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The Seifert Matrices of Periodic Links with Rational Quotients

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Abstract. In this paper, we characterize the Seifert matrices of p-periodic links whose quotients are 2-bridge links $C(2, n_1, -2, n_2, \cdots, n_r, (-2)^r)$ and give formulas for the signatures and determinants of the 3-periodic links of these kind in terms of $n_1, n_2, \cdots, n_r$.

1. Introduction

A link L in S^3 is called a *p-periodic link* ($p \geq 2$ an integer) if there exists an orientation preserving auto-homeomorphism h of S^3 such that $h(L) = L$, h is of order p and the set of fixed points of h is a circle disjoint from L. In this paper, we are interested in a special class of periodic knots and links.

A link in S^3 is called a *p-periodic link with rational quotient* if it is obtained as the preimage of one component of a 2-bridge link in S^3 by the p-fold branched cyclic covering branched along the other component. In [5], the authors introduced a special kind of Conway's normal form $C(2, n_1, -2, n_2, \cdots, n_r, (-1)^r 2)$ of a 2-bridge link with two components and studied the excellent component of the character variety of periodic knots in S^3 with rational quotient. In [10], the authors re-examined this presentation to study the Alexander polynomials of 2-bridge links and periodic links in S^3 with rational quotients in terms of $n_1, n_2, \cdots, n_r$. In [7, 11], the authors gave formulas for the Casson knot invariant and the Δ-unknotting number of p-periodic knots with rational quotients via $n_1, n_2, \cdots, n_r$.

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The purpose of this paper is to give a characterization of the Seifert matrices of periodic links with rational quotients and to study the properties of numerical invariants of the Seifert matrices. In Section 2, we review presentations of 2-bridge links and p-periodic links with rational quotients. In Section 3, we show that the Seifert matrices of $(p+1)$-periodic links with rational quotient $C(2, n_1, -2, n_2, \cdots, n_r, (-1)^r 2)$ is S-equivalent to a $p \times p$ block tridiagonal matrix in which each block is also a $r \times r$ tridiagonal matrix whose entries are completely determined by the integers $n_1, n_2, \cdots, n_r$. In Section 4, we give formulas for the signature and determinant of a 3-periodic link with rational quotient $C(2, n_1, -2, n_2, \cdots, n_r, (-1)^r 2)$ in terms of $n_1, n_2, \cdots, n_r$.

2. Periodic links with rational quotients

To each pair (α, β) of two co-prime integers subject to the condition that β is odd and $0 < |\beta| < \alpha$, Schubert[14] associated an oriented diagram on the 2-sphere S^2 of an oriented 2-bridge knot(α odd) or link(α even) L in S^3, now called the *Schubert normal form* of L and denoted by $S(\alpha, \beta)$, and showed that any (oriented) 2-bridge knots and links in S^3 can be represented in this way. Two such pairs of integers (α, β) and (α', β') define an equivalent oriented (resp. unoriented) knot or link if and only if

$$\alpha = \alpha' \text{ and } \beta^{\pm 1} \equiv \beta' \bmod 2\alpha \text{ (resp. mod } \alpha),$$

where β^{-1} denotes the integers with the properties $0 < \beta^{-1} < 2\alpha$ and $\beta\beta^{-1} \equiv 1 \bmod 2\alpha$.

Let $[a_1, a_2, \cdots, a_n]$ denote the continued fraction of α/β:

$$[a_1, a_2, \cdots, a_n] \equiv a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_n}}} = \frac{\alpha}{\beta}.$$

Then $L = S(\alpha, \beta)$ has also a diagram $C(a_1, a_2, \cdots, a_n)$, called *Conway normal form* of L, as shown in Figure 1, depending on whether n is even or odd [1]. The integral tangles in Figure 1, which are rectangles labeled a_i, are the 2-braids with $|a_i|$ crossings as shown in Figure 2. It is well known that $L = S(\alpha, \beta)$ admits a diagram $C(2b_1, 2b_2, \cdots, 2b_m)$, which is equivalent to $C(a_1, a_2, \cdots, a_n)$[6].

It is known [5], [10] that the 2-bridge link $L = S(\alpha, \beta)$(α even) can also be represented by Conway diagram of the form $C(2, n_1, -2, n_2, \cdots, n_r, (-1)^r 2)$ as shown in Figure 3. We choose an orientation of the 2-bridge link $C(2, n_1, -2, n_2, \cdots, n_r, (-1)^r 2)$ as shown in Figure 3. Then it is easy to see that the diagram shown in Figure 3 can be deformed to the diagrams in Figure 4 by using Reidemeister moves. Throughout this paper, an oriented 2-bridge link L in S^3 represented by the Conway normal form $C(2, n_1, -2, n_2, \cdots, n_r, (-1)^r 2)$ is denoted by $L = \overrightarrow{C}[[n_1, n_2, \cdots, n_r]]$.

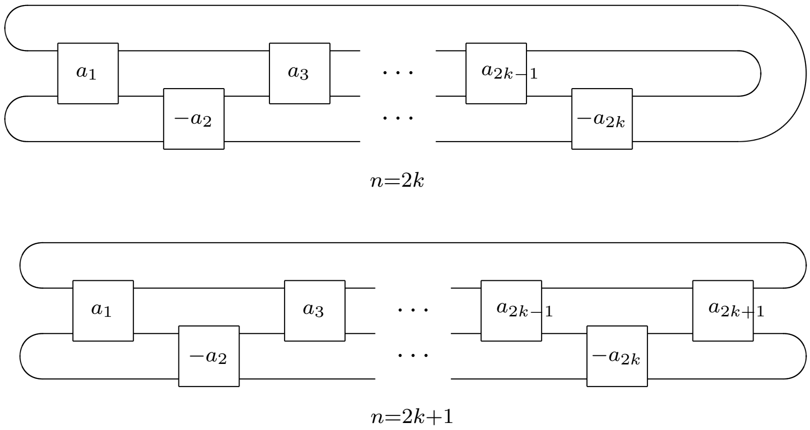


Figure 1:

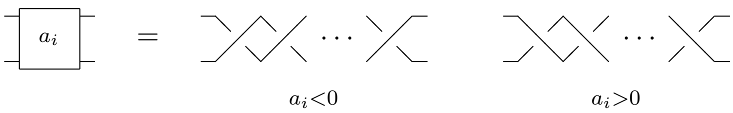


Figure 2:

A link L in S^3 is called a *p-periodic link* ($p \geq 2$ an integer) if there exists an orientation preserving auto-homeomorphism h of S^3 such that $h(L) = L$, h is of order p and the set $\mathrm{Fix}(h)$ of fixed points of h is a circle disjoint from L. In this case, the link $L/\langle h\rangle \cup \mathrm{Fix}(h)$ in the orbit space $S^3/\langle h\rangle \cong S^3$ is called the *quotient link* of L. Let K be an oriented link in S^3 and U an oriented trivial knot with $K \cap U = \emptyset$. For any integer $p \geq 2$, let $\phi_U^p : \Sigma^3 \to S^3$ be a p-fold branched cyclic covering branched along U. Then Σ^3 is homeomorphic to the 3-sphere S^3. Then $(\phi_U^p)^{-1}(K)$ is a p-periodic link in Σ^3 with $L = K \cup U$ as its quotient link. We give an orientation to $(\phi_U^p)^{-1}(K)$ induced by the orientation of K. Note that any periodic knot or link in S^3 arises in this manner.

Definition 2.1. A link $\tilde{L}$ in S^3 is called a *p-periodic link with rational quotient* if it is a p-periodic link whose quotient link is a 2-bridge link, or equivalently, if there exists a 2-bridge link $L = U_1 \cup U_2$ in S^3 such that $\tilde{L}$ is equivalent to the preimage $(\phi_{U_2}^p)^{-1}(U_1)$ of the component U_1 of L by a p-fold cyclic covering $\phi_{U_2}^p : \Sigma^3 \to S^3$ branched along the component U_2 of L.

Note that each component U_1 and U_2 of L is a trivial knot and they can be interchanged each other by an orientation preserving homeomorphism of

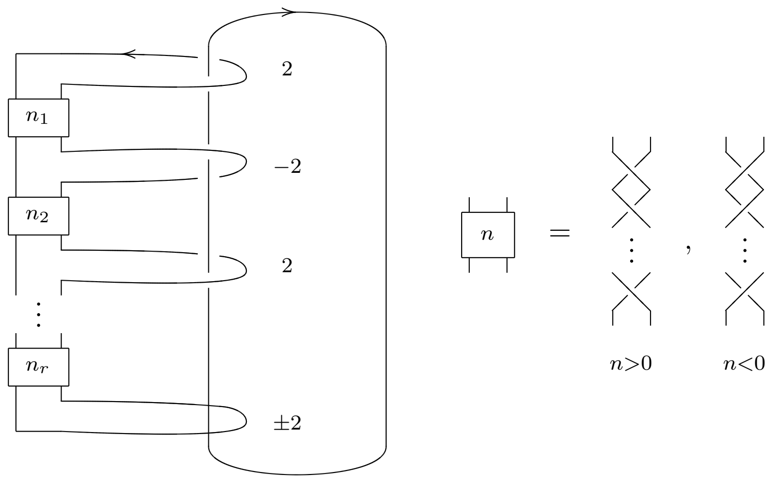


Figure 3:

S^3[13]. This implies that $(\phi^p_{U_2})^{-1}(U_1)$ is equivalent to $(\phi^p_{U_1})^{-1}(U_2)$. Now let $L = \overrightarrow{C}[[n_1, n_2, \cdots, n_r]] = U_1 \cup U_2$ be an oriented 2-bridge link as shown Figure 4. Then the diagram, $D^{(p)}$, shown in Figure 5 is a canonical oriented p-periodic diagram of the oriented p-periodic link $(\phi^p_{U_2})^{-1}(U_1)$ with rational quotient $L = \overrightarrow{C}[[n_1, n_2, \cdots, n_r]]$. In what follows, we shall denote the oriented p-periodic link $(\phi^p_{U_2})^{-1}(U_1)$ by $L^{(p)}$ or $\overrightarrow{C}[[n_1, n_2, \cdots, n_r]]^{(p)}$ for our convenience. Then any p-periodic link with rational quotient can be represented by $\overrightarrow{C}[[n_1, n_2, \cdots, n_r]]^{(p)}$ for some nonzero integers $n_1, n_2, \cdots, n_r$[7], [10].

3. Seifert matrices

We begin with a brief review of Seifert matrix of a link in S^3 from Chapter 5 in [13].

A *Seifert surface* for a link in S^3 is a connected compact orientable surface embedded in S^3 with as its boundary L. In [15], Seifert proved the existence of Seifert surface for a link L applying L an algorithm, called *Seifert's algorithm*, on a diagram of L. Let L be a link and F its Seifert surface. There is an embedding $F \times [-1, 1] \to S^3$ such that $b(F \times \{0\}) = F$ and $b(F \times \{1\})$ lies on the positive side of F. For any simple closed curve $x \in F$, let $x^+ = b(x \times \{1\})$ and $x^- = b(x \times \{-1\})$. Since $H_1(F)$ is a free abelian group of finite rank n and is generated by simple closed oriented curves $x_1, \cdots, x_n$, we can define a bilinear form $\phi : H_1(F) \times H_1(F) \to \mathbb{Z}$

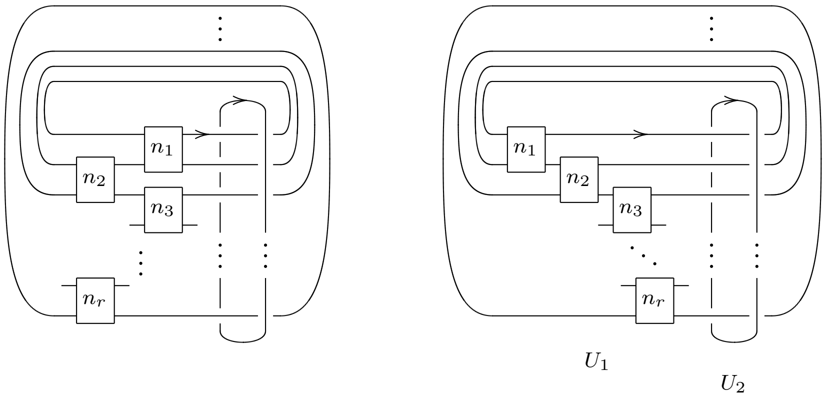


Figure 4:

by

$$\phi(x_i, x_j) = lk(x_i, x_j^+), i, j = 1, 2, \cdots, n.$$

This is called the *Seifert pairing* or *linking form* of F. The $n \times n$ matrix $M = (m_{i,j})$ defined by

$$m_{i,j} = \phi(x_i, x_j)$$

is called a *Seifert matrix* of L associated to F. The Seifert matrix of L depends on the Seifert surface F and the choice of generators of $H_1(F)$.

Theorem 3.1([13]). *Two Seifert matrices obtained from two equivalent links can be changed from one to the other by applying, a finite number of times, the following two operations Λ_1 and Λ_2, and their inverses:*

Λ_1 : $M_1 \longrightarrow PM_1P^T$, *where P is an invertible matrix with* $\det P = \pm 1$ *and P^T denotes the transpose matrix of P.*

$$\Lambda_2 : M_1 \longrightarrow M_2 = \begin{pmatrix} M_1 & \mathbf{v} & \mathbf{0} \\ \mathbf{0} & 0 & 1 \\ \mathbf{0} & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} M_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{v} & 0 & 0 \\ \mathbf{0} & 1 & 0 \end{pmatrix},$$

where $\mathbf{v}$ *denotes an arbitrary integral row or column vector, and* $\mathbf{0}$ *the row or column zero vector.*

Two square matrices M and M' are said to be *S-equivalent* if one is obtained from the other by applying the operations Λ_1, Λ_2 and the inverse Λ_2^{-1} a finite number of times.

For any real number y, let $\lfloor y \rfloor$ denote the largest integer less than or equal to y.

Theorem 3.2. *For given nonzero integers $n_1, n_2, \cdots, n_r$ $(r \geq 1)$ and a positive*

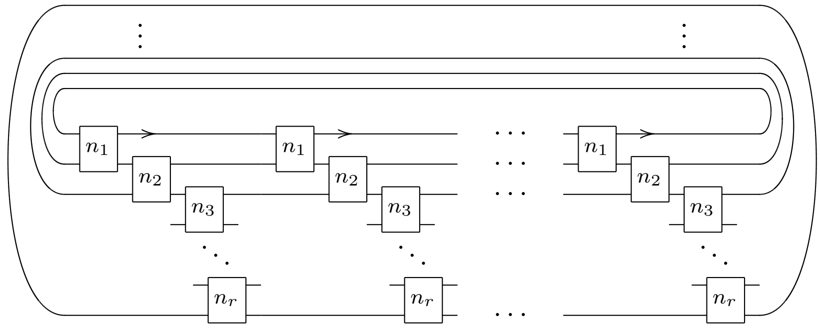


Figure 5:

integer $p \geq 1$, let A, B and C be $r \times r$ tridiagonal matrices with integral entries given by

$$A = \begin{bmatrix} \alpha_1 & \epsilon_1 - 1 & & & \\ & \alpha_2 & \epsilon_2 - 1 & & \\ & & \ddots & \ddots & \\ & & & \alpha_{r-1} & \epsilon_{r-1} - 1 \\ & & & & \alpha_r \end{bmatrix},$$

$$B = \begin{bmatrix} \beta_1 & & & & \\ -\epsilon_1 & \beta_2 & & & \\ & -\epsilon_2 & \ddots & & \\ & & \ddots & \beta_{r-1} & \\ & & & -\epsilon_{r-1} & \beta_r \end{bmatrix},$$

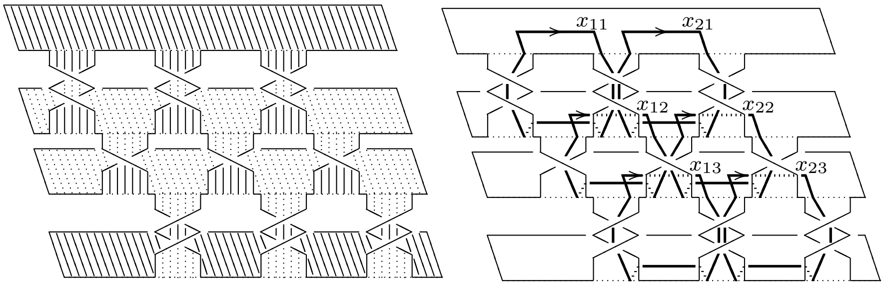
$$C = \begin{bmatrix} -n_1 & 1-\epsilon_1 & & & \\ \epsilon_1 & -n_2 & 1-\epsilon_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \epsilon_{r-2} & -n_{r-1} & 1-\epsilon_{r-1} \\ & & & \epsilon_{r-1} & -n_r \end{bmatrix},$$

where $\alpha_i = \left\lfloor \frac{n_i+1-\epsilon_{i-1}}{2} \right\rfloor$, $\beta_i = \left\lfloor \frac{n_i+\epsilon_{i-1}}{2} \right\rfloor$ and $\epsilon_i = 1$ if $n_1 + n_2 + \cdots + n_i + i$ is even and $\epsilon_i = 0$ otherwise. Suppose that $L^{(p+1)}$ is the $(p+1)$-periodic link in S^3 with rational quotient $L = \overrightarrow{C}[[n_1, n_2, \cdots, n_r]]$. Then a Seifert matrix of $L^{(p+1)}$ is

S-equivalent to the $p \times p$ *block tridiagonal matrix*

$$M = \begin{bmatrix} C & B & & & & \\ A & C & B & & & \\ & A & C & B & & \\ & & \ddots & \ddots & \ddots & \\ & & & A & C & B \\ & & & & A & C \end{bmatrix}.$$

Proof. Let $D^{(p+1)}$ be the diagram of $L^{(p+1)}$ as shown in Figure 5 and let F be the Seifert surface of $L^{(p+1)}$ obtained by applying Seifert algorithm to $D^{(p+1)}$. Let $\{x_{i,j} \mid 1 \le i \le p, 1 \le j \le r\}$ be the set of simple closed curves which represent the generators of $H_1(F)$. We assign the clockwise orientation to each curves $x_{i,j}$. For example, see Figure 6. In Figure 6, there are the Seifert surface F of $\overrightarrow{C}[[2,1,-2]]^{(3)}$ obtained by applying Seifert algorithm and simple closed curves representing the generators of F. For each 4-tuple (i,j,k,l) with $1 \le i,k \le p$ and $1 \le j,l \le r$, we



A Seifert surface F of $\overrightarrow{C}[[2,1,-2]]^{(3)}$. Generators of $H_1(F)$.

Figure 6:

can calculate that

$$(3.1) \qquad lk(x_{i,j}, x^{+}_{k,l}) = \begin{cases} -n_j & \text{if } k=i, l=j, \\ \epsilon_{j-1} & \text{if } k=i, l=j-1, \\ -\epsilon_{j-1} & \text{if } k=i+1, l=j-1, \\ 1-\epsilon_j & \text{if } k=i, l=j+1, \\ -1+\epsilon_j & \text{if } k=i-1, l=j+1, \\ \alpha_j & \text{if } k=i-1, l=j, \\ \beta_j & \text{if } k=i+1, l=j, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the simple closed curves $x_{1,1}, x_{1,2}, \cdots, x_{1,r}$, $x_{2,1}, x_{2,2}, \cdots, x_{2,r}$, $\cdots$, $x_{p,1}, x_{p,2}, \cdots, x_{p,r}$ and let $M' = (m'_{a,b})$ be the $rp \times rp$ Seifert matrix defined by

$$m'_{a,b} = lk(x_{i,j}, x^{+}_{k,l})$$

where $a = r(i-1)+j$ and $b = r(k-1)+l$. We can partition the matrix M' into $r \times r$ submatrices of M' as follows:

$$M' = (M'_{i,j}),\ M'_{i,j} = (m''_{k,l})$$

where $m''_{k,l} = m'_{r(i-1)+k,r(j-1)+l}$. From (3.1), we can see that

$$M'_{i,j} = \begin{cases} C & \text{if } j = i, \\ A & \text{if } j = i-1, \\ B & \text{if } j = i+1, \\ O & \text{otherwise}, \end{cases}$$

where O is the $r \times r$ zero matrix. Hence $M' = M$. This completes the proof. $\square$

Example 3.3. Let $L^{(3)}$ be the 3-periodic link with rational quotient $L = \overrightarrow{C}[[2,1,-2]]$. Let F be a Seifert surface obtained by applying Seifert algorithm to a diagram as described in Figure 6. Consider the simple closed curves representing the generator of $H_1(F)$ as depicted in Figure 6. Then any Seifert matrix M of $L^{(3)}$ is S-equivalent to the matrix of the form:

$$M = \left[\begin{array}{ccc|ccc} -2 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & -1 \\ \hline 1 & -1 & 0 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{array}\right].$$

4. Invariants of Seifert matrices

For a real symmetric matrix A, there exists an invertible matrix P such that $PAP^T = B$ is a diagonal matrix. By Sylvester's Theorem in Linear Algebra, the sum of signs of the entries in the diagonal of B, called the *signature* of A and is denoted by $\sigma(A)$, is independent on the diagonalization. It is well known that two S-equivalent symmetric matrices have the same signature. Now let M be a Seifert matrix of a link L. Then the *signature* $\sigma(L)$ of L is defined by

$$\sigma(L) = \sigma(M + M^T).$$

Note that $\sigma(L)$ is a link invariant [13].

For given nonzero integers $n_1, n_2, \cdots, n_r$ $(r \geq 1)$, let $L^{(p+1)}$ be the $(p+1)$-periodic link in $S^3 (p \geq 1)$ with rational quotient $L = \overrightarrow{C}[[n_1, n_2, \cdots, n_r]]$. Let M be the Seifert matrix of $L^{(p+1)}$ given by Theorem 3.2 above and $S = M + M^T$. Then

S is the $p \times p$ symmetric block tridiagonal matrix given by

$$S = \begin{bmatrix} E & F^T & & & & \\ F & E & F^T & & & \\ & F & E & F^T & & \\ & & \ddots & \ddots & \ddots & \\ & & & F & E & F^T \\ & & & & F & E \end{bmatrix}, \tag{4.2}$$

where E and F are $r \times r$ tridiagonal matrices given by

$$E = \begin{bmatrix} -2n_1 & 1 & & & \\ 1 & -2n_2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2n_{r-1} & 1 \\ & & & 1 & -2n_r \end{bmatrix},$$

$$F = \begin{bmatrix} n_1 & -1 & & & \\ & n_2 & -1 & & \\ & & \ddots & \ddots & \\ & & & n_{r-1} & -1 \\ & & & & n_r \end{bmatrix}.$$

4.1. Signatures of 2-periodic links with rational quotients. For given nonzero integers $n_1, n_2, \cdots, n_r$ $(r \geq 1)$, let $L^{(2)}$ be the 2-periodic link in S^3 with rational quotient $L = \overrightarrow{C}[[n_1, n_2, \cdots, n_r]]$. Let M denote the Seifert matrix of $L^{(2)}$ given by Theorem 3.2 above and set $S = M + M^T$. From (4.2), we know that $S = E$. In [7], the authors show that $L^{(2)}$ is the 2-bridge knot with Conway normal form $C(-2n_1, -2n_2, ..., -2n_r)$. For each $k = 1, 2, \cdots, r$, we define a rational number $\langle k \rangle$ by

$$\langle k \rangle = \begin{cases} -2n_1 & \text{if } k = 1, \\ -2n_k - \frac{1}{\langle k-1 \rangle} & \text{if } k = 2, 3, \cdots, r. \end{cases}$$

We know that all $\langle k \rangle$ is not equal to zero. We can calculate that

$$S_2 = VDV^T,$$

where D is the diagonal matrix with diagonal entries $\langle 1 \rangle, \langle 2 \rangle, \cdots, \langle r \rangle$ and V is the

$r \times r$ tridiagonal matrices given by

$$V = \begin{bmatrix} 1 & & & & & \\ \frac{1}{\langle 1 \rangle} & 1 & & & & \\ & \frac{1}{\langle 2 \rangle} & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \frac{1}{\langle r-2 \rangle} & 1 & \\ & & & & \frac{1}{\langle r-1 \rangle} & 1 \end{bmatrix}.$$

Since all n_k are nonzero, it follows that $0 < |\frac{1}{\langle k-1 \rangle}| < 1$ and hence the sign of $\langle k \rangle$ is opposite to the sign of n_k. Therefore the signature of 2-periodic link $L^{(2)}$ with rational quotient $L = \overrightarrow{C}[[n_1, n_2, \cdots, n_r]]$ is given by

$$\sigma(L^{(2)}) = -\sum_{i=1}^{r} \frac{n_i}{|n_i|}. \tag{4.3}$$

4.2. Signatures of 3-periodic links with rational quotients. For given nonzero integers $n_1, n_2, \cdots, n_r$ $(r \geq 1)$, let $L^{(3)}$ be the 3-periodic link in S^3 with rational quotient $L = \overrightarrow{C}[[n_1, n_2, \cdots, n_r]]$. Let M denote the Seifert matrix of $L^{(3)}$ given by Theorem 3.2 above and set $S = M + M^T$. From (4.2), we know that S is given by

$$S = \begin{bmatrix} E & F^T \\ F & E \end{bmatrix}.$$

For given nonzero integers $n_1, n_2, \cdots, n_r$, we define the rational numbers $d_1, d_2, \cdots, d_r$, $w_1, w_2, \cdots, w_{r-2}$ by

$$\begin{aligned} d_1 &= -\frac{3n_1}{2} + \frac{1}{2n_2}, \\ d_2 &= \frac{1}{2n_1} - \frac{3n_2}{2} + \frac{1}{2n_3}, \\ d_i &= \frac{1}{2n_{i-1}} - \frac{3n_i}{2} + \frac{1}{2n_{i+1}} - \frac{1}{4n_{i-1}^2 d_{i-2}}, \ i = 3, 4, \cdots, r-1, \\ d_r &= \frac{1}{2n_{r-1}} - \frac{3n_r}{2} - \frac{1}{4n_{r-1}^2 d_{r-2}}, \\ w_j &= \frac{\tau_j}{2n_{j+1} d_j}, \ j = 1, 2, \cdots, r-2, \end{aligned}$$

where $\tau_j = -1$ if $j - 1 \equiv 0 \pmod 3$ and $\tau_j = 1$ otherwise. Note that if $d_i = 0$, then w_j and d_{j+2} are not defined for all $j = i, i+1, \cdots, r-2$.

Thmorem 4.1. *Let $n_1, n_2, \cdots, n_r$ be given nonzero integers($r \geq 1$) and let $L^{(3)}$*

be the 3*-periodic link in* S^3 *with rational quotient* $L = \overrightarrow{C}[[n_1, n_2, \cdots, n_r]]$*. Let* M *be the Seifert matrix of* $L^{(3)}$ *given by Theorem 3.2 above and set* $S = M + M^T$*. Suppose that* $d_i \neq 0$ *for all* $i = 1, 2, \cdots, r$*. Then there exists an invertible matrix* P *such that* $\det P = \pm 1$ *and*

$$S = PDP^T,$$

where D *is the* $2r \times 2r$ *diagonal matrix with diagonal entries* $-2n_1, -2n_2, \cdots, -2n_r$, $d_1, d_2, \cdots, d_r$*.*

Proof. Let D_1 and D_2 be the $r \times r$ diagonal matrices with diagonal entries $-2n_1, -2n_2, \cdots, -2n_r$ and $d_1, d_2, \cdots, d_r$, respectively, and let $G = (g_{ij})$ be the $r \times r$ tridiagonal matrix given by

$$g_{ij} = \begin{cases} -n_i & \text{if } j = i, \\ -1 & \text{if } j = i+1 \text{ and } i \not\equiv 0 \pmod 3, \\ 1 & \text{if } j = i+1 \text{ and } i \equiv 0 \pmod 3, \\ 1 & \text{if } j = i-1 \text{ and } j \not\equiv 0 \pmod 3, \\ -1 & \text{if } j = i-1 \text{ and } j \equiv 0 \pmod 3, \\ 0 & \text{otherwise.} \end{cases}$$

Then $D = D_1 \oplus D_2$ and we have that

$$\begin{bmatrix} U_1 & U_3 \\ U_2 & U_1 \end{bmatrix} S \begin{bmatrix} U_1 & U_3 \\ U_2 & U_1 \end{bmatrix}^T = \begin{bmatrix} D_1 & G^T \\ G & D_1 \end{bmatrix}, \tag{4.4}$$

where $U_3 = U_1 - U_2$, $U_1 = (u_{ij})$ and $U_2 = (v_{ij})$ are $r \times r$ diagonal matrices with entries

$$u_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } i \not\equiv 0 \pmod 3, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$v_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } i \not\equiv 1 \pmod 3, \\ 0 & \text{otherwise.} \end{cases}$$

Now let $W = (w_{ij})$ be the $r \times r$ matrix given by

$$w_{ij} = \begin{cases} 1 & \text{if } j = i, \\ w_j & \text{if } j = i-2, \\ 0 & \text{otherwise.} \end{cases}$$

By elementary calculations, we obtain that

$$D_1 = GD_1^{-1}G^T + WD_2W^T.$$

Hence it follows that

$$\begin{bmatrix} D_1 & G^T \\ G & D_1 \end{bmatrix} = \begin{bmatrix} I & O \\ GD_1^{-1} & W \end{bmatrix} \begin{bmatrix} D_1 & O \\ O & D_2 \end{bmatrix} \begin{bmatrix} I & O \\ GD_1^{-1} & W \end{bmatrix}^T. \tag{4.5}$$

From (4.4) and (4.5), we have $S = PDP^T$, where

$$P = \begin{bmatrix} U_1 & U_3 \\ U_2 & U_1 \end{bmatrix}^{-1} \begin{bmatrix} I & O \\ GD_1^{-1} & W \end{bmatrix}.$$

This completes the proof. □

Corollary 4.2. *Let $n_1, n_2, \cdots, n_r$ be given nonzero integers($r \geq 1$) and let $L^{(3)}$ be the 3-periodic link in S^3 with rational quotient $L = \overrightarrow{C}[[n_1, n_2, \cdots, n_r]]$. Suppose that $d_i \neq 0$ for all $i = 1, 2, \cdots, r$. Then*

$$\sigma(L^{(3)}) = \sum_{i=1}^{r} \left(\frac{d_i}{|d_i|} - \frac{n_i}{|n_i|} \right).$$

Proof. The result follows from Theorem 4.1 at once. □

Corollary 4.3. *Let $n_1, n_2, \cdots, n_r$ be given nonzero integers($r \geq 1$) and let $L^{(3)}$ be the 3-periodic link in S^3 with rational quotient $L = \overrightarrow{C}[[n_1, n_2, \cdots, n_r]]$. Suppose that $|n_i n_{i+1} n_{i+2} n_{i+3}| \geq 2$ for each $i = 1, 2, \cdots, r-3$. Then the signature $\sigma(L^{(3)})$ of $L^{(3)}$ is given by*

$$\sigma(L^{(3)}) = -2 \sum_{i=1}^{r} \frac{n_i}{|n_i|} = 2\sigma(L^{(2)}).$$

Proof. We will claim that the sign of d_i is opposite of the sign of n_i and the absolute value of d_i is greater than or equal to $\frac{1}{4}$ for all $i = 1, 2, \cdots, r$.

Since n_1 and n_2 are nonzero integers and $d_1 = -\frac{3n_1}{2} + \frac{1}{2n_2}$, the sign of d_1 is opposite of the sign of n_1 and

$$|d_1| \geq \frac{3}{2} - \frac{1}{2} = 1 \geq \frac{1}{4}.$$

Since n_1, n_2 and n_3 are nonzero integers and $d_2 = \frac{1}{2n_1} - \frac{3n_2}{2} + \frac{1}{2n_3}$, the sign of d_2 is opposite of the sign of n_2 and

$$|d_2| \geq \frac{3}{2} - \frac{1}{2} - \frac{1}{2} = \frac{1}{2} \geq \frac{1}{4}.$$

Suppose that the sign of d_i is opposite of the sign of n_i and $|d_i| \geq \frac{1}{4}$ for all $i = 1, 2, \cdots, k$. Now we claim that the sign of d_{k+1} is opposite of the sign of n_{k+1} and $|d_{k+1}| \geq \frac{1}{4}$. We recall that, for $2 \leq k \leq r-2$,

$$d_{k+1} = \frac{1}{2n_k} - \frac{3n_{k+1}}{2} + \frac{1}{2n_{k+2}} - \frac{1}{4n_k^2 d_{k-1}}.$$

If $|n_{k+1}| \geq 2$, then $|\frac{3n_{k+1}}{2}| \geq 3$. Since $|d_{k-1}| \geq \frac{1}{4}$, $|\frac{1}{2n_k} + \frac{1}{2n_{k+2}} - \frac{1}{4n_k^2 d_{k-1}}| \leq \frac{1}{2} + \frac{1}{2} + 1 = 2$. Hence the sign of d_{k+1} is opposite of the sign of n_{k+1} and

$$|d_{k+1}| \geq 3 - 2 = 1 \geq \frac{1}{4}.$$

If $|n_{k+1}| = 1$ and $|n_k| \geq 2$, then $|\frac{3n_{k+1}}{2}| = \frac{3}{2}$ and $|\frac{1}{2n_k} + \frac{1}{2n_{k+2}} - \frac{1}{4n_k^2 d_{k-1}}| \leq \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$. Hence the sign of d_{k+1} is opposite of the sign of n_{k+1} and

$$|d_{k+1}| \geq \frac{3}{2} - 1 = \frac{1}{2} \geq \frac{1}{4}.$$

If $|n_{k+1}| = 1$, $|n_k| = 1$ and $|n_{k-1}| \geq 2$, then $|d_{k-1}| \geq 1$. Since $|\frac{3n_{k+1}}{2}| = \frac{3}{2}$ and $|\frac{1}{2n_k} + \frac{1}{2n_{k+2}} - \frac{1}{4n_k^2 d_{k-1}}| \leq \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{5}{4}$, the sign of d_{k+1} is opposite of the sign of n_{k+1} and

$$|d_{k+1}| \geq \frac{3}{2} - \frac{5}{4} = \frac{1}{4}.$$

If $|n_{k+1}| = 1$, $|n_k| = 1$ and $|n_{k-1}| = 1$, then $|n_{k-2}| \geq 2$ and $|n_{k+2}| \geq 2$. If $4 \leq k \leq r-2$, then $|d_{k-1}| \geq |\frac{3n_{k-1}}{2}| - |\frac{1}{2n_{k-2}} + \frac{1}{2n_k} - \frac{1}{4n_{k-2}^2 d_{k-3}}| \geq \frac{3}{2} - (\frac{1}{4} + \frac{1}{2} + \frac{1}{4}) = \frac{1}{2}$ and hence $|\frac{1}{2n_k} + \frac{1}{2n_{k+2}} - \frac{1}{4n_k^2 d_{k-1}}| \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{2} = \frac{5}{4}$. If $k = 3$, then $|d_{k-1}| = |d_2| \geq \frac{1}{2}$ and hence $|\frac{1}{2n_k} + \frac{1}{2n_{k+2}} - \frac{1}{4n_k^2 d_{k-1}}| \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{2} = \frac{5}{4}$. If $k = 2$, then $|d_{k-1}| = |d_1| \geq 1$ and hence $|\frac{1}{2n_k} + \frac{1}{2n_{k+2}} - \frac{1}{4n_k^2 d_{k-1}}| \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$. Since $|\frac{3n_{k+1}}{2}| = \frac{3}{2}$, the sign of d_{k+1} is opposite of the sign of n_{k+1} and

$$|d_{k+1}| \geq \frac{3}{2} - \frac{5}{4} = \frac{1}{4}.$$

If $k+1 = r$, then we can also see that the sign of d_r is opposite of the sign of n_r and $|d_r| \geq \frac{1}{4}$ by the similar argument.

Therefore d_i is nonzero and the sign of d_i is opposite of the sign of n_i for all $i = 1, 2, \cdots, r$. From Corollary 4.2, the signature $\sigma(L^{(3)})$ of $L^{(3)}$ is given by

$$\sigma(L^{(3)}) = -2 \sum_{i=1}^{r} \frac{n_i}{|n_i|}.$$

From (4.3), $\sigma(L^{(3)}) = 2(-\sum_{i=1}^{r} \frac{n_i}{|n_i|}) = 2\sigma(L^{(2)})$. This completes the proof. $\square$

Corollary 4.4. *Let $n_1, n_2, \cdots, n_r$ be given nonzero integers($r \geq 1$) and let $L^{(3)}$ be the 3-periodic link in S^3 with rational quotient $L = \overrightarrow{C}[[n_1, n_2, \cdots, n_r]]$. Suppose that $d_i \neq 0$ for all $i = 1, 2, \cdots, r$. Then*

$$\det(L^{(3)}) = |\Delta_{L^{(3)}}(-1)| = 2^r \left| n_1 n_2 \cdots n_r d_1 d_2 \cdots d_r \right|.$$

Proof. The result follows from Theorem 4.1 at once. $\square$

Example 4.5. The symmetric matrix S of the 3-periodic link $L^{(3)}$ with rational quotient $L = \vec{C}[[2, 1, -2]]$ is given by

$$S = \left[\begin{array}{ccc|ccc} -4 & 1 & 0 & 2 & 0 & 0 \\ 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 1 & 4 & 0 & -1 & -2 \\ \hline 2 & -1 & 0 & -4 & 1 & 0 \\ 0 & 1 & -1 & 1 & -2 & 1 \\ 0 & 0 & -2 & 0 & 1 & 4 \end{array}\right].$$

We have that $S = PDP^T$, where $D = \mathrm{diag}(-4, -2, 4, -\frac{5}{2}, -\frac{3}{2}, \frac{18}{5})$ and

$$P = \left[\begin{array}{ccc|ccc} \frac{1}{2} & -\frac{1}{2} & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{5} & 0 & 1 \\ \hline \frac{1}{2} & \frac{1}{2} & 0 & 1 & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{array}\right].$$

This implies that $\sigma(L^{(3)}) = -2 = -2\sum_{i=1}^{3} \frac{n_i}{|n_i|} = 2\sigma(L^{(2)})$ (cf. Corollary 4.3) and $\det(L^{(3)}) = 432$.

Remarks 4.6.

(1) In order to generalize Theorem 4.1 for the case $p \geq 4$, we need to diagonalize the symmetric matrix $S = M + M^T$ in (4.2) of a p-periodic link with rational quotient $\overrightarrow{C}[[n_1, n_2, \cdots, n_r]]$ so that the diagonal entries are completely expressed as the integer $n_1, n_2, \cdots, n_r$. The authors have no such a diagonalization of S and so we leave this an open question.

It should be note that if $|n_i n_{i+1} n_{i+2} n_{i+3}| = 1$ for some $i = 1, 2, \cdots, r-3$, Corollary 4.3 may not holds. For example, if $n_1 = n_2 = n_3 = n_4 = n_6 = n_7 = n_8 = n_9 = 1$ and $n_5 = 2$, then $d_9 = 0$.

(2) The signatures of more general periodic knots and links in S^3 have been studied by several authors, for example, see [2], [3], [4], [8], [9], [12].

(3) It is well known that the Alexander polynomial is given by the formula $\Delta_K(t) \doteq \det\left(M - tM^T\right)$. In [10], with Fox's free differential calculus, Lee and Seo gave a recurrence formula for calculating the Alexander polynomials of 2-bridge links by using a special type of Conway diagram as shown in Figure 3 and the reduced Alexander polynomials of p-periodic links with rational quotient $\overrightarrow{C}[[n_1, n_2, \cdots, n_r]]$ in terms of $n_1, n_2, \cdots, n_r$ and p.

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