KYUNGPOOK Math. J. 47(2007), 253-265

Lipschitz Estimates for Multilinear Singular Integral Operators on Hardy and Herz Type Spaces

Liu Lanzhe
College of Mathematics, Changsha University of Science and Technology, Changsha 410077, P. R. of China
e-mail : lanzheliu@163.com

Abstract. In this paper, the boundedness for some multilinear operators generated by singular integralv operators and Lipschitz functions on Hardy and Herz-type spaces are obtained.

1. Introduction

Let T be a Calderón-Zygmund operator, a well-known result of Coifman, Rochberg and Weiss (see [6]) states that the commutator $[b, T](f) = T(bf) - bT(f)$(where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$; Chanillo(see [1]) proves a similar result when T is replaced by the fractional integral operator. However, it was observed that $[b, T]$ is not bounded, in general, from $H^p(R^n)$ to $L^p(R^n)$ for $0 < p \leq 1$. But, the boundedness hold if b belongs to the Lipschitz spaces $Lip_\beta(R^n)$(see[11]). This show the difference of $b \in BMO(R^n)$ and $b \in Lip_\beta(R^n)$. In [10][14], the $L^p(p > 1)$-boundedness of the commutators when b is the Lipschitz function are obtained. The purpose of this paper is to establish the boundedness properties for some multilinear singular integral operators generated by some singular integral operators and Lipschitz functions on Hardy and Herz-type spaces.

2. Preliminaries and results

In this paper, we will study a class of multilinear operators related to some singular integral operators, whose definitions are following. Let m be a positive integer and A be a function on R^n. Set

$$R_{m+1}(A; x, y) = A(x) - \sum_{|\gamma| \leq m} \frac{1}{\gamma!} D^\gamma A(y)(x-y)^\gamma$$

Received January 25, 2006.
2000 Mathematics Subject Classification: 42B20, 42B25.
Key words and phrases: singular integral operator, multilinear operator, Lipschitz function, Hardy space, Herz space.

and

$$Q_{m+1}(A;x,y) = R_m(x) - \sum_{|\gamma|=m} D^\gamma A(x)(x-y)^\gamma.$$

Fixed $\delta \geq 0$ and $\varepsilon > 0$. Let $T_\delta : S \to S'$ be a linear operator. T_δ is called a singular integral operator if there exists a locally integrable function $K(x,y)$ on $R^n \times R^n$ such that

$$T_\delta(f)(x) = \int_{R^n} K(x,y)f(y)dy$$

for every bounded and compactly supported function f. Where K satisfies:

$$|K(x,y)| \leq C|x-y|^{-n+\delta}$$

and

$$|K(y,x) - K(z,x)| + |K(x,y) - K(x,z)| \leq C|y-z|^\varepsilon |x-z|^{-n-\varepsilon+\delta}$$

if $2|y-z| \leq |x-z|$. The multilinear operator related to the singular integral operator T_δ is defined by

$$T_\delta^A(f)(x) = \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^m} K(x,y)f(y)dy.$$

We also consider the variant of T_δ^A, which is defined by

$$\tilde{T}_\delta^A(f)(x) = \int_{R^n} \frac{Q_{m+1}(A;x,y)}{|x-y|^m} K(x,y)f(y)dy.$$

Note that when $m = 0$, T_δ^A is just the commutator of the singular integral operators T_δ and A (see [1], [6], [10], [11], [14]), while when $m > 0$, it is non-trivial generalizations of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors when A has derivatives of order m in $BMO(R^n)$(see [3], [4], [5], [7]). In [2], author obtains the $L^p(p > 1)$-boundedness of multilinear singular integral operators generated by singular integrals and Lipschitz functions. The main purpose of this paper is to discuss the boundedness properties of the multilinear singular integral operators on Hardy and Herz-type spaces. Let us first introduce some definitions (see [8], [9], [12], [13], [15]). Throughout this paper, $M(f)$ will denote the Hardy-Littlewood maximal function of f, Q will denote a cube of R^n with side parallel to the axes. Denote the Hardy spaces by $H^p(R^n)$. It is well known that $H^p(R^n)(0 < p \leq 1)$ has the atomic decomposition characterization(see [8], [15]). For $\beta > 0$, the Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that(see [14])

$$\|f\|_{Lip_\beta} = \sup_{x,h \in R^n,\ h>0} |f(x+h) - f(x)|/|h|^\beta < \infty.$$

Definition 1. Let $0 < p, q < \infty$, $\alpha \in R$ and $B_k = \{x \in R^n : |x| \le 2^k\}$, $C_k = B_k \setminus B_{k-1}$ for $k \in Z$. For a measurable function f on R^n, let $m_k(\lambda, f) = |\{x \in A_k : |f(x)| > \lambda\}|$ for $k \in Z$ and $\tilde{m}_k(\lambda, f) = m_k(\lambda, f)$ for $k \in N$ and $\tilde{m}_0(\lambda, f) = |\{x \in B_0 : |f(x)| > \lambda\}|$.

(1) The homogeneous Herz space is defined by

$$\cdot K_q^{\alpha,p}(R^n) = \{f \in L^q_{loc}(R^n \setminus \{0\}) : ||f||_{\cdot K_q^{\alpha,p}} < \infty\},$$

where

$$\|f\|_{\cdot K_q^{\alpha,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}||f\chi_k||^p_{L^q}\right]^{1/p};$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n) = \{f \in L^q_{loc}(R^n) : ||f||_{K_q^{\alpha,p}} < \infty\},$$

where

$$\|f\|_{K_q^{\alpha,p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p}||f\chi_k||^p_{L^q} + ||f\chi_{B_0}||^p_{L^q}\right]^{1/p}.$$

(3) The homogeneous weak Herz space is defined by

$$W \cdot K_q^{\alpha,p}(R^n) = \{f \text{ is measurable on } R^n : ||f||_{W \cdot K_q^{\alpha,p}} < \infty\},$$

where

$$\|f\|_{W \cdot K_q^{\alpha,p}} = \sup_{\lambda>0} \lambda \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} m_k(\lambda, f)^{p/q}\right]^{1/p};$$

(4) The nonhomogeneous weak Herz space is defined by

$$WK_q^{\alpha,p}(R^n) = \{f \text{ is measurable on } R^n : ||f||_{WK_q^{\alpha,p}} < \infty\},$$

where

$$\|f\|_{WK_q^{\alpha,p}} = \sup_{\lambda>0} \lambda \left[\sum_{k=0}^{\infty} 2^{k\alpha p} \tilde{m}_k(\lambda, f)^{p/q}\right]^{1/p}.$$

Definition 2. Let $\alpha \in R$, $0 < p, q < \infty$.

(1) The homogeneous Herz type Hardy space is defined by

$$H \cdot K_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in \cdot K_q^{\alpha,p}(R^n)\},$$

and

$$\|f|\_H \cdot K_q^{\alpha,p} = \|G(f)\|_{\cdot K_q^{\alpha,p}};$$

(2) The nonhomogeneous Herz type Hardy space is defined by

$$HK_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in K_q^{\alpha,p}(R^n)\},$$

and

$$\|f\|_{HK_q^{\alpha,p}} = \|G(f)\|_{K_q^{\alpha,p}};$$

where $G(f)$ is the grand maximal function of f. The Herz type Hardy spaces have the atomic decomposition characterization.

Definition 3. Let $\alpha \in R$, $1 < q < \infty$. A function $a(x)$ on R^n is called a central (α, q)-atom (or a central (a, q)-atom of restrict type), if

(1) Supp$a \subset B(0, r)$ for some $r > 0$ (or for some $r \geq 1$),

(2) $||a||_{L^q} \leq |B(0, r)|^{-\alpha/n}$,

(3) $\int_{R^n} a(x)x^\gamma dx = 0$ for $|\gamma| \leq [\alpha - n(1 - 1/q)]$.

Lemma 1([13]). *Let $0 < p < \infty$, $1 < q < \infty$ and $\alpha \geq n(1-1/q)$. A temperate distribution f belongs to $H\dot{K}_q^{\alpha,p}(R^n)$(or $HK_q^{\alpha,p}(R^n)$) if and only if there exist central (α, q)-atoms (or central (α, q) -atoms of restrict type) a_j supported on $B_j = B(0, 2^j)$ and constants λ_j, $\sum_j |\lambda_j|^p < \infty$ such that $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$(or $f = \sum_{j=0}^{\infty} \lambda_j a_j$)in the $S'(R^n)$ sense, and*

$$\|f\|_{H\dot{K}_q^{\alpha,p}}(\text{ or } ||f||_{HK_q^{\alpha,p}}) \sim \left(\sum_j |\lambda_j|^p\right)^{1/p}.$$

Now we can state our results as following.

Theorem 1. *Let $0 < \beta \leq 1$, $0 \leq \delta < n - \beta$ and $D^\gamma A \in Lip_\beta(R^n)$ for all γ with $|\gamma| = m$. Suppose that T_δ^A is bounded from $L^p(R^n)$ to $L^q(R^n)$ for any $p, q \in (1, +\infty)$ and $1/q = 1/p - (\delta + \beta)/n$. Then*

(a) *If $\max(n/(n + \beta), n/(n + \varepsilon)) < p \leq 1$, $1/p - 1/q = (\delta + \beta)/n$, then T_δ^A is bounded from $H^p(R^n)$ to $L^q(R^n)$;*

(b) *If $0 < \beta < \min(1, \varepsilon)$, then $\tilde{T}_\delta^A$ is bounded from $H^{n/(n+\beta)}(R^n)$ to $L^{n/(n-\delta)}(R^n)$;*

(c) *If $0 < \beta < \min(1, \varepsilon)$, then T_δ^A is bounded from $H^{n/(n+\beta)}(R^n)$ to weak $L^{n/(n-\delta)}(R^n)$.*

Theorem 2. *Let $0 < \beta \leq 1$, $0 \leq \delta < n - \beta$, $0 < p < \infty$, $1 < q_1, q_2 < \infty$, $1/q_1 - 1/q_2 = (\delta + \beta)/n$ and $D^\gamma A \in Lip_\beta(R^n)$ for all $|\gamma| = m$ with $|\gamma| = m$. Suppose that T_δ^A is bounded from $L^p(R^n)$ to $L^q(R^n)$ for any $p, q \in (1, +\infty)$ and $1/q = 1/p - \delta/n$. Then*

(a) *If* $n(1-1/q_1) \le \alpha < \min(n(1-1/q_1)+\beta, n(1-1/q_1)+\varepsilon)$, *then* T_δ^A *is bounded from* $H\dot{K}_{q_1}^{\alpha,p}(R^n)$ *to* $\dot{K}_{q_2}^{\alpha,p}(R^n)$;

(b) *If* $0 < p \le 1$ *and* $0 < \beta < \min(1,\varepsilon)$, *then* $\tilde{T}_\delta^A$ *is bounded from* $H\dot{K}_{q_1}^{n(1-1/q_1)+\beta,p}(R^n)$ *to* $\dot{K}_{q_2}^{n(1-1/q_1)+\beta,p}(R^n)$;

(c) *If* $0 < p \le 1$ *and* $0 < \beta < \min(1,\varepsilon)$, *then* T_δ^A *is bounded from* $H\dot{K}_{q_1}^{n(1-1/q_1)+\beta,p}(R^n)$ *to* $W\dot{K}_{q_2}^{n(1-1/q_1)+\beta,p}(R^n)$.

Remark. Theorem 2 also holds for the nonhomogeneous Herz and Herz type Hardy space.

3. Proofs of theorems

We begin with a preliminary lemma.

Lemma 2([5]). *Let* A *be a function on* R^n *such that* $D^\gamma A \in L^q_{loc}(R^n)$ *for* $|\gamma| = m$ *and some* $q > n$. *Then*

$$|R_m(A;x,y)| \le C|x-y|^m \sum_{|\gamma|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\gamma A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x,y)$ *is the cube centered at* x *and having side length* $5\sqrt{n}|x-y|$.

Proof of Theorem 1. (a) It suffices to show that there exists a constant $C > 0$ such that for every H^p-atom a, there is

$$\|T_\delta^A(a)\|_{L^q} \le C.$$

Let a be a H^p-atom, that is that a supported on a cube $Q = Q(x_0, r)$, $||a||_{L^\infty} \le |Q|^{-1/p}$ and $\int_{R^n} a(x)x^\gamma dx = 0$ for $|\gamma| \le [n(1/p-1)]$. We write

$$\int_{R^n} |T_\delta^A(a)(x)|^q dx = \left(\int_{|x-x_0|\le 2r} + \int_{|x-x_0|>2r} \right) |T_\delta^A(a)(x)|^q dx = I + II.$$

For I, taking $q_1 > q$ and $1 < p_1 < n/(\delta+\beta)$ such that $1/p_1 - 1/q_1 = (\delta+\beta)/n$, by Hölder's inequality and the (L^{p_1}, L^{q_1})-boundedness of T_δ^A, we have

$$I \le C\|T_\delta^A(a)\|_{L^{q_1}}^q |2Q|^{1-q/q_1} \le C\|a\|_{L^{p_1}}^q |Q|^{1-q/q_1} \le C.$$

To obtain the estimate of II, we need to estimate $T_\delta^A(a)(x)$ for $x \in (2Q)^c$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\gamma|=m} \frac{1}{\gamma!}(D^\gamma A)_{\tilde{Q}} x^\gamma$. Then $R_m(A;x,y) = R_m(\tilde{A};x,y)$

and $D^\gamma \tilde{A}(y) = D^\gamma A(y) - (D^\gamma A)_Q$. We write, by the vanishing moment of a,

$$\begin{aligned} T_\delta^A(a)(x) &= \int_{R^n} \left[\frac{K(x,y)R_m(\tilde{A};x,y)}{|x-y|^m} - \frac{K(x,x_0)R_m(\tilde{A};x,x_0)}{|x-x_0|^m} \right] a(y)dy \\ &\quad - \sum_{|\gamma|=m} \frac{1}{\gamma!} \int_{R^n} \frac{K(x,y)(x-y)^\gamma D^\gamma \tilde{A}(y)}{|x-y|^m} a(y)dy. \end{aligned}$$

By Lemma 2 and the following inequality

$$|b(x) - b_Q| \le \frac{1}{|Q|} \int_Q \|b\|_{Lip_\beta} |x-y|^\beta dy \le \|b\|_{Lip_\beta} (|x-x_0| + r)^\beta,$$

we get

$$|R_m(\tilde{A};x,y)| \le \sum_{|\gamma|=m} ||D^\gamma A||_{Lip_\beta} (|x-y| + r)^{m+\beta}.$$

By the formula (see [5]):

$$R_m(\tilde{A};x,y) - R_m(\tilde{A};x,x_0) = \sum_{|\eta|<m} \frac{1}{\eta!} R_{m-|\eta|}(D^\eta \tilde{A}; x_0, y)(x-x_0)^\eta,$$

and note that $|x-y| \sim |x-x_0|$ for $y \in Q$ and $x \in R^n \setminus Q$, we obtain

$$\begin{aligned} |T_\delta^A(a)(x)| &\le C \sum_{|\alpha|=m} ||D^\alpha A||_{Lip_\beta} \int_Q \left[\frac{|y-x_0|}{|x-x_0|^{n+1-\delta-\beta}} + \frac{|y-x_0|^\varepsilon}{|x-x_0|^{n+\varepsilon-\delta-\beta}} \right. \\ &\quad \left. + \sum_{|\eta|<m} \frac{|y-x_0|^{m+\beta-|\eta|}}{|x-x_0|^{n+m-|\eta|-\delta}} + \frac{|y-x_0|^\beta}{|x-x_0|^{n-\delta}} \right] |a(y)|dy \\ &\le C \sum_{|\gamma|=m} ||D^\gamma A||_{Lip_\beta} \left[\frac{|Q|^{\beta/n+1-1/p}}{|x-x_0|^{n-\delta}} + \frac{|Q|^{\varepsilon/n+1-1/p}}{|x-x_0|^{n+\varepsilon-\delta-\beta}} \right]. \end{aligned}$$

Thus

$$\begin{aligned} II &\le \sum_{k=1}^\infty \int_{2^{k+1}Q \setminus 2^k Q} |T_\delta^A(a)(x)|^q dx \\ &\le C \left(\sum_{|\gamma|=m} \|D^\gamma A\|_{Lip_\beta} \right)^q \sum_{k=1}^\infty \left[2^{kqn(1/p-(n+\beta)/n)} + 2^{kqn(1/p-(n+\varepsilon)/n)} \right] \\ &\le C \left(\sum_{|\gamma|=m} \|D^\gamma A\|_{Lip_\beta} \right)^q, \end{aligned}$$

which together with the estimate for I yields the desired result.
(b) We need only prove that there exists a constant $C > 0$ such that for every $H^{n/(n+\beta)}$-atom a supported on $Q = Q(x_0, r)$, there is

$$\|\tilde{T}_\delta^A(a)\|_{L^{n/(n-\delta)}} \le C.$$

We write

$$\int_{R^n} |\tilde{T}_\delta^A(a)(x)|^{n/(n-\delta)} dx = \left[\int_{|x-x_0|\le 2r} + \int_{|x-x_0|>2r}\right] |\tilde{T}_\delta^A(a)(x)|^{n/(n-\delta)} dx := J + JJ.$$

For J, by the following equality

$$Q_{m+1}(A; x, y) = R_{m+1}(A; x, y) - \sum_{|\gamma|=m} \frac{1}{\gamma!}(x-y)^\gamma (D^\gamma A(x) - D^\gamma A(y)),$$

we have

$$|\tilde{T}_\delta^A(a)(x)| \le |T_\delta^A(a)(x)| + C \sum_{|\gamma|=m} \int_{R^n} \frac{|D^\gamma A(x) - D^\gamma A(y)|}{|x-y|^{n-\delta}} |a(y)| dy,$$

thus, $\tilde{T}_\delta^A$ is (L^p, L^q)-bounded by [10][14], where $1 < p < n/(\delta+\beta)$ and $1/q = 1/p - (\delta+\beta)/n$. We see that

$$J \le C \|\tilde{T}_\delta^A(a)\|_{L^q}^{n/(n-\delta)} |2Q|^{1-n/((n-\delta)q)} \le C ||a||_{L^p}^{n/(n-\delta)} |Q|^{1-n/((n-\delta)q)} \le C.$$

To obtain the estimate of JJ, we denote that $\tilde{A}(x) = A(x) - \sum_{|\gamma|=m} \frac{1}{\gamma!}(D^\gamma A)_{2Q} x^\gamma$. Then $Q_m(A; x, y) = Q_m(\tilde{A}; x, y)$. We write, by the vanishing moment of a and $Q_{m+1}(A; x, y) = R_m(A; x, y) - \sum_{|\gamma|=m} \frac{1}{\gamma!}(x-y)^\gamma D^\gamma A(x)$, for $x \in (2Q)^c$,

$$\begin{aligned}
\tilde{T}_\delta^A(a)(x) &= \int_{R^n} \frac{K(x,y) R_m(\tilde{A}; x, y)}{|x-y|^m} a(y) dy \\
&\quad - \sum_{|\gamma|=m} \frac{1}{\gamma!} \int_{R^n} \frac{K(x,y) D^\gamma \tilde{A}(x)(x-y)^\gamma}{|x-y|^m} a(y) dy \\
&= \int_{R^n} \left[\frac{K(x,y) R_m(\tilde{A}; x, y)}{|x-y|^m} - \frac{K(x,x_0) R_m(\tilde{A}; x, x_0)}{|x-x_0|^m}\right] a(y) dy \\
&\quad - \sum_{|\gamma|=m} \frac{1}{\gamma!} \int_{R^n} \left[\frac{K(x,y)(x-y)^\gamma}{|x-y|^m} - \frac{K(x,x_0)(x-x_0)^\gamma}{|x-x_0|^m}\right] D^\gamma \tilde{A}(x) a(y) dy.
\end{aligned}$$

Then, similar to the proof of (a), we obtain, for $x \in (2Q)^c$

$$\begin{aligned}
|\tilde{T}_\delta^A(a)(x)| &\le C|Q|^{-\beta/n} \sum_{|\gamma|=m} \left[||D^\gamma A||_{Lip_\beta} \left(\frac{|Q|^{1/n}}{|x-x_0|^{n+1-\delta-\beta}} + \frac{|Q|^{\varepsilon/n}}{|x-x_0|^{n+\varepsilon-\delta-\beta}} \right) \right. \\
&\quad \left. + |D^\gamma \tilde{A}(x)| \left(\frac{|Q|^{1/n}}{|x-x_0|^{n+1-\delta}} + \frac{|Q|^{\varepsilon/n}}{|x-x_0|^{n+\varepsilon-\delta}} \right) \right].
\end{aligned}$$

Thus

$$JJ \le C\left(\sum_{|\gamma|=m} ||D^{\gamma}A||_{Lip_{\beta}}\right)^{n/(n-\delta)} \sum_{k=1}^{\infty}\left[2^{kn(\beta-1)/(n-\delta)} + 2^{kn(\beta-\varepsilon)/(n-\delta)}\right] \le C,$$

which together with the estimate for J yields the desired result.
(c) By the following equality

$$R_{m+1}(A;x,y) = Q_{m+1}(A;x,y) + \sum_{|\gamma|=m} \frac{1}{\gamma!}(x-y)^{\gamma}(D^{\gamma}A(x) - D^{\gamma}A(y)),$$

we get

$$|T_{\delta}^{A}(f)(x)| \le |\tilde{T}_{\delta}^{A}(f)(x)| + C\sum_{|\gamma|=m}\int_{R^n} \frac{|D^{\gamma}A(x) - D^{\gamma}A(y)|}{|x-y|^{n-\delta}}|f(y)|dy.$$

By (b) and [11, Theorem 3.1], we obtain

$$\begin{aligned} & |\{x \in R^n : T_{\delta}^{A}(f)(x) > \lambda\}| \\ \le\ & |\{x \in R^n : \tilde{T}_{\delta}^{A}(f)(x) > \lambda/2\}| \\ & + \left|\left\{x \in R^n : \sum_{|\gamma|=m}\int_{R^n} \frac{|D^{\gamma}A(x) - D^{\gamma}A(y)|}{|x-y|^{n-\delta}}|f(y)|dy > C\lambda\right\}\right| \\ \le\ & C(||f||_{H^{n/(n+\beta)}}/\lambda)^{n/(n-\delta)}. \end{aligned}$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. (a) Let $f \in H\dot{K}_{q_1}^{\alpha,p}(R^n)$ and $f(x) = \sum_{j=-\infty}^{\infty}\lambda_j a_j(x)$ be the atomic decomposition for f as in Lemma 1. We write

$$\begin{aligned} \|T_{\delta}^{A}(f)\|_{\dot{K}_{q_2}^{\alpha,p}}^{p} &\le \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|\|T_{\delta}^{A}(a_j)\chi_k\|_{L^{q_2}}\right)^{p} \\ &\quad + \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|||T_{\delta}^{A}(a_j)\chi_k||_{L^{q_2}}\right)^{p} \\ &= L_1 + L_2. \end{aligned}$$

For L_2, by the (L^{q_1}, L^{q_2}) boundedness of T_δ^A, we have

$$\begin{aligned} L_2 &\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| ||a_j||_{L^{q_1}} \right)^p \\ &\le \begin{cases} C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p} \right), & 0 < p \le 1 \\ C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha} \right) \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha} \right)^{p/p'}, & p > 1 \end{cases} \\ &\le C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \le C ||f||^p_{H\dot{K}^{\alpha,p}_{q_1}}. \end{aligned}$$

For L_1, similar to the proof of Theorem 1(a), we have, for $x \in C_k$, $j \le k-3$,

$$\begin{aligned} |T_\delta^A(a_j)(x)| &\le C \left(\frac{|B_j|^{\beta/n}}{|x|^{n-\delta}} + \frac{|B_j|^{\varepsilon/n}}{|x|^{n+\varepsilon-\delta-\beta}} \right) \int_{R^n} |a_j(y)| dy \\ &\le C \left(2^{j(\beta+n(1-1/q_1)-\alpha)} |x|^{\delta-n} + 2^{j(\varepsilon+n(1-1/q_1)-\alpha)} |x|^{\delta+\beta-n-\varepsilon} \right). \end{aligned}$$

Thus

$$||T_\delta^A(a_j)\chi_k||_{L^{q_2}} \le C 2^{-k\alpha} \left(2^{(j-k)(\beta+n(1-1/q_1)-\alpha)} + 2^{(j-k)(\varepsilon+n(1-1/q_1)-\alpha)} \right)$$

and

$$\begin{aligned} L_1 &\le C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{k-3} |\lambda_j| (2^{(j-k)(\beta+n(1-1/q_1)-\alpha)} + 2^{(j-k)(1/2+n(1-1/q_1)-\alpha)} \\ &\qquad + 2^{(j-k)(\gamma+n(1-1/q_1)-\alpha)} \Big)^p \\ &\le \begin{cases} C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+3}^{\infty} \big(2^{(j-k)(\beta+n(1-1/q_1)-\alpha)} \\ \quad + 2^{(j-k)(1/2+n(1-1/q_1)-\alpha)} + 2^{(j-k)(\gamma+n(1-1/q_1)-\alpha)} \big)^p, & 0 < p \le 1 \\ C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \Big[\sum_{k=j+3}^{\infty} \big(2^{(j-k)(\beta+n(1-1/q_1)-\alpha)} \\ \quad + 2^{(j-k)(1/2+n(1-1/q_1)-\alpha)} + 2^{(j-k)(\gamma+n(1-1/q_1)-\alpha)} \big) \Big], & p > 1 \end{cases} \\ &\le C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \le C ||f||^p_{H\dot{K}^{\alpha,p}_{q_1}}. \end{aligned}$$

These yield the desired result.

(b) Let $f \in H\dot{K}^{n(1-1/q_1)+\beta,p}_{q_1}(R^n)$ and $f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x)$ be the atomic decomposition for f as in Lemma 1. We write

$$\begin{aligned}\|\tilde{T}_\delta^A(f)\|^p_{\dot{K}_{q_2}^{n(1-1/q_1)+\beta,p}} &\leq \sum_{k=-\infty}^{\infty} 2^{kp(n(1-1/q_1)+\beta)}\left(\sum_{j=-\infty}^{k-3}|\lambda_j\|\tilde{T}_\delta^A(a_j)\chi_k\|_{L^{q_2}}\right)^p \\ &\quad + \sum_{k=-\infty}^{\infty} 2^{kp(n(1-1/q_1)+\beta)}\left(\sum_{j=k-2}^{\infty}|\lambda_j|\|\tilde{T}_\delta^A(a_j)\chi_k\|_{L^{q_2}}\right)^p \\ &= M_1+M_2.\end{aligned}$$

For M_2, by the (L^{q_1}, L^{q_2}) boundedness of $\tilde{T}_\delta^A$, we have

$$\begin{aligned}M_2 &\leq C\sum_{k=-\infty}^{\infty} 2^{kp(n(1-1/q_1)+\beta)}\left(\sum_{j=k-2}^{\infty}|\lambda_j|\|a_j\|_{L^{q_1}}\right)^p \\ &\leq C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=-\infty}^{j+2}2^{(k-j)p(n(1-1/q_1)+\beta)}\right) \\ &\leq C\sum_{j=-\infty}^{\infty}|\lambda_j|^p \leq C\|f\|^p_{H\dot{K}_{q_1}^{n(1-1/q_1)+\beta,p}}.\end{aligned}$$

For M_1, similar to the proof of Theorem 1(b), we have, for $x\in C_k$, $j\leq k-3$,

$$\begin{aligned}|\tilde{T}_\delta^A(a_j)(x)| &\leq C\sum_{|\gamma|=m}||D^\gamma A||_{Lip_\beta}\left(\frac{|B_j|^{1/n}}{|x|^{n+1-\delta-\beta}}+\frac{|B_j|^{\varepsilon/n}}{|x|^{n+\varepsilon-\delta-\beta}}\right)\int_{R^n}|a_j(y)|dy \\ &\quad + C\sum_{|\gamma|=m}|D^\gamma\tilde{A}(x)|\left(\frac{|B_j|^{1/n}}{|x|^{n+1-\delta-\beta}}+\frac{|B_j|^{\varepsilon/n}}{|x|^{n+\varepsilon-\delta-\beta}}\right)\int_{R^n}|a_j(y)|dy \\ &\leq C\sum_{|\gamma|=m}\left[||D^\gamma A||_{Lip_\beta}\left(\frac{2^{j(1-\beta)}}{|x|^{n+1-\delta-\beta}}+\frac{2^{j(\varepsilon-\beta)}}{|x|^{n+\varepsilon-\delta-\beta}}\right)\right. \\ &\quad \left.+|D^\alpha\tilde{A}(x)|\left(\frac{2^{j(1-\beta)}}{|x|^{n+1-\delta-\beta}}+\frac{2^{j(\varepsilon-\beta)}}{|x|^{n+\varepsilon-\delta-\beta}}\right)\right].\end{aligned}$$

Thus

$$\begin{aligned}M_1 &\leq C\sum_{|\gamma|=m}||D^\gamma A||_{Lip_\beta} \\ &\quad \cdot\sum_{k=-\infty}^{\infty}2^{kp(n(1-1/q_1)+\beta)}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|^p\frac{2^{j(1-\beta)}}{2^{k(n+1-\delta-\beta)}}+\frac{2^{j(\varepsilon-\beta)}}{2^{k(n+\varepsilon-\delta-\beta)}}\right)^p 2^{knp/q_2}\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{|\gamma|=m}||D^\gamma A||_{Lip_\beta}\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}\left(2^{p(1-\beta)(j-k)}+2^{p(\varepsilon-\beta)(j-k)}\right)\\
&\le C\sum_{|\gamma|=m}||D^\gamma A||_{Lip_\beta}\sum_{j=-\infty}^{\infty}|\lambda_j|^p\le C||f||^p_{H\dot{K}_{q_1}^{n(1-1/q_1)+\beta,p}}.
\end{aligned}
$$

These yield the desired result.

(c) We know the following inequality

$$
|T_\delta^A(f)(x)|\le|\tilde{T}_\delta^A(f)(x)|+C\sum_{|\gamma|=m}\int_{R^n}\frac{|D^\gamma A(x)-D^\gamma A(y)|}{|x-y|^{n-\delta}}|f(y)|dy.
$$

Thus, by (b) and [11, Theorem 3.2.], we get

$$
\begin{aligned}
&||T_\delta^A(f)||_{WK_q^{n(1-1/q_1)+\beta,p}}\\
\le\ &||\tilde{T}_\delta^A(f)||_{WK_q^{n(1-1/q_1)+\beta,p}}\\
&+C\sum_{|\gamma|=m}\left|\left|\int_{R^n}\frac{|D^\gamma A(x)-D^\gamma A(y)|}{|x-y|^{n-\delta}}|f(y)|dy\right|\right|_{WK_q^{n(1-1/q_1)+\beta,p}}\\
\le\ &C||f||_{H\dot{K}_{q_1}^{n(1-1/q_1)+\beta,p}}.
\end{aligned}
$$

This finishes the proof of Theorem 2. □

4. Examples

1. Calderón-Zygmund singular integral operator.
Let T be the Calderón-Zygmund operator defined by(see [8], [15])

$$
T(f)(x)=\int_{R^n}K(x,y)f(y)dy,
$$

the multilinear operator related to T is defined by

$$
T^A(f)(x)=\int_{R^n}\frac{R_{m+1}(A;x,y)}{|x-y|^m}K(x,y)f(y)dy.
$$

Then it is easily to see that T satisfies the conditions in Theorem 1 and 2 with $\delta=0$.

2. Fractional integral operator with rough kernel.
For $0\le\delta<n$, let T_δ be the fractional integral operator with rough kernel defined by(see [2], [7])

$$
T_\delta(f)(x)=\int_{R^n}\frac{\Omega(x-y)}{|x-y|^{n-\delta}}f(y)dy,
$$

the multilinear operator related to T_δ is defined by

$$T_\delta^A(f)(x) = \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^{m+n-\delta}} \Omega(x-y) f(y)dy,$$

where Ω is homogeneous of degree zero on R^n, $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$ and $\Omega \in Lip_\varepsilon(S^{n-1})$ for $0 < \varepsilon \leq 1$, that is there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \leq M|x-y|^\varepsilon$. Then T_δ satisfies the conditions in Theorem 1 and 2.

3. Riesz potential operator.
Let $0 \leq \delta < n$, the Riesz potential operator is defined by(see [1], [14])

$$I_\delta(f)(x) = \int_{R^n} \frac{f(y)}{|x-y|^{n-\delta}} dy,$$

the multilinear operator related to I_δ is defined by

$$I_\delta^A(f)(x) = \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^{m+n-\delta}} f(y)dy,$$

Then I_δ satisfies the conditions in Theorem 1 and 2.

References

[1] S. Chanillo, *A note on commutators*, Indiana Univ. Math. J., **31**(1982), 7-16.

[2] W. G. Chen, *Besov estimates for a class of multilinear singular integrals*, Acta Math. Sinica, **16**(2000), 613-626.

[3] J. Cohen, *A sharp estimate for a multilinear singular integral on R^n*, Indiana Univ. Math. J., **30**(1981), 693-702.

[4] J. Cohen and J. Gosselin, *On multilinear singular integral operators on R^n*, Studia Math., **72**(1982), 199-223.

[5] J. Cohen and J. Gosselin, *A BMO estimate for multilinear singular integral operators*, Illinois J. Math., **30**(1986), 445-465.

[6] R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math., **103**(1976), 611-635.

[7] Y. Ding and S. Z. Lu, *Weighted boundedness for a class rough multilinear operators*, Acta Math. Sinica, **3**(2001), 517-526.

[8] J. Garcia-Cuerva and J. L. Rubio de Francia, Weighted norm inequalities and related topics, North-Holland Math.16, Amsterdam, 1985.

[9] J. Garcia-Cuerva and M. L. Herrero, *A theory of Hardy spaces associated to the Herz spaces*, Proc. London Math. Soc., **69**(1994), 605-628.

[10] S. Janson, *Mean oscillation and commutators of singular integral operators*, Ark. Math., **16**(1978), 263-270.

[11] S. Z. Lu, Q. Wu and D. C. Yang, *Boundedness of commutators on Hardy type spaces*, Sci.in China(ser.A), **32**(2002), 232-244.

[12] S. Z. Lu and D. C. Yang, *The decomposition of the weighted Herz spaces and its applications*, Sci. in China (ser.A), **38**(1995), 147-158.

[13] S. Z. Lu and D. C. Yang, *The weighted Herz type Hardy spaces and its applications*, Sci. in China(ser.A), **38**(1995), 662-673.

[14] M. Paluszynski, *Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss*, Indiana Univ. Math. J., **44**(1995), 1-17.

[15] E. M. Stein, Harmonic Analysis: real variable methods, orthogonality and oscillatory integrals, Princeton Univ. Press, Princeton NJ, 1993.