ_i (a \circ_j b) \\ &= (((c \circ_j b) \circ_i a) \circ_j (a \circ_i b)) \circ_i (a \circ_j b) \\ &= ((c \circ_j b) \circ_i a) \circ_j (b \circ_i a) = (c \circ_j b) \circ_i a. \end{aligned}$$

(3) Equation (3) follows from (2) by the substitution $b = a \circ_j b$.
(4) From (2),

$$\begin{aligned} a \circ_i b \circ_i (a \circ_j c) &= a \circ_i (b \circ_i (a \circ_j c)) \\ &= a \circ_i (b \circ_i (b \circ_j c) \circ_i (a \circ_j c)) \\ &= a \circ_i (b \circ_j c) \circ_i b \circ_i (a \circ_j c) \\ &= b \circ_i (a \circ_i (a \circ_j c) \circ_i (b \circ_j c)) \\ &= b \circ_i (a \circ_i (b \circ_j c)) \\ &= a \circ_i b \circ_i (b \circ_j c)\,. \end{aligned}$$

(5) Using (4) and the weak distributivity, we obtain

$$\begin{aligned} a \circ_i (a \circ_j b) \circ_i (a \circ_j c) &= a \circ_i (a \circ_j b) \circ_i (a \circ_j b \circ_j c) \\ &= a \circ_i ((a \circ_j b) \circ_j ((a \circ_j b) \circ_i c)) \\ &= a \circ_i ((a \circ_j b) \circ_j (b \circ_i c) \circ_j ((a \circ_j b) \circ_i c)) \\ &= a \circ_i ((a \circ_j b) \circ_j (b \circ_i c) \circ_j ((b \circ_i c) \circ_i c)) \\ &= a \circ_i ((a \circ_j b) \circ_j (b \circ_i c)) \\ &= a \circ_i (a \circ_j (b \circ_i c)) \circ_i ((a \circ_j b) \circ_j (b \circ_i c)) \\ &= a \circ_i (a \circ_j (b \circ_i c)) \circ_i (a \circ_j b \circ_j (b \circ_i c)) \\ &= a \circ_i (a \circ_j (b \circ_i c))\,. \end{aligned}$$

(6) Using (1) and (2), we have

$$\begin{aligned} a \circ_i (a \circ_j (b \circ_i (b \circ_j c))) &= a \circ_i (a \circ_j b \circ_j (b \circ_i c)) \\ &= a \circ_i (a \circ_j (b \circ_i c)) \circ_i (a \circ_j b \circ_j (b \circ_i c)) \\ &= ((a \circ_i b \circ_i c) \circ_j a) \circ_i (a \circ_j b \circ_j (b \circ_i c)) \\ &= a \circ_j (a \circ_i b \circ_i c)\,, \end{aligned}$$

and similarly, $a \circ_i (a \circ_j (c \circ_i (c \circ_j b))) = a \circ_j (a \circ_i b \circ_i c)$. It is easy to show that equation (7) hold using (1) and (5). Equations (8) and (9) follow from (4) by the substitution $b = a \circ_i b$ and (1), respectively. This completes the proof. □

Note that a weak distributive n-semilattice $W = (W, F)$ is an algebra of type n. Then we may denote the operations of W by $\circ_1, \circ_2, \cdots, \circ_n$. We observe that for any $k \in [n]$, there is a subsequence $K = (i_1, i_2, \cdots, i_k)$ of the sequence $I = (1, 2, \cdots, n)$.

In the following, we denote $a \circ_{i_1} \left(a \circ_{i_2} \left(\cdots \left(a \circ_{i_{k-1}} (a \circ_{i_k} b)\right) \cdots\right)\right)$ by $f_{i_1,i_2,\cdots,i_k}(a,b)$ or $f_K(a,b)$ for the convenience.

Lemma 2.4. *If $W = (W,F)$ is a weak distributive n-semilattice, then for any $i \in [n]$, we have the following equations*:

(1) $f_I(a, b \circ_i c) = f_I(a,b) \circ_i f_I(a,c)$ *and* $f_I(a \circ_i b, c) = f_I(a,c) \circ_i f_I(b,c)$,

(2) $f_I(a \circ_i b, a) = a \circ_i b = f_I(a \circ_i b, b)$,

(3) $f_K(f_K(a,b), c) = f_K(a, f_K(b,c)) = f_K(a, f_K(c,b))$ *for any nonempty subsequence K of I.*

Proof. (1) For any $i, k \in [n]$, we denote the subsequences $(1,2,\cdots,i)$ and $(1,2,\cdots,k-1,k+1,\cdots,i)$ of the sequence $I = (1,2,\cdots,n)$ by I_i and $I_i - \{k\}$, respectively. Using (5) and (8) of Lemma 2.3., we have

$$\begin{aligned}
f_I(a, b \circ_i c) &= f_{I_{n-1}}(a, a \circ_n (b \circ_i c)) \\
&= f_{I_{n-1}-\{i\}}(a, a \circ_i (a \circ_n (b \circ_i c))) \\
&= f_{I_{n-1}-\{i\}}(a, a \circ_i (a \circ_n b) \circ_i (a \circ_n c)) \\
&= f_{I_{n-2}}(a, (a \circ_{n-1} (a \circ_n b)) \circ_i (a \circ_{n-1} (a \circ_n c))) \\
&= f_{I_{n-2}}(a, f_{n-1,n}(a,b) \circ_i f_{n-1,n}(a,c)) \\
&= f_{I_{n-3}}(a, a \circ_{n-2} (f_{n-1,n}(a,b) \circ_i f_{n-1,n}(a,c))) \\
&= f_{I_{n-3}}(a, f_{n-2,n-1,n}(a,b) \circ_i f_{n-2,n-1,n}(a,c)) \\
&= a \circ_i f_{I-\{i\}}(a,b) \circ_i a \circ_i f_{I-\{i\}}(a,c) \\
&= f_I(a,b) \circ_i f_I(a,c)
\end{aligned}$$

and the second part is proved from (8) of Lemma 2.3 and idempotence ;

$$\begin{aligned}
f_I(a \circ_i b, c) &= f_{I_{n-1}-\{i\}}(a \circ_i b, (a \circ_i b) \circ_i ((a \circ_i b) \circ_n c)) \\
&= f_{I_{n-1}-\{i\}}(a \circ_i b, (a \circ_i b) \circ_i (a \circ_n c) \circ_i (a \circ_i b) \circ_i (b \circ_n c)) \\
&= f_{I_{n-1}}(a \circ_i b, (a \circ_n c) \circ_i (b \circ_n c)) \\
&= f_{I_{n-1}}(a \circ_i b, a \circ_n c) \circ_i f_{I_{n-1}}(a \circ_i b, b \circ_n c) \\
&= f_{I_{n-2}}(a \circ_i b, (a \circ_i b) \circ_{n-1} (a \circ_n c)) \circ_i f_{I_{n-2}}(a \circ_i b, (a \circ_i b) \circ_{n-1} (b \circ_n c)) \\
&= f_{I_{n-2}-\{i\}}(a \circ_i b, a \circ_i b \circ_i ((a \circ_i b) \circ_{n-1} (a \circ_n c))) \\
&\qquad \circ_i f_{I_{n-2}-\{i\}}(a \circ_i b, a \circ_i b \circ_i ((a \circ_i b) \circ_{n-1} (b \circ_n c))) \\
&= f_{I_{n-2}-\{i\}}(a \circ_i b, a \circ_i b \circ_i (a \circ_{n-1} (a \circ_n c))) \\
&\qquad \circ_i f_{I_{n-2}-\{i\}}(a \circ_i b, a \circ_i b \circ_i (b \circ_{n-1} (b \circ_n c))) \\
&= f_{I_{n-3}}(a \circ_i b, (a \circ_i b) \circ_{n-2} (a \circ_{n-1} (a \circ_n c))) \\
&\qquad \circ_i f_{I_{n-3}}(a \circ_i b, a \circ_i b \circ_{n-2} (b \circ_{n-1} (b \circ_n c)))
\end{aligned}$$

$$
\begin{aligned}
&= f_{I_{n-3}-\{i\}}\left(a \circ_i b, (a \circ_i b) \circ_i (a \circ_{n-2} (a \circ_{n-1} (a \circ_n c)))\right) \\
&\qquad\qquad \circ_i f_{I_{n-3}-\{i\}}\left(a \circ_i b, (a \circ_i b) \circ_i (b \circ_{n-2} (b \circ_{n-1} (b \circ_n c)))\right) \\
&\qquad\qquad\qquad \vdots \\
&= f_{1,i}\left(a \circ_i b, f_{I-\{1,i\}}(a,c)\right) \circ_i f_{1,i}\left(a \circ_i b, f_{I-\{1,i\}}(b,c)\right) \\
&= (a \circ_i b) \circ_i \left((a \circ_i b) \circ_1 f_{I-\{1,i\}}(a,c)\right) \\
&\qquad\qquad \circ_i (a \circ_i b) \circ_i \left((a \circ_i b) \circ_1 f_{I-\{1,i\}}(b,c)\right) \\
&= \left(a \circ_i b \circ_i a \circ_1 f_{I-\{1,i\}}(a,c)\right) \circ_i \left(a \circ_i b \circ_i b \circ_1 f_{I-\{1,i\}}(b,c)\right) \\
&= a \circ_i \left(a \circ_1 f_{I-\{1,i\}}(a,c)\right) \circ_i b \circ_i \left(b \circ_1 f_{I-\{1,i\}}(b,c)\right) \\
&= f_I(a,c) \circ_i f_I(b,c).
\end{aligned}
$$

(2) From the weak distributivity and idempotence, we have

$$
\begin{aligned}
f_I(a \circ_i b, b) &= f_{1,2,\cdots,n-1}(a \circ_i b, (a \circ_i b) \circ_n b) \\
&= f_{1,2,\cdots,i-1,i+1,\cdots,n}(a \circ_i b, (a \circ_i b) \circ_i b) \\
&= a \circ_i b.
\end{aligned}
$$

Interchange roles of a and b, $f_I(a \circ_i b, a) = a \circ_i b$ holds.
(3) First, we show that for any nonempty subsequence $K = (i_1, i_2, \cdots, i_k)$ of $(1, 2, \cdots, n)$,

$$
\begin{aligned}
f_{i_1,i_2,\cdots,i_k}(a, f_{i_1,i_2,\cdots,i_k}(b,c)) &= a \circ_{i_1} \left(a \circ_{i_2} \left(\cdots \left(a_{i_{k-1}} (a \circ_{i_k} b \circ_{i_k} c)\right) \cdots\right)\right) \\
&= f_{i_1,i_2,\cdots,i_k}(a, b \circ_{i_k} c).
\end{aligned}
$$

We use the induction on k. If $k = 2$, then by (5), (2) of Lemma 2.3.,

$$
\begin{aligned}
f_{i_1,i_2}(a, f_{i_1,i_2}(b,c)) &= a \circ_{i_1} (a \circ_{i_2} (b \circ_{i_1} (b \circ_{i_2} c))) \\
&= a \circ_{i_1} (a \circ_{i_2} b) \circ_{i_1} (a \circ_{i_2} b \circ_{i_2} c) = a \circ_{i_1} (a \circ_{i_2} b \circ_{i_2} c) \\
&= f_{i_1,i_2}(a, b \circ_{i_2} c).
\end{aligned}
$$

Assume that the above statement is true for all sequences of indices with the length$\leq$ $k-1$. Let $K = (i_1, i_2, \cdots, i_k)$ and $J = (i_1, i_2, \cdots, i_{k-1})$. Then by induction hypothesis, (5) and (3) of Lemma 2.3, we have

$$
\begin{aligned}
f_K(a, f_K(b,c)) &= f_K(a, f_J(b, b \circ_{i_k} c)) \\
&= a \circ_{i_k} f_J(a, f_J(b, b \circ_{i_k} c)) \\
&= a \circ_{i_k} f_J\left(a, b \circ_{i_{k-1}} (b \circ_{i_k} c)\right) \\
&= f_J\left(a, a \circ_{i_k} \left(b \circ_{i_{k-1}} (b \circ_{i_k} c)\right)\right) \\
&= f_J\left(a, (a \circ_{i_k} b) \circ_{i_{k-1}} (a \circ_{i_k} b \circ_{i_k} c)\right) \\
&= f_J(a, a \circ_{i_k} b \circ_{i_k} c) \\
&= f_K(a, b \circ_{i_k} c).
\end{aligned}
$$

Hence $f_K(a, f_K(b,c)) = f_K(a, b \circ_{i_k} c) = f_K(a, f_K(c,b))$. Also, we show that

$$\begin{aligned} f_K(f_K(a,b), c) &= a \circ_{i_1} \left(a \circ_{i_2} \left(\cdots \left(a \circ_{i_{k-1}} (a \circ_{i_k} b \circ_{i_k} c)\right) \cdots\right)\right) \\ &= f_K(a, b \circ_{i_k}, c). \end{aligned}$$

First, we claim that for index $J = (i_1, i_2, \cdots, i_n)$ $(2 \le n \le k-1)$

$$\begin{aligned} f_J(f_K(a,b), c) &= a \circ_{i_1} (a \circ_{i_2} (\cdots (a \circ_{i_n} f_{K-J}(a,b) \circ_{i_n} c) \cdots)) \\ &= f_J(a, f_{K-J}(a,b) \circ_{i_n} c). \end{aligned}$$

We use the induction on n. Let $S = (i_2, i_3, \cdots, i_k)$ and $T = (i_3, \cdots, i_k)$. If $n = 2$, then by (7), (5) and (3) of Lemma 2.3,

$$\begin{aligned} f_{i_1,i_2}(f_K(a,b), c) &= f_K(a,b) \circ_{i_1} (f_K(a,b) \circ_{i_2} c) \\ &= \circ_{i_1} f_S(a,b) \circ_{i_1} ((a \circ_{i_1} f_S(a,b)) \circ_{i_2} c) \\ &= a \circ_{i_1} (a \circ_{i_2} f_T(a,b)) \circ_{i_1} ((a \circ_{i_1} (a \circ_{i_2} f_T(a,b))) \circ_{i_2} c) \\ &= a \circ_{i_1} (a \circ_{i_2} (f_T(a,b) \circ_{i_1} (f_T(a,b) \circ_{i_2} c))) \\ &= a \circ_{i_1} (a \circ_{i_2} f_T(a,b)) \circ_{i_1} (a \circ_{i_2} f_T(a,b) \circ_{i_2} c) \\ &= a \circ_{i_1} (a \circ_{i_2} f_T(a,b) \circ_{i_2} c) \\ &= a \circ_{i_1} \left(a \circ_{i_2} f_{K-\{i_1,i_2\}}(a,b) \circ_{i_2} c\right) \\ &= f_{i_1,i_2}\left(a, f_{K-\{i_1,i_2\}}(a,b) \circ_{i_2} c\right). \end{aligned}$$

Assume that

$$\begin{aligned} f_J(f_K(a,b), c) &= a \circ_{i_1} (a \circ_{i_2} (\cdots (a \circ_{i_n} f_{K-J}(a,b) \circ_{i_n} c) \cdots)) \\ &= f_J(a, f_{K-J}(a,b) \circ_{i_n} c). \end{aligned}$$

holds for all $n \le k-2$. Then by the induction hypothesis, (1) and (3) of Lemma 2.3, we have

$$\begin{aligned} f_{i_1,\cdots,i_{k-1}}(f_K(a,b), c) &= f_{i_1,\cdots,i_{k-2}}(f_K(a,b), c) \circ_{i_{k-1}} f_K(a,b) \\ &= f_K(a,b) \circ_{i_{k-1}} f_{i_1,\cdots,i_{k-2}}\left(a, f_{K-J}(a,b) \circ_{i_{k-2}} c\right) \\ &= a \circ_{i_{k-1}} f_{i_1,\cdots,i_{k-2}}(a, a \circ_{i_k} b) \circ_{i_{k-1}} f_{i_1,\cdots,i_{k-2}}\left(a, f_{K-J}(a,b) \circ_{i_{k-2}} c\right) \\ &= a \circ_{i_{k-1}} f_{i_1,\cdots,i_{k-2}}\left(a, (a \circ_{i_k} b) \circ_{i_{k-1}} \left(f_{K-J}(a,b) \circ_{i_{k-2}} c\right)\right) \\ &= f_{i_1,\cdots,i_{k-2}}\left(a, a \circ_{i_{k-1}} (a \circ_{i_k} b) \circ_{i_{k-1}} \left(\left(a \circ_{i_{k-1}} (a \circ_{i_k} b)\right) \circ_{i_{k-2}} c\right)\right) \\ &= f_{i_1,\cdots,i_{k-2}}\left(a, \left(a \circ_{i_{k-1}} (a \circ_{i_k} b)\right) \circ_{i_{k-2}} \left(a \circ_{i_{k-1}} (a \circ_{i_k} b) \circ_{i_{k-1}} c\right)\right) \\ &= f_{i_1,\cdots,i_{k-2}}\left(a, a \circ_{i_{k-1}} (a \circ_{i_k} b) \circ_{i_{k-1}} c\right) \\ &= f_{i_1,\cdots,i_{k-1}}\left(a, f_{K-\{i_1,i_2,\cdots,i_{k-1}\}}(a,b) \circ_{i_{k-1}} c\right). \end{aligned}$$

Using the above claim, we have

$$f_{i_1,i_2,\cdots,i_k}(f_K(a,b), c) = f_K(a, b \circ_{i_k} c).$$

This completes the proof. □

3. wDn-SLatt and wDn-Latt

In this section, we prove that a weak distributive n-semilattice has a partition consisting of weak distributive n-lattices and which form a direct system in the category **wDn-Latt** of weak distributive n-lattices and homomorphisms. Furthermore, we show that the direct limit of this direct system gives to the reflection. Firstly, for a weak distributive n-semilattice W, we have a partition of weak distributive n-lattices of W by the following equivalence relation.

Proposition 3.1. *Let $W = (W, F)$ be a weak distributive n-semilattice. Define a binary relation θ on W as follows*:

$$(a, b) \in \theta \text{ if and only if } f_I\,(a, b) = a \text{ and } f_I\,(b, a) = b,$$

where $I = (1, 2, \cdots, n)$. Then θ is an equivalence relation and each equivalence class $\theta(x)$ of x is a subalgebra of W. Moreover, each $\theta(x)$ is a weak distributive n-lattice.

Proof. Clearly, θ is reflexive and symmetric. Let $(a, b), (b, c) \in \theta$. Then

$$f_I(a, b) = a, f_I(b, a) = b = f_I(b, c) \text{ and } f_I(c, b) = c.$$

Using (3) of Lemma 2.4, we have $(a, c) \in \theta$; θ is transitive. Then θ is an equivalence relation. It remains to show that each $\theta(x)$ is a subalgebra which is a weak distributive n-lattice. Take any $a, b \in \theta(x)$. Then

$$f_I(a, x) = a, f_I(x, a) = x = f_I(x, b) \text{ and } f_I(b, x) = b.$$

Thus for any $j \in [n]$,

$$\begin{aligned} f_I(a \circ_j b, x) &= f_I(a, x) \circ_j f_I(b, x) = a \circ_j b, \\ f_I(x, a \circ_j b) &= f_I(x, a) \circ_j f_I(x, b) = x \circ_j x = x; \end{aligned}$$

$a \circ_j b \in \theta(x)$. So $\theta(x)$ is a subalgebra of W. By the definition of θ and Lemma 2.4., $\theta(x)$ satisfies the generalized absorption law and thus each $\theta(x)$ is a weak distributive n-lattice. □

Proposition 3.1 amounts to saying that for a weak distributive n-semilattice $W = (W, F)$ we have a partition $\{W_\alpha \mid \alpha \in S\}$ of subalgebras of W which are weak distributive n-lattices. Here we consider a binary relation $\leq$ on the set S of indices of the set $\{W_\alpha \mid \alpha \in S\}$ defined as follows :

$\alpha \leq \beta$ if and only if there are $a \in W_\alpha, b \in W_\beta$ such that $f_I\,(b, a) = b$.

Then $(S, \leq)$ is a join semilattice.

For $\alpha \leq \beta$ let $\varphi_{\alpha,\beta} : W_\alpha \longrightarrow W_\beta$ be the map defined by $\varphi_{\alpha,\beta}\,(a) = f_I\,(a, b)$, where b is an arbitrary element of W_β. Thus we have a family of homomorphisms

$\{\varphi_{\alpha,\beta} \mid \alpha \leq \beta\}$. Moreover, for $\alpha \leq \beta$ and $\beta \leq \gamma$, $\varphi_{\alpha,\beta}(a) = f_I(a,b)$ and $\varphi_{\beta,\gamma}(b) = f_I(b,c)$, where $b \in W_\beta$ and $c \in W_\gamma$, and thus

$$\begin{aligned}\varphi_{\beta,\gamma} \circ \varphi_{\alpha,\beta}(a) &= \varphi_{\beta,\gamma}(f_I(a,b)) = f_I(f_I(a,b),c) \\ &= f_I(a, f_I(b,c)) = f_I(a, f_I(c,b)) \\ &= f_I(a,c) = \varphi_{\alpha,\gamma}(a)\end{aligned}$$

and

$$\varphi_{\alpha,\alpha}(a) = f_I(a,a) = a = 1_{W_\alpha}(a).$$

Then we obtain a direct system(see [4]) $((S,\leq), \{W_\alpha \mid \alpha \in S\}, \{\varphi_{\alpha,\beta} \mid \alpha \leq \beta\})$ of weak distributive n-lattices, where $\{W_\alpha \mid \alpha \in S\}$ is the partition of the given weak distributive n-semilattice W, given by Proposition 3.1.
Let $S(W) = \left(\bigcup_{\alpha\in S} W_\alpha, *_1, *_2, \cdots, *_n\right)$ be an algebra with n binary operations such that for $x \in W_\alpha$, $y \in W_\beta$, $x *_i y = \varphi_{\alpha,\gamma}(x) \circ_i \varphi_{\beta,\gamma}(y)$, where $\gamma = \alpha \vee \beta$ in the join semilattice $(S, \leq)$. Then one has the following Proposition :

Proposition 3.2. *For any weak distributive n-semilattice $W = (W, \circ_1, \circ_2, \cdots, \circ_n)$, W and $S(W)$ are identical.*

Proof. For any $x, y \in W$, assume that $x \in W_\alpha$, $y \in W_\beta$ and let $\gamma = \alpha \vee \beta$, then $z = x \circ_i y \in W_\gamma$, by the above argument. Then $x *_i y = \varphi_{\alpha,\gamma}(x) \circ_i \varphi_{\beta,\gamma}(y) = f_I(x,z) \circ_i f_I(y,z) = f_I(x \circ_i y, z) = f_I(z,z) = z = x \circ_i y$ for all $i \in [n]$. □

From the definition of homomorphism $\varphi_{\alpha,\beta}$ and Proposition 3.2, we have the following theorem:

Theorem 3.3. *For a weak distributive n-semilattice $W = (W, \circ_1, \circ_2, \cdots, \circ_n)$, let $S(W) = \left(\bigcup_{\alpha\in S} W_\alpha, *_1, *_2, \cdots, *_n\right) = (W, \circ_1, \circ_2, \cdots, \circ_n)$ in the above Proposition 3.2. Define a binary relation Λ on $S(W)$ by $(x,y) \in \Lambda$ if and only if $x \in W_\alpha$, $y \in W_\beta$ for some $\alpha, \beta \in S$ and there exists $\gamma \in S$ such that $\alpha \leq \gamma$, $\beta \leq \gamma$, $\varphi_{\alpha,\gamma}(x) = \varphi_{\beta,\gamma}(y)$, i.e., $f_I(x,z) = f_I(y,z)$, where z is an arbitrary element of W_γ. Then the relation Λ is a congruence on $S(W)$.*

The following theorem follows from Theorem 3.3.

Theorem 3.4. *The quotient algebra $(S(W)/\Lambda, *_1, *_2, \cdots, *_n)$ of $S(W)$ is a weak distributive n-lattice.*

Proof. It is enough to show that $S(W)/\Lambda$ satisfies the generalized absorption law. Let $x \in W_\alpha$ and $y \in W_\beta$ for some $\alpha, \beta \in S$, then there exists $\gamma \in S$ such that $\alpha, \beta \leq \gamma$. So

$$\begin{aligned}&[x]_\Lambda *_1 ([x]_\Lambda *_2 (\cdots ([x]_\Lambda *_{n-1} ([x]_\Lambda *_n [y]_\Lambda)) \cdots)) \\ =\ &[\varphi_{\alpha,\gamma}(x) \circ_1 (\varphi_{\alpha,\gamma}(x) \circ_2 (\cdots (\varphi_{\alpha,\gamma}(x) \circ_{n-1} (\varphi_{\alpha,\gamma}(x) \circ_n \varphi_{\beta,\gamma}(y))) \cdots))]_\Lambda \\ =\ &[\varphi_{\alpha,\gamma}(x)]_\Lambda = [x]_\Lambda.\end{aligned}$$

Hence $S(W)/\Lambda$ is a weak distributive n-lattice. $\square$

As the following terminologies are refer to [4], we obtain the following facts:

Remark 3.5. For a weak distributive n-semilattice $W = (W, \circ_1, \circ_2, \cdots, \circ_n)$, W and $S(W)$ are identical. Thus, $S(W)/\Lambda$ may be viewed as a quotient algebra of W. In fact, W/Λ is the direct limit of the direct system

$$((S, \leq), \{W_\alpha \mid \alpha \in S\}, \{\varphi_{\alpha,\beta} \mid \alpha \leq \beta\}).$$

The class of weak distributive n-semilattices and homomorphisms between them forms a category, which will be denoted by **wDn-SLatt**, and the class of weak distributive n-lattices forms a full subcategory of **wDn-SLatt**, which will be denoted by **wDn-Latt**.

Theorem 3.6. *The category* **wDn-Latt** *is a reflective subcategory of the category* **wDn-SLatt**.

Proof. For a weak distributive n-semilattice $W = (W, \circ_1, \circ_2, \cdots, \circ_n)$, let $q : W \longrightarrow W/\Lambda$ be the quotient homomorphism, where Λ is the congruence given in Theorem 3.3. Then $(q, W/\Lambda)$ is the **wDn-Latt**-reflection of $W \in$ **wDn-SLatt**. In fact, take any $L = (L, *_1, *_2, \cdots, *_n) \in$ **wDn-Latt** and any homomorphism $f : W \longrightarrow L$, then $\ker(q) \subseteq \ker(f)$. For any $(x, y) \in \ker(q)$, there are $\alpha, \beta \in S$ such that $x \in W_\alpha$, $y \in W_\beta$. Then there is $\gamma \in S$ such that $\alpha \leq \gamma$, $\beta \leq \gamma$, $q(x) = [\varphi_{\alpha,\gamma}(x)]_\Lambda = [\varphi_{\beta,\gamma}(y)]_\Lambda = q(y)$ so, $x \circ_1 (x \circ_2 (\cdots (x \circ_{n-1} (x \circ_n z)) \cdots)) = y \circ_1 (y \circ_2 (\cdots (y \circ_{n-1} (y \circ_n z)) \cdots))$, where z is an arbitrary element of W_γ. Since f is a homomorphism and each element of L satisfies the generalized absorption law, $f(x) = f(x) *_1 (f(x) *_2 (\cdots (f(x) *_{n-1} (f(x) *_n f(z))) \cdots)) = f(y) *_1 (f(y) *_2 (\cdots (f(y) *_{n-1} (f(y) *_n f(z))) \cdots)) = f(y)$; therefore $(x, y) \in \ker(f)$. So by the Fundamental Theorem of Factorization, there is a unique homomorphism $\bar{f} : W/\Lambda \longrightarrow L$ with $\bar{f} \circ q = f$. Hence **wDn-Latt** is a reflective subcategory of **wDn-SLatt**. $\square$

Corollary 3.7. *The category* **wDn-Latt** *is closed under the formation of limits in the category* **wDn-SLatt**.

Note that W/Λ is the direct limit of the following direct system

$$((S, \leq), \{W_\alpha \mid \alpha \in S\}, \{\varphi_{\alpha,\beta} \mid \alpha \leq \beta\}).$$

Then we have the following corollary, directly.

Corollary 3.8. *If $W = (W, F)$ is a finite weak distributive n-semilattice, then $W_p \cong W/\Lambda$, where p is the largest element of $(S, \leq)$.*

References

[1] J. Adámek, H. Herrlich and G. E. Strecker, Abstract and Concrete Categories, John Wiley Sons, Inc, New York, 1990.

[2] B. Davey and H. Priestley, Introdution to Lattices and Order, Combridge University Press, New York, 1990.

[3] J. Galuszka, *Generalized absorption laws in bisemilattices*, Algebra Universalis, **19**(1984), 304-318.

[4] G. Grätzer, Universal Algebra, 2nd ed. Springer-Verlag, New York, 1970.

[5] S. S. Hong and Y. H. Hong, Abstract Algebra, Towers, Seoul, 1976.

[6] A. Knoeble and A. Romanowska, *Distributive multisemilattices*, Dissertationes Mathematicae, **CCCIX**(1991), 4-42.

[7] R. Padmanabhan, *regular identities in lattices*, Trans. Amer. Math. Soc., **158**(1971), 179-188.

[8] J. Plonka, *On distributive quasilattices*, Fund. Math., **60**(1967), 191-200.

[9] J. Plonka, *On distributive n-lattices and n-quasilattices*, Fund. Math., **62**(1967), 293-300.

[10] J. Plonka, *Some remarks on sums of direct systems of algebras*, Fund. Math., **62**(1968), 301-308.