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Near λ-lattices

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Abstract. By a near λ-lattice is meant an upper λ-semilattice where is defined a partial binary operation $x \wedge y$ with respect to the induced order whenever x, y has a common lower bound. Alternatively, a near λ-lattice can be described as an algebra with one ternary operation satisfying nine simple conditions. Hence, the class of near λ-lattices is a quasi-variety. A λ-semilattice $\mathcal{A} = (A; \vee)$ is said to have sectional (antitone) involutions if for each $a \in A$ there exists an (antitone) involution on $[a, 1]$, where 1 is the greatest element of $\mathcal{A}$. If this antitone involution is a complementation, $\mathcal{A}$ is called an ortho λ-semilattice. We characterize these near λ-lattices by certain identities.

Nearlattices were studied (under different names) by several authors. Some essential results are collected in [3] where is given also a characterization of nearlattices as algebras with one ternary operation. The concept of a lattice was generalized by V. Snášel [5] by dropping out associativity. The resulting algebra $\mathcal{A} = (A; \vee, \wedge)$ satisfying idempotency for $\vee, \wedge$, commutativity for $\vee, \wedge$, the absorption laws and the so-called skew associativity

(SA) $\quad x \vee ((x \vee y) \vee z) = (x \vee y) \vee z, \quad x \wedge ((x \wedge y) \wedge z) = (x \wedge y) \wedge z$

is called a **λ-lattice**. Applying this concept instead of a lattice in the definition of nearlattice, we obtain a near λ-lattice. This is the subject of our next considerations.

In the sequel, we equip these near λ-lattices with the so-called sectional involutions to obtain structures analogous to ortholattices (see [2]). They can be characterized by a new binary operation which is derived "as implication" similarly as it was done by J. C. Abbott [1] for boolean near-lattices.

Definition 1. An **upper λ-semilattice** (or a **commutative directoid** in [4]) is an algebra $\mathcal{A} = (A; \vee)$ of type (2) satisfying the identities

(A1) $x \vee x = x$ (*idempotency*);

(A2) $x \vee y = y \vee x$ (*commutativity*);

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(A3) $x \vee ((x \vee y) \vee z) = (x \vee y) \vee z$ (*skew associativity*).

Lemma 1. *Let* $(A; \vee)$ *be an upper* λ*-semilattice. If we define*

$$x \leq y \quad \textit{if and only if} \quad x \vee y = y,$$

then the relation $\leq$ *is a partial order on* A.

Proof. Clearly $x \leq x$ for each $x \in A$ by (A1). Further, if $x \leq y$ and $y \leq x$, then $y = x \vee y = y \vee x = x$ by (A2). Finally, if $x \leq y$, $y \leq z$, then, by (A3),

$$x \vee z = x \vee (y \vee z) = x \vee ((x \vee y) \vee z) = (x \vee y) \vee z = y \vee z = z,$$

thus $x \leq z$. □

Let $(A; \leq)$ be an ordered set. Denote by

$U(a, b) = \{x \in A; a \leq x \text{ and } b \leq x\}$ and

$L(a, b) = \{x \in A; x \leq a \text{ and } x \leq b\}$ for $a, b \in A$.

Definition 2. A partial binary operation $\wedge$ on an upper λ-semilattice $\mathcal{A} = (A; \vee)$ will be called the **associated operation**, if the following properties hold for all $x, y, z \in A$:

i) $x \wedge y$ is defined if and only if $L(x, y) \neq \emptyset$ and

   a) $x \wedge y \in L(x, y)$;

   b) $x \leq y$ implies $x \wedge y = x$;

ii) If $x \wedge y$ is defined then $y \wedge x$ and $x \vee (x \wedge y)$ are defined and

   a) $x \wedge y = y \wedge x$;

   b) $x \vee (x \wedge y) = x$;

iii) If $(x \wedge y) \wedge z$ is defined then $x \wedge ((x \wedge y) \wedge z)$ is defined and $x \wedge ((x \wedge y) \wedge z) = (x \wedge y) \wedge z$.

Remark 1. It is clear from the definition that the associated operation $\wedge$ is idempotent, i.e., for each $x \in A$, $x \wedge x$ exists and $x \wedge x = x$. Further, the associated operation $\wedge$ satisfies the identity $x \wedge (x \vee y) = x$, since $x \leq x \vee y$.

Definition 3. An upper λ-semilattice $\mathcal{A} = (A; \vee)$ is called a **near λ-lattice**, if there is defined the associated operation $\wedge$ on A.

Remark 2. If $\mathcal{A} = (A; \vee)$ is a near λ-lattice then it does not mean that for each $a \in A$ the interval $[a)$ is a λ-lattice, see e.g. the following example:

Example 1. Consider the ordered set $(\{a, b, c, d, 1\}, \leq)$ as shown in Fig. 1. If we define $a \vee b = c$, $c \vee d = 1$ and trivially for comparable elements then $\mathcal{A} = (\{a, b, c, d, 1\}, \vee)$ is an upper λ-semilattice. To convert it into a near λ-lattice, we have two choices for non-comparable elements, namely $c \wedge d = a$ or $c \wedge d = b$. Let e.g. $c \wedge d = b$ and $x \wedge y = x$ whenever $x \leq y$. Then $\mathcal{A}$ is a near λ-lattice but the interval $[a, 1] = \{a, c, d, 1\}$ is not a λ-lattice because $c \wedge d$ is not defined in it. On the contrary $[b, 1]$ is a λ-lattice as one can easily verify.

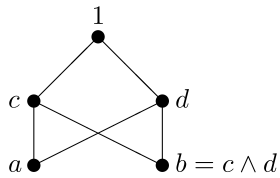

Fig. 1

Now, we show that near λ-lattices can be considered equivalently as algebras with one ternary operation.

Theorem 1. *Let $\mathcal{M} = (M; \vee)$ be a near λ-lattice and $\wedge$ its associated operation. Define a ternary operation $w(x, y, z) = (x \vee z) \wedge (y \vee z)$ on M. Then $w(x, y, z)$ is an everywhere defined operation and the following conditions are satisfied :*

(C) *for every $p, q \in L(x, y)$,* $w(x, y, p) = w(x, y, q)$;

(P1) $w(x, y, x) = x$;

(P2) $w(x, x, y) = w(y, y, x)$;

(P3) $w(x, x, w(w(x, x, y), w(x, x, y), z)) = w(w(x, x, y), w(x, x, y), z)$;

(P4) $w(x, y, z) = w(y, x, z)$;

(P5) $w(x, w(w(x, y, z), v, z), z) = w(w(x, y, z), v, z)$;

(P6) $w(x, w(y, y, x), z) = w(x, x, z)$;

(P7) $w(w(x, x, z), w(x, x, z), w(y, x, z)) = w(x, x, z)$;

(P8) $w(w(x, x, z), w(y, y, z), z) = w(x, y, z)$.

Proof. Clearly $z \leq x \vee z$, $z \leq y \vee z$, hence $L(x \vee z, y \vee z) \neq \emptyset$, thus $(x \vee z) \wedge (y \vee z)$ is an everywhere defined operation on M. To prove the condition (C) we suppose $p, q \in L(x, y)$. Then $L(x, y) \neq \emptyset$ and hence $x \wedge y$ is defined. This yields

$$w(x, y, p) = (x \vee p) \wedge (y \vee p) = x \wedge y = (x \vee q) \wedge (y \vee q) = w(x, y, q).$$

Prove the identities (P1)–(P8) :

(P1) $w(x, y, x) = (x \vee x) \wedge (y \vee x) = x \wedge (y \vee x) = x$;

(P2)

$$\begin{aligned} w(x,x,y) &= (x \vee y) \wedge (x \vee y) = x \vee y \\ &= y \vee x = (y \vee x) \wedge (y \vee x) \\ &= w(y,y,x); \end{aligned}$$

(P3)

$$\begin{aligned} w(x,x,w(w(x,x,y),w(x,x,y),z)) &= x \vee w(w(x,x,y),w(x,x,y),z) \\ &= x \vee (w(x,x,y) \vee z) \\ &= x \vee ((x \vee y) \vee z) \\ &= (x \vee y) \vee z \\ &= w(x,x,y) \vee z \\ &= w(w(x,x,y),w(x,x,y),z); \end{aligned}$$

(P4) $w(x,y,z) = (x \vee z) \wedge (y \vee z) = (y \vee z) \wedge (x \vee z) = w(y,x,z)$;

(P5)

$$\begin{aligned} w(x,w(w(x,y,z),v,z),z) &= (x \vee z) \wedge (w(w(x,y,z),v,z) \vee z) \\ &= (x \vee z) \wedge (((w(x,y,z) \vee z) \wedge (v \vee z)) \vee z) \\ &= (x \vee z) \wedge ((w(x,y,z) \vee z) \wedge (v \vee z)) \\ &= (x \vee z) \wedge ((((x \vee z) \wedge (y \vee z)) \vee z) \wedge (v \vee z)) \\ &= (x \vee z) \wedge (((x \vee z) \wedge (y \vee z)) \wedge (v \vee z)) \\ &= ((x \vee z) \wedge (y \vee z)) \wedge (v \vee z) \\ &= (((x \vee z) \wedge (y \vee z)) \vee z) \wedge (v \vee z) \\ &= w((x \vee z) \wedge (y \vee z),v,z) \\ &= w(w(x,y,z),v,z); \end{aligned}$$

(P6)

$$\begin{aligned} w(x,w(y,y,x),z) &= w(x,y \vee x,z) = (x \vee z) \wedge ((y \vee x) \vee z) \\ &= (x \vee z) \wedge (y \vee (x \vee z)) = x \vee z \\ &= w(x,x,z); \end{aligned}$$

(P7)

$$\begin{aligned} w(w(x,x,z),w(x,x,z),w(y,x,z)) &= w(x \vee z, x \vee z, w(y,x,z)) \\ &= (x \vee z) \vee w(y,x,z) \\ &= (x \vee z) \vee ((y \vee z) \wedge (x \vee z)) \\ &= x \vee z = w(x,x,z); \end{aligned}$$

(P8)

$$\begin{aligned} w(w(x,x,z),w(y,y,z),z) &= w(x\vee z, y\vee z, z) \\ &= ((x\vee z)\vee z)\wedge((y\vee z)\vee z) \\ &= (x\vee z)\wedge(y\vee z) \\ &= w(x,y,z). \end{aligned}$$

□

We are going to prove the converse. For this, let us state the following

Lemma 2. *Let $\mathcal{M}=(M;w)$ be an algebra of type* (3) *satisfying the identities* (P1), (P2), *and* (P3). *Define $x\vee y = w(x,x,y)$. Then $(M;\vee)$ is an upper λ-semilattice.*

Proof. Idempotency: by (P1), we have $x\vee x = w(x,x,x) = x$.

Commutativity : by (P2), $x\vee y = w(x,x,y) = w(y,y,x) = y\vee x$.

Skew associativity : applying (P3), we infer

$$\begin{aligned} x\vee((x\vee y)\vee z) &= w(x,x,(x\vee y)\vee z) \\ &= w(x,x,w(x\vee y, x\vee y, z)) \\ &= w(x,x,w(w(x,x,y),w(x,x,y),z)) \\ &= w(w(x,x,y),w(x,x,y),z) \\ &= w(x\vee y, x\vee y, z) = (x\vee y)\vee z. \end{aligned}$$

□

Due to Lemma 2, we can introduce an order $\leq$ on an algebra $\mathcal{M}=(M;w)$ as follows :

$$x\leq y \quad \text{if and only if} \quad w(x,x,y)=y.$$

This order will be called the **induced order** of $\mathcal{M}$.

Theorem 2. *Let $\mathcal{M}=(M;w)$ be an algebra of type* (3) *satisfying* (C), (P1) – (P7), *and let $\leq$ be the induced order. Then for $x\vee y = w(x,x,y)$, $(M;\vee)$ is an upper λ-semilattice. For $x,y,p\in M$, such that $p\leq x,y$ we define*

$$x\wedge y = w(x,y,p).$$

Then $(M;\vee)$ is a near λ-lattice where $\wedge$ is the associated operation.

If $\mathcal{M}=(M;w)$ satisfies moreover (P8), *then the correspondence between near λ-lattices and algebras $(M;w)$ satisfying* (C), (P1) – (P8) *is one-to-one.*

Proof. By Lemma 2, $(M;\vee)$ is an upper λ-semilattice. Further, for each $x\in M$ we have $x\in L(x,x)$ and hence

$$x\wedge x = w(x,x,x) = x\vee x = x.$$

Suppose now $L(x,y)\neq\emptyset$, i.e., there exists $p\in L(x,y)$. By (P4) we get

$$x\wedge y = w(x,y,p) = w(y,x,p) = y\wedge x.$$

Since $(x \wedge y) \wedge z$ is defined, we have $L(L(x,y),z) \neq \emptyset$, and thus exist p,q such that $p \in L(x,y)$ and $q \in L(L(x,y),z)$. Hence also $q \in L(x,y)$, by (C), $w(x,y,p) = w(x,y,q)$, and by (P5)

$$\begin{aligned}
x \wedge ((x \wedge y) \wedge z) &= x \wedge (w(x,y,p) \wedge z) = x \wedge (w(w(x,y,p),z,q)) \\
&= w(x, w(w(x,y,p),z,q),q) = w(w(x,y,q),z,q) \\
&= (w(x,y,q) \vee q) \wedge (z \vee q) \\
&= (((x \vee q) \wedge (y \vee q)) \vee q) \wedge (z \vee q) \\
&= ((x \vee q) \wedge (y \vee q)) \wedge (z \vee q) \\
&= (x \wedge y) \wedge z.
\end{aligned}$$

It remains to show the absorption laws. Since $x \leq y \vee x$, we have $x \in L(x, y \vee x)$ and hence $x \wedge (y \vee x)$ is defined and, by (P6), we have

$$\begin{aligned}
x \wedge (y \vee x) &= x \wedge w(y,y,x) = w(x, w(y,y,x), x) \\
&= w(x,x,x) = x.
\end{aligned}$$

To prove the second absorption law, suppose $L(x,y) \neq \emptyset$ and $p \in L(x,y)$. Then $y \wedge x$ is defined, and applying (P7),

$$\begin{aligned}
x \vee (y \wedge x) &= x \vee w(y,x,p) = (x \vee p) \vee w(y,x,p) \\
&= w(x,x,p) \vee w(y,x,p) \\
&= w(w(x,x,p), w(x,x,p), w(y,x,p)) \\
&= w(x,x,p) = x \wedge x = x.
\end{aligned}$$

Hence, $(M, \vee)$ is a near λ-lattice.

If $(M; w)$ is an algebra of type (3) satisfying (C), (P1) – (P8), $x \vee y := w(x,x,y)$ for all $x, y \in M$ and $x \wedge y := w(x,y,p)$ for all $p \in M$ and all $x,y \in M$ with $x, y \geq p$ then $(x \vee z) \wedge (y \vee z) = w(w(x,x,z), w(y,y,z), z) = w(x,y,z)$ for all $x,y,z \in M$.

Thus the correspondence between near λ-lattices and induced algebras $\mathcal{M} = (M; w)$ is one-to-one. □

Example 2. Let $\mathcal{M} = (M; \vee)$ be a near λ-lattice depicted in Fig. 2, such that $x \wedge y = p_2$, $p_1 \wedge p_2 = p_3$, $p_1 \vee p_2 = x$ and $p_3 \vee p_4 = p_2$. Then $L(x,y) = \{p_1, p_2, p_3, p_4\}$ and, by condition (C) from Theorem 1, we have :

$$w(x,y,p_i) = w(x,y,p_j) \text{ for all } i,j \in \{1,2,3,4\}.$$

Note that $p_3 \wedge p_4$ is not defined, because $L(p_3, p_4) = \emptyset$.

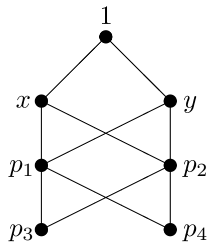

Fig. 2

Remark 3. Because of Theorems 1 and 2, near λ-lattices can be alternatively considered as algebras $\mathcal{M} = (M; w)$ of type (3) satisfying (C), (P1) – (P8) and $\leq$ will be referred to as the induced order of $\mathcal{M} = (M; w)$.

Since (P1) – (P8) are identities and (C) is a quasi-identity, we have

Corollary. *The class of all near λ-lattices* (*considered as ternary algebras*) *is a quasivariety $\mathcal{N}$.*

For varieties which are subquasivarieties of $\mathcal{N}$, we can prove

Theorem 3. *Every variety of near λ-lattices is congruence distributive.*

Proof. Take $n = 4$, and $t_0(x, y, z) = x$, $t_4(x, y, z) = z$ and $t_1(x, y, z) = w(z, y, x)$, $t_2(x, y, z) = w(x, x, z)$, $t_3(x, y, z) = w(x, y, z)$.

Then $t_0(x, y, x) = x$
$t_1(x, y, x) = w(x, y, x) = x$
$t_2(x, y, x) = w(x, x, x) = x$
$t_3(x, y, x) = w(x, y, x) = x$
$t_4(x, y, x) = x$

i even: $t_0(x, x, y) = x = w(x, y, x) = w(y, x, x) = t_1(x, x, y)$
$t_2(x, x, y) = w(x, x, y) = t_3(x, x, y)$

i odd: $t_1(x, y, y) = w(y, y, x) = w(x, x, y) = t_2(x, y, y)$
$t_3(x, y, y) = w(x, y, y) = w(y, x, y) = y = t_4(x, y, y)$.

Then $t_0, \cdots, t_4$ are Jónsson's terms and hence the variety is congruence distributive. □

Near λ-lattices with sectional antitone involutions

Let $\mathcal{A} = (A; \vee)$ be a λ-semilattice with a greatest element 1. We say that $\mathcal{A}$ is **with sectional involutions** if for each $a \in A$ there is a mapping f_a of $[a, 1]$ into itself such that $f_a(f_a(x)) = x$ for each $x \in [a, 1]$ and $f_a(a) = 1$, $f_a(1) = a$. We say that $\mathcal{A}$ is **with sectional antitone involutions** if for each $a \in A$, the mapping f_a is antitone, i.e. if $x, y \in [a, 1]$ with $x \leq y$ then $f_a(y) \leq f_a(x)$.

For the sake of brevity, we will denote $f_a(x) = x^a$.

Example 3. Consider the near λ-lattice $\mathcal{A}$ from Fig. 1. Define e.g.

$$c^a = c,\ d^a = d,\ a^a = 1,\ 1^a = a,\ c^b = d,\ d^b = c,\ b^b = 1,\ 1^b = b$$

and trivially for 2-element intervals. One can easily check that $\mathcal{A}$ is a near λ-lattice with sectional antitone involutions.

Let $\mathcal{A} = (A; \vee)$ be a near λ-lattice with sectional involutions. Introduce new binary operation $\circ$ on A as follows :

$$x \circ y = (x \vee y)^y.$$

Since $x \vee y \in [y, 1]$, $\circ$ is everywhere defined operation on A.

Lemma 3. *Let $\mathcal{A} = (A; \vee)$ be a near λ-lattice with sectional involutions. Then $x \circ y = 1$ if and only if $x \leq y$.*

Proof. If $x \leq y$ then $x \circ y = (x \vee y)^y = y^y = 1$. Conversely, suppose $x \circ y = 1$. Then $(x \vee y)^y = 1$. Since the involution is a bijection with $y^y = 1$, we conclude $x \vee y = y$ thus also $x \leq y$. □

Theorem 4. *Let $\mathcal{A} = (A; \vee)$ be a near λ-lattice with sectional involutions. Then the operation $\circ$ satisfies the following identities:*

(I1) $x \circ 1 = 1,\ 1 \circ x = x,\ x \circ x = 1$;

(I2) $(x \circ y) \circ y = (y \circ x) \circ x$;

(I3) $((x \circ y) \circ y) \circ y = x \circ y$;

(I4) $x \circ ((((x \circ y) \circ y) \circ z) \circ z) = 1$;

(I5) $x \circ (y \circ x) = 1$.

In this case we have $x \vee y = (x \circ y) \circ y$.

If, moreover, the sectional involutions are antitone then $\circ$ satisfies also

(I6) $(((((x \circ y) \circ y) \circ z) \circ z) \circ x) \circ (y \circ x) = 1$.

Proof.

(I1) :

$$\begin{aligned} x \circ 1 &= (x \vee 1)^1 = 1^1 = 1; \\ 1 \circ x &= (1 \vee x)^x = 1^x = x; \\ x \circ x &= (x \vee x)^x = x^x = 1. \end{aligned}$$

(I2) : $(x \circ y) \circ y = ((x \vee y)^y \vee y)^y = (x \vee y)^{yy} = x \vee y$ thus also $(y \circ x) \circ x = y \vee x = x \vee y = (x \circ y) \circ y$.

(I3) : By the previous we have

$$((x \circ y) \circ y) \circ y = (x \vee y) \circ y = ((x \vee y) \vee y)^y = (x \vee y)^y = x \circ y.$$

(I4) : Since $\mathcal{A}$ is a near λ-lattice, it satisfies the identity (A3) whence $x \leq (x \vee y) \vee z$. Applying the previous result $x \vee y = (x \circ y) \circ y$, we obtain $x \leq (((x \circ y) \circ y) \circ z) \circ z$. Due to Lemma 3 we get (I4).

(I5) : $x \circ (y \circ x) = (x \vee (y \vee x)^x)^{(y \vee x)^x} = ((y \vee x)^x)^{(y \vee x)^x} = 1$.
Suppose now that the sectional involutions are antitone. Evidently $x \leq y \vee x$, $x \vee y \leq (x \vee y) \vee z$ and $x \leq x \vee ((x \vee y) \vee z) = (x \vee y) \vee z$ thus

$$\begin{aligned}((((x \circ y) \circ y) \circ z) \circ z) \circ x &= ((x \vee y) \vee z) \circ x = (((x \vee y) \vee z) \vee x)^x \\ &= (x \vee ((x \vee y) \vee z))^x = ((x \vee y) \vee z)^x \leq (x \vee y)^x \\ &= (y \vee x)^x = y \circ x.\end{aligned}$$

By Lemma 3 we obtain (I6). □

Remark 4. The third simple identity in (I1), namely $x \circ x = 1$, can be derived by the other two remaining and (I2) as follows

$$x \circ x = (1 \circ x) \circ x = (x \circ 1) \circ 1 = 1.$$

We are wonder if our operation $\circ$ determines also the near λ-lattice with sectional involutions. We can state

Theorem 5. *Let $\mathcal{A} = (A; \circ, 1)$ be an algebra of type $(2,0)$ satisfying the identities* (I1) - (I5). *Define*

$$x \leq y \quad \textit{if and only if} \quad x \circ y = 1.$$

Then $(A; \leq)$ is an ordered set with the greatest element 1 which is an upper λ-semilattice for

$$x \vee y = (x \circ y) \circ y.$$

The involution on each $[a, 1]$ is defined by $x^a = x \circ a$ for $x \in [a, 1]$.
If $\mathcal{A}$ satisfies, moreover, (I6) *then for each $p \in A$ the involution on $[p, 1]$ is antitone and $([p, 1]; \leq)$ is a λ-lattice whose operations are $\vee$ and $\wedge_p$ defined by $x \wedge_p y = (x^p \vee y^p)^p$.*

Proof. By (I1), the relation $\leq$ is reflexive and $x \leq 1$ for each $x \in A$. If $x \leq y$ and $y \leq x$ then, by (I2), $x = 1 \circ x = (y \circ x) \circ x = (x \circ y) \circ y = \circ y = y$ thus $\leq$ is antisymmetrical. Suppose $x \leq y$ and $y \leq z$. Then, applying (I1) and (I4) we have

$$\begin{aligned}x \circ z &= x \circ (1 \circ z) = x \circ ((y \circ z) \circ z) \\ &= x \circ (((1 \circ y) \circ z) \circ z) \\ &= x \circ ((((x \circ y) \circ y) \circ z) \circ z) = 1\end{aligned}$$

whence $x \leq z$. Thus $\leq$ is transitive and hence an order on A.
Put $x \vee y = (x \circ y) \circ y$. By (I5) and (I2) we have $x \leq (y \circ x) \circ x = (x \circ y) \circ y$ and, by (I5), $y \leq (x \circ y) \circ y$ thus $(x \circ y) \circ y \in U(x, y)$. If $x \leq y$ then $x \circ y = 1$ thus $(x \circ y) \circ y = 1 \circ y = y$.
Hence, $(A; \vee)$ is an upper λ-semilattice with the greatest element 1.

Let $x \in [a, 1]$ and define $x^a = x \circ a$. Then $x^{aa} = (x \circ a) \circ a = x \vee a = x$, $a^a = a \circ a = 1$ and $1^a = 1 \circ a = a$ thus it is an involution on $[a, 1]$ for each $a \in A$. Suppose that $\mathcal{A} = (A; \circ, 1)$ satisfies also (I6). Then for $x, y, z \in A$, $x \leq y \leq z$ (i.e., $y, z \in [x, 1]$) we have by (I6) $((x \vee y) \vee z) \circ x \leq y \circ x$, i.e., $z^x = z \circ x = ((x \vee y) \vee z) \circ x \leq y \circ x = y^x$, i.e., every involution on each $[x, 1]$ is antitone. In this case, define for $a, b \in [p, 1]$

$$a \wedge_p b = (a^p \vee b^p)^p.$$

Since a^p, $b^p \leq a^p \vee b^p$, we have

$$\begin{aligned} a &= a^{pp} \geq (a^p \vee b^p)^p = a \wedge_p b \\ b &= b^{pp} \geq (a^p \vee b^p)^p = a \wedge_p b \end{aligned}$$

thus $a \wedge_p b \in L(a, b)$. If $a \leq b$ then $a^p \geq b^p$ thus $a \wedge_p b = (a^p \vee b^p)^p = a^{pp} = a$, i.e., $\wedge_p$ satisfies (i) of Definition 2. Of course, $x \wedge_p y = y \wedge_p x$. Since $x \wedge_p y \leq x$, we have $x \vee (y \wedge_p x) = x$ thus also (ii) of Definition 2 is satisfied; (iii) is clear. Hence, $\wedge_p$ is the associated operation and $([p, 1]; \vee, \wedge_p)$ is a λ-lattice. □

Example 4. The structure derived from $\mathcal{A} = (A; \circ, 1)$ as shown in Theorem 5 need not be a near λ-lattice. Consider the near λ-lattice from Example 2. Then we have $c \wedge_a d = (c^a \vee d^a)^a = (c \vee d)^a = 1^a = a$ in $[a, 1]$ but in $[b, 1]$ we have $c \wedge_b d = (c^b \vee d^b)^b = (d \vee c)^b = 1^b = b \neq a$. Hence, $\wedge_p$ cannot serve as an associated operation of $(A; \vee)$.

On the contrary, if $\mathcal{A}$ is a near λ-lattice, we can prove :

Theorem 6. *Let $\mathcal{A} = (A; \vee)$ be a near λ-lattice with sectional antitone involutions and $\wedge$ be its associated operation. If for $b \in A$ the section $([b, 1], \vee, \wedge)$ is a λ-lattice then*

$$x \wedge y = (((x \circ b) \circ (y \circ b)) \circ (y \circ b)) \circ b$$

for all $x, y \in [b, 1]$.

Proof. Since the section $[b, 1]$ is a λ-lattice, $x \wedge y$ is uniquely determined for each $x, y \in [b, 1]$. By Theorem 5 (in the section $[b, 1]$), we have $x \wedge y = (x^b \vee y^b)^b$. Moreover, also by Theorem 5, for each $x, y \in [b, 1]$ it holds:

$$\begin{aligned} (x^b \vee y^b)^b &= (x^b \vee y^b) \circ b = ((x \circ b) \vee (y \circ b)) \circ b \\ &= (((x \circ b) \circ (y \circ b)) \circ (y \circ b)) \circ b. \end{aligned}$$

□

Ortho λ-semilattices

By an **ortholattice** is meant an algebra $\mathcal{L} = (L; \vee, \wedge, ^\perp, 0, 1)$ where $(L; \vee, \wedge, 0, 1)$ is a bounded lattice, $x^{\perp\perp} = x$, $x \leq y \Rightarrow y^\perp \leq x^\perp$ and $x \wedge x^\perp = 0$ (which is equivalent to $x \vee x^\perp = 1$).
Hence, it is a complemented lattice where the operation $^\perp$ of complementation is an antitone involution on L. We can generalize this concept as follows:

Definition 4. By an **ortho λ-lattice** is meant an algebra $\mathcal{L} = (L; \vee, \wedge, ^\perp, 0, 1)$ such that $(L; \vee, \wedge, 0, 1)$ is a bounded λ-lattice and $x \mapsto x^\perp$ is an antitone involution satisfying $x \vee x^\perp = 1$, $x \wedge x^\perp = 0$.
By an **ortho λ-semilattice** is meant a λ-semilattice with sectional antitone involutions $(A; \vee)$ where all sections are ortho λ-lattices, i.e., for each $p \in A$ $([p, 1]; \leq)$ is an ortho λ-lattice, such that x^p is the orthocomplement of $x \in [p, 1]$ in this section.

Example 5. The following λ-lattice is an ortho λ-lattice and ortho λ-semilattice as well.

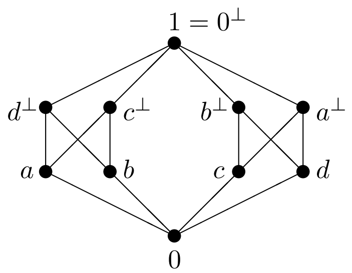

Fig. 3

The orthocomplementation in intervals $[x, 1]$ for $x \neq 0$ is determined uniquelly and for $x = 0$ it is pointed in the diagram.

Theorem 7. *Let $\mathcal{A} = (A; \vee)$ be a λ-semilattice with sectional antitone involutions. Then $\mathcal{A}$ is an ortho λ-semilattice if and only if the derived operation $x \circ y = (x \vee y)^y$ satisfies the identity*

$$(((x \circ y) \circ y) \circ (x \circ y)) \circ (x \circ y) = 1. \qquad (*)$$

Proof. Obviously,

$$\begin{aligned}(((x \circ y) \circ y) \circ (x \circ y)) \circ (x \circ y) &= ((x \circ y) \circ y) \vee (x \circ y)\\ &= ((x \circ y) \circ y) \vee (((x \circ y) \circ y) \circ y)\\ &= (x \vee y) \vee (x \vee y)^y,\end{aligned}$$

hence the identity $(*)$ can be rewritten as

$$(x \vee y) \vee (x \vee y)^y = 1.$$

Trivially, $x \vee y \in [y, 1]$ thus it is clear that in this case $a \vee a^y = 1$ for each $a \in [y, 1]$. Since $y \in L(a, a^y)$, we have for the operation $\wedge_y$

$$a \wedge_y a^y = (a^y \vee a^{yy})^y = (a^y \vee a)^y = 1^y = y$$

thus a^y is an orthocomplement of a in $[y, 1]$.
Conversely, if $\mathcal{A}$ is an ortho λ-semilattice and $x, y \in A$ then $x \vee y \in [y, 1]$ and hence

$$(x \vee y) \vee (x \vee y)^y = 1$$

whence the identity is evident. □

Due to Theorem 7, the class $\mathcal{O}$ of ortho λ-semilattices (considered in the signature $(\circ, 1)$) forms a variety.

Theorem 8. *The variety $\mathcal{O}$ of ortho λ-semilattices is weakly regular.*

Proof. Let $t_1(x, y) = x \circ y$ and $t_2(x, y) = y \circ x$. Then $t_1(x, x) = t_2(x, x) = x \circ x = 1$ and conversely, if $t_1(x, y) = t_2(x, y) = 1$ then $x \circ y = 1 = y \circ x$ thus $x \leq y$ and $y \leq x$ whence $x = y$. Hence, t_1, t_2 are Csákány's terms for weak regularity and hence $\mathcal{O}$ is weakly regular. □

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