Vortex Filament Equation and Non-linear Schrödinger Equation in $S^3$

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Abstract. In 1906, da Rios, a student of Leivi-Civita, wrote a master’s thesis modeling the motion of a vortex in a viscous fluid by the motion of a curve propagating in $R^3$, in the direction of its binormal with a speed equal to its curvature. Much later, in 1971 Hasimoto showed the equivalence of this system with the non-linear Schrödinger equation (NLS)

$$q_t = i(q_{ss} + \frac{1}{2}|q|^2 q).$$

In this paper, we use the same idea as Terng used in her lecture notes but different technique to extend the above relation to the case of $S^3$, and obtained an analogous equation that

$$q_t = i[q_{ss} + (\frac{1}{2}|q|^2 + 1)q].$$

1. Introduction

The material of this section was taken from [2] with a minor modification.

1.1. A special orthogonal frame field on $S^3$

$S^3$ is the unit sphere in $R^4$ i.e.,

$$S^3 = \{x \in R^4||x| = 1\}.$$ (1.1)

For any $x, y \in S^3$, the distance $d(x, y)$ between $x$ and $y$ is defined by

$$\cos d(x, y) = x \cdot y,$$ (1.2)
where $x \cdot y$ is the inner product of $x$ and $y$. For any constant $a, a \in (0, 1)$, there exists $A \in O(4)$ such that
\begin{equation}
(1.3) \quad d(x, Ax) = a \forall x \in S^3.
\end{equation}

For example we may take
\begin{equation}
(1.4) \quad A = \begin{pmatrix}
a & -b & -c & -d \\
b & a & -d & c \\
c & d & a & -b \\
d & -c & b & a
\end{pmatrix} \text{ or } A = \begin{pmatrix}
-a & b & c & d \\
-b & a & -d & c \\
-c & d & a & -b \\
-d & -c & b & a
\end{pmatrix},
\end{equation}

and $a^2 + b^2 + c^2 + d^2 = 1$.

We can also regard $S^3$ as a set of all the unit quaternions and regard $R^4$ as a non-commutative division algebra. Its unit element is $1=(1,0,0,0)$, and its generators are $i=(0,1,0,0)$, $j=(0,0,1,0)$, $k=(0,0,0,1)$, where $i, j, k$ satisfy
\begin{equation}
(1.5) \quad \begin{cases}
i \cdot j = k = -j \cdot i \\
j \cdot k = i = -k \cdot j \\
k \cdot i = j = -i \cdot k \\
i^2 = j^2 = k^2 = -1.
\end{cases}
\end{equation}

Define the module of a quaternion $x = x_1i + x_2j + x_3j + x_4k \in R^4$ by
\begin{equation}
(1.6) \quad |x|^2 = \sum_{i=1}^{4} x_i^2,
\end{equation}

and the product of two quaternions has the property:
\begin{equation}
(1.7) \quad |x \cdot y| = |x| \cdot |y|,
\end{equation}

for any $x, y \in R^4$. So the set of all the unit quaternions i.e $S^3$ is a non-commutative Lie group. The two matrices in (1.4) just correspond to the left and right translation of $a_1 + bi + cj + dk \in S^3$. That is to say, for $g = a_1 + bi + cj + dk \in S^3$ we have
\begin{equation}
(1.8) \quad Lg, Rg : S^3 \rightarrow S^3 \quad \text{for } x \in S^3.
\end{equation}

The mapping

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\end{equation}
Vortex Filament Equation and Non-linear Schrödinger Equation in $S^3$ gives an isomorphism from $S^3$ to a subgroup of $O(4)$ corresponding to the left translation which is determined by the element in $S^3$. It will be convenient to regard $S^3$ as this subgroup for computation.

In the following we’ll find the tangent space of $S^3$ at the unit element. It is spaned by $x_1 = (0,1,0,0), x_2 = (0,0,1,0), x_3 = (0,0,0,1)$. Notice that $x_1$ is the tangent vector of the curve $c(t) = (\cos t, \sin t, 0, 0) \in S^3$ at $1=(1,0,0,0)$. Since

\[
(1.10) \quad c(t) = \cos t \cdot 1 + \sin t \cdot i = \begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix},
\]

so we can regard $x_1$ as

\[
(1.11) \quad \frac{d}{dt} c(t)|_{t=0} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in O(4),
\]

similarly regard $x_2, x_3$ as

\[
(1.12) \quad \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
\]

respectively. It’s easy to verify that

\[
(1.13) \quad [x_1, x_2] = 2x_3, [x_2, x_3] = 2x_1, [x_3, x_1] = 2x_2.
\]

If we ignore the first component of $x_i, i=1,2,3$, and regard it as the vector of $\mathbb{R}^3$ then

\[
(1.14) \quad \begin{cases} x_1 \times x_2 = x_3 \\ x_2 \times x_3 = x_1 \\ x_3 \times x_1 = x_2 \end{cases}
\]

or use the usual inner product and orientation in $S^3$, we can also define the above relation.

Let $\tilde{x}_i$ be the vector field which is obtained by the left translation of $x_i$ similarly we have

\[
(1.15) \quad \begin{cases} \tilde{x}_1 \times \tilde{x}_2 = \tilde{x}_3 \\ \tilde{x}_2 \times \tilde{x}_3 = \tilde{x}_1 \\ \tilde{x}_3 \times \tilde{x}_1 = \tilde{x}_2 \end{cases}
\]

The cross product $\times$ in the tangent space at each point in $S^3$ is defined by ordinary inner product and orientation.
1.2. The Frenet frame of curves on $S^3$

In this section we want to build the Frenet frame of curves in $S^3$. The theory of curves in $S^3$ has a special treatment. In other words we can use left invariant vector field $\tilde{x}_i$ to express all the tangent vector fields on $S^3$.

Let $c : [0, l] \to S^3$ be a curve and parametrized by its arc length. Its tangent vector is

$$\frac{d}{ds} c(s) = t(s),$$

as $c(s) \in S^3$, then

$$c(s) \cdot c(s) = 1.$$

Differentiating both sides of (1.17), we get

$$\frac{d}{ds} c(s) \cdot c(s) = 0.$$

So $t(s) = \frac{d}{ds} c(s)$ is the tangent vector field on $S^3$ along $c(s)$, it can be expressed as

$$t(s) = \sum_{i=1}^{3} f_i(s) \tilde{x}_i(c(s)),$$

where $f_i(s)$ are some smooth functions on $c(s)$. As $c(s)$ is parametrized by its arc length, so

$$\sum_{i=1}^{3} f_i^2(s) = 1.$$

Differentiating both sides of (1.20), we get

$$\sum_{i=1}^{3} f_i(s)f_i'(s) = 0.$$

Let $\nabla'$ denotes covariant differentiation on $S^3$. Any vector fields along $c(s)$ can be expressed as

$$\sum_{i=1}^{3} h_i(s) \tilde{x}_i(c(s)).$$
Then

\[
\frac{\nabla'}{ds}\{\sum_{i=1}^{3} h_i(s)\tilde{x}_i(c(s))\} = \sum_{i=1}^{3} h_i'(s)\tilde{x}_i(c(s)) + \sum_{i=1}^{3} h_i(s)\frac{\nabla'}{ds}\tilde{x}_i(c(s))
\]

\[
= \sum_{i=1}^{3} h_i'(s)\tilde{x}_i(c(s)) + \sum_{i=1}^{3} h_i(s)\sum_{j=1}^{3} f_j(s)\nabla'\tilde{x}_j(c(s))
\]

\[
= \sum_{i=1}^{3} h_i'(s)\tilde{x}_i(c(s)) + \frac{3}{2}\sum_{i=1}^{3} h_i(s)f_j(s)[\tilde{x}_j, \tilde{x}_i](c(s))
\]

\[
= \sum_{i=1}^{3} h_i'(s)\tilde{x}_i(c(s)) + \det\left(\begin{array}{ccc}
\tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \\
1 & f_1 & f_2 \\
h_1 & h_2 & h_3
\end{array}\right).
\]

In particular

\[
\frac{\nabla'}{ds}t(s) = \sum_{i=1}^{3} f_i'(s)\tilde{x}_i(c(s)).
\]

Define curvature function of curve \( c(s) \) by

\[
k = \left| \frac{\nabla'}{ds}t(s) \right| = (\sum_{i=1}^{3} f_i'^2(s))^\frac{3}{2}.
\]

Assume that \( k \neq 0 \) then the normal vector field along \( c(s) \) is defined by

\[
n = \frac{1}{k} \frac{\nabla'}{ds}t(s) = \frac{1}{k} \sum_{i=1}^{3} f_i'(s)\tilde{x}_i(c(s)).
\]

Then \( n \) is a unit vector of the tangent space of \( S^3 \) at \( c(s) \), and \( n \) is perpendicular to \( t \).

Binormal vector field along \( c(s) \) is given by:

\[
b = t \times n = \frac{1}{k} \det\left(\begin{array}{ccc}
\tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \\
f_1 & f_2 & f_3 \\
h_1 & h_2 & h_3
\end{array}\right) = \frac{1}{k} \sum_{i=1}^{3} g_i(s)\tilde{x}_i(c(s)).
\]

So \( b \) is still a unit vector of the tangent space of \( S^3 \) at \( c(s) \), and \( b \) is perpendicular to both \( t \) and \( n \). By (1.26), (1.24) can be written as

\[
\frac{\nabla'}{ds}t(s) = kn.
\]
Let
\[(1.29) \quad \tau = \frac{\nabla'}{ds} n(s) \cdot b(s),\]
the function \(\tau\) is called the torsion of the curve \(c(s)\). By direct computation we get
\[(1.30) \quad \frac{\nabla'}{ds} n(s) = -kt(s) + \tau b(s),\]
\[(1.31) \quad \frac{\nabla'}{ds} b(s) = -\tau n(s).\]

So along the curve \(c(s)\) there is an orthogonal frame field \(\{c(s); t(s), n(s), b(s)\}\) which is called Frenet frame of curves on \(S^3\). (1.28), (1.30), (1.31) are called Frenet formula. We rewrite it in the matrix form
\[(1.32) \quad \frac{\nabla'}{ds} \begin{pmatrix} t \\ n \\ b \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix}.\]

1.3. The parallel frame of curves on \(S^3\)

We want to change the Frenet frame \((t, n, b)^T\) to \((e_1, e_2, e_3)^T\) so that the 2,3-th entry of the coefficient matrix of \(\frac{\nabla'}{ds}(e_1, e_2, e_3)^T\) is zero. To do this, we follow the method as described in [1]. Rotate the Frenet frame \((n, b)\) by an angle \(\beta(s)\) satisfy that
\[(1.33) \quad \beta'(s) = -\tau(s),\]
then
\[(1.34) \quad \frac{\nabla'}{ds} e_2 \cdot e_3 = 0.\]

So that we get the new o.n frame \((e_1, e_2, e_3)^T\), and it satisfies
\[(1.35) \quad \frac{\nabla'}{ds} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix},\]
where
\[(1.36) \quad \begin{cases} k_1 = k \cos \beta(s) \\ k_2 = -k \sin \beta(s) \end{cases}.\]

The o.n frame \((e_1, e_2, e_3)^T\) is called parallel frame along \(c(s)\). The function \(k_1(s), k_2(s)\) are called the principal curvatures along \(e_2, e_3\) respectively. However,
the choice of parallel frame is not unique, because we can replace \( \beta(s) \) by \( \beta(s) \) plus a constant \( \beta_0 \), get another o.n frame \((e_1, v_2, v_3)^T \). It is again parallel, but the principal curvature \( \hat{k}_1, \hat{k}_2 \) along \( v_2, v_3 \) are satisfies
\[
\frac{\nabla' ds}{ds} e_1 = k_1 e_2 + k_2 e_3 = \hat{k}_1 v_2 + \hat{k}_2 v_3.
\]

2. Vortex filament equation and the NLS in \( S^3 \)

In 1906, da.Rios, a graduate student of Levi-Civita, wrote a master degree thesis, in which he modeled the movement of a thin vortex in a vicious fluid by the motion of a curve propagating in \( R^3 \) in the direction of its binormal with a speed equal to its curvature according to
\[
\gamma_t = \gamma_s \times \gamma_{ss}.
\]
This is called the vortex filament equation or smoke ring equation, and it can be regarded as a dynamical system on the space of curves in \( R^3 \). Much later, in 1971, Hasimoto showed the equivalence of this system with the NLS
\[
q_t = i(q_{ss} + \frac{1}{2} |q|^2 q).
\]

In this section we’ll build the vortex filament equation in \( S^3 \) similar to that as in \( R^3 \), and study the relationship between vortex filament equation and the NLS in \( S^3 \). For any \( \gamma(s,t) \) belongs to \( S^3 \), \( \gamma(s,t) \) is a surface in \( S^3 \) so
\[
\gamma(s,t) \cdot \gamma(s,t) = 1.
\]
Differentiating both sides of (2.3) with respect to \( s \) and \( t \) respectively
\[
\begin{cases}
\gamma_s(s,t) \cdot \gamma(s,t) = 0 \\
\gamma_t(s,t) \cdot \gamma(s,t) = 0.
\end{cases}
\]
So both \( \gamma_s(s,t) \) and \( \gamma_t(s,t) \) are the vectors of the tangent space of \( S^3 \) at \( \gamma(s,t) \).

We can use left invariant vector fields \( \tilde{x}_i \) to express all the tangent vector fields on \( S^3 \), let
\[
\gamma_s(s,t) = \sum_{i=1}^{3} f_i(s,t) \tilde{x}_i(s,t),
\]
then similar to (1.24),
\[
\frac{\nabla' ds}{ds} \gamma_s(s,t) = \sum_{i=1}^{3} \frac{d}{ds} f_i(s,t) \tilde{x}_i(\gamma(s,t)).
\]
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So $\gamma_s(s, t) \times \frac{\nabla'}{ds} \gamma_s(s, t)$ is still a vector of tangent space of $S^3$ at $\gamma(s, t)$. If $\gamma(s, t)$ satisfy

\[(2.7) \quad \gamma_t(s, t) = \gamma_s(s, t) \times \frac{\nabla'}{ds} \gamma_s(s, t),\]

we call $\gamma(s, t)$ is the vortex filament surface in $S^3$. (2.7) is called the vortex filament equation on $S^3$.

Equation (2.7) has the property:

**Proposition.** If $\gamma_s(s, t)$ is a solution of (2.7) and $|\gamma_s(s, 0)| = 1$ for all $s$, then $|\gamma_s(s, t)| = 1$ for all $(s, t)$. In other words if $\gamma(\cdot, 0)$ is parametrized by arc length then so is $\gamma(\cdot, t)$ for all $t$.

**Proof.** It suffices to prove that

\[(2.8) \quad \frac{d}{dt} \langle \gamma_s(s, t), \gamma_s(s, t) \rangle = 0\]

Remark: $\langle , \rangle$ is the inner product in $S^3$. To see (2.8), we compute directly to get:

\[(2.9) \quad \frac{1}{2} \frac{d}{dt} \langle \gamma_s(s, t), \gamma_s(s, t) \rangle = \langle \frac{\nabla'}{dt} \gamma_s, \gamma_s \rangle = \langle \frac{d}{ds} \gamma_s - a \gamma, \gamma_s \rangle = \langle \frac{d}{ds} \gamma_s, \gamma_s \rangle - \langle \frac{\nabla'}{ds} \gamma_s, \gamma_s \rangle = \langle \frac{d}{ds} \gamma_s \times \frac{\nabla'}{ds} \gamma_s, \gamma_s \rangle = \langle \frac{d}{ds} \gamma_s, \gamma_s \rangle + \langle \frac{\nabla'}{ds} \gamma_s, \gamma_s \rangle = \langle \frac{d}{ds} \gamma_s, \gamma_s \rangle + \langle \frac{\nabla'}{ds} \gamma_s, \gamma_s \rangle = \langle \frac{\nabla'}{ds} \gamma_s, \gamma_s \rangle.
\]

Both $a$ and $b$ are some functions on $\gamma(s, t)$. So for a solution $\gamma(s, t)$ of (2.7), we may assume that $\gamma(\cdot, t)$ is parametrized by arc length for all $t$.

Next we will explain the geometric meaning of the evolution equation on the space of curves in $S^3$. Let $(t, n, b)(\cdot, t)$ denote the Frenet frame of the curve $\gamma(\cdot, t)$. Since $\gamma_s = t$, and $\frac{\nabla'}{ds} \gamma_s = \frac{\nabla'}{ds} t = kn$ then the curve flow (2.7) becomes:

\[(2.10) \quad \gamma_t = kt \times n = kb.\]

In other words, the curve flow (2.7) moves in the direction of binormal with curvature as its speed in $S^3$. In the following, we write equation (2.7) in terms of parallel frame $(e_1, e_2, e_3)^T$. Recall that if we rotate the Frenet frame $(n, b)$ by an angle $\beta(s, t)$ satisfy

\[(2.11) \quad \frac{d}{ds} \beta(s, t) = -\tau(s, t),\]
then

\[(2.12)\]

\[b = \sin \beta e_2 + \cos \beta e_3.\]

Hence

\[(2.13)\]

\[\gamma_t = kb = k(\sin \beta e_2 + \cos \beta e_3) = -k_2 e_2 + k_1 e_3.\]

In fact vortex filament equation (2.7) and NLS

\[q_t = i[q_{ss} + (1/2|q|^2 + 1)]q\]

shown the same motion equation. We will give the demonstration below.

Suppose \(\gamma(s, t)\) is a solution of (2.7), choose a parallel frame \((e_1, e_2, e_3)\) for each curve \(\gamma(\cdot, t)\). Let \(k_1(\cdot, t)\) and \(k_2(\cdot, t)\) denote the principal curvature of \(\gamma(\cdot, t)\) along \(e_2(\cdot, t), e_3(\cdot, t)\) respectively. Then we get

\[(2.14)\]

\[\nabla' ds \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.\]

We want to compute \(\nabla' dt (e_1, e_2, e_3)^T\), so we first compute \(\nabla' dt e_1(\cdot, t)\),

\[(2.15)\]

\[\nabla' dt e_1 = \nabla' dt \gamma_s = \frac{\nabla' ds}{ds} \gamma_t = -(k_2)e_2 + (k_1)e_3.\]

Since \(e_i(s, t)\) are orthogonal, there exists a function \(u(s, t)\) so that

\[(2.16)\]

\[\nabla' dt \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & -(k_2)_s & (k_1)_s \\ (k_2)_s & 0 & u \\ -(k_1)_s & -u & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.\]

Now, how to compute the function \(u(s, t)\) is the key step. If we can find \(u(s, t)\), then we can get the coefficient matrix of \(\nabla' dt (e_1, e_2, e_3)^T\). Before computing \(u(s, t)\) we give some preparative knowledge.

(1): Since

\[(2.17)\]

\[\nabla' \nabla' ds dt - \nabla' dt ds = \nabla_{\alpha_2} \nabla_{\alpha_3} - \nabla_{\alpha_3} \nabla_{\alpha_2} - \nabla_{\alpha_2} \nabla_{\alpha_3} + \nabla_{\alpha_2} \nabla_{\alpha_3} + \nabla_{\alpha_2} \nabla_{\alpha_3} + \nabla_{\alpha_2} \nabla_{\alpha_3} = R(\partial / \partial s + \partial / \partial t).\]
where $\nabla_{[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}]} = 0$, so

\[
\nabla' = \frac{\nabla'}{dt} \frac{\nabla'}{ds} = \nabla' \frac{\partial}{\partial s} \frac{\partial}{\partial t} = R(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}).
\]

$R(\frac{\partial}{\partial s}, \frac{\partial}{\partial t})$ is the curvature operator of $S^3$.

(2): Let $M$ be a Riemannian manifold of constant curvature $K$, for any $X, Y, Z, W \in T_pM$ we have

\[
R(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle = -K\langle\langle X, Z \rangle\langle Y, W \rangle - \langle X, W \rangle\langle Y, Z \rangle \rangle.
\]

This can refer to [3]. Since $S^3$ is a space of constant curvature, and the sectional curvature $K=1$. So

\[
\langle R(\frac{\partial}{\partial s}, \frac{\partial}{\partial t})e_2, e_3 \rangle = \langle R(\frac{\partial}{\partial s}, \frac{\partial}{\partial t})e_1, e_2 \rangle = k_2,
\]

\[
\langle R(\frac{\partial}{\partial s}, \frac{\partial}{\partial t})e_1, e_3 \rangle = -k_1
\]

by (2.18), (2.20), (2.21), (2.22) we get

\[
\langle \nabla' \frac{\nabla'}{dt} \frac{\nabla'}{ds} e_2, e_3 \rangle = \langle \nabla' \frac{\nabla'}{dt} \frac{\nabla'}{ds} e_2, e_3 \rangle,
\]

\[
\langle \nabla' \frac{\nabla'}{dt} \frac{\nabla'}{ds} e_1, e_2 \rangle = \langle \nabla' \frac{\nabla'}{dt} \frac{\nabla'}{ds} e_1, e_2 \rangle - k_2,
\]

\[
\langle \nabla' \frac{\nabla'}{dt} \frac{\nabla'}{ds} e_1, e_3 \rangle = \langle \nabla' \frac{\nabla'}{dt} \frac{\nabla'}{ds} e_1, e_3 \rangle + k_1.
\]

In the following we begin to compute $u(s, t)$,

\[
\langle \nabla' \frac{\nabla'}{dt} \frac{\nabla'}{ds} e_2, e_3 \rangle = \langle \nabla' \frac{\nabla'}{dt} (-k_1e_1), e_3 \rangle
\]

\[
= \langle -(k_1)e_1, e_3 \rangle
\]

\[
= \langle -k_1(-k_2)e_2 + (k_1)e_3, e_3 \rangle
\]

\[
= -k_1(k_1)s
\]
similarly

\[(2.27) \quad \langle \nabla^\prime (\frac{\nabla^\prime}{dt} e_2), e_3 \rangle = (k_2)_s k_2 + u_s, \]

by \((2.23)\)

\[(2.28) \quad -k_1(k_1)_s = (k_2)_s k_2 + u_s. \]

So

\[(2.29) \quad u = -\frac{1}{2}(k_1^2 + k_2^2) + c(t), \]

for some smooth function \(c(t)\). Remember that for each fixed \(t\), we can rotate \((e_2, e_3)(\cdot, t)\) by a constant angle \(\theta(t)\) to another parallel normal frame \((v_2, v_3)(\cdot, t)\) of \(\gamma(\cdot, t)\). If we choose \(\theta(t)\) so that \(\theta'(t) = -c(t)\), then the new parallel frame satisfies

\[(2.30) \quad \frac{\nabla^\prime}{dt} \begin{pmatrix} e_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -(k_2)_s \\ (k_1)_s \end{pmatrix} - \frac{(k_1)_s}{k_1^2 + k_2^2} \begin{pmatrix} k_1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} e_1 \\ v_2 \\ v_3 \end{pmatrix}. \]

So we have proved the first part of the following theorem:

**Theorem.** Suppose \(\gamma(s,t)\) is a solution of the vortex filament equation (2.7) and \(|\gamma(s,0)| = 1\) for all \(s\). Then

1. there exists a parallel normal frame \((e_1, e_2, e_3)^T(\cdot, t)\) for each curve \(\gamma(\cdot, t)\) so that

\[(2.31) \quad \frac{\nabla^\prime}{ds} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}; \]

\[(2.32) \quad \frac{\nabla^\prime}{dt} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & -(k_2)_s & (k_1)_s \\ (k_2)_s & 0 & -\frac{k_1^2 + k_2^2}{2} \\ -(k_1)_s & \frac{k_1^2 + k_2^2}{2} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \]

where \(k_1(\cdot, t)\) and \(k_2(\cdot, t)\) are the principal curvatures of \(\gamma(\cdot, t)\) along \(e_2(\cdot, t)\) and \(e_3(\cdot, t)\) respectively.

2. \(q = k_1 + ik_2\) is a solution of the NLS

\[q_t = i[q_{ss} + (\frac{1}{2}|q|^2 + 1)q].\]
Proof. We have proved (1). For (2), we use (2.31),(2.32) to compute the evolution of \((k_1)_t\) and \((k_2)_t\).

\[(2.33) \quad (k_1)_t = \frac{\nabla'}{dt} k_1 = \frac{\nabla'}{dt} (\nabla' \langle e_1, e_2 \rangle)
\]

\[= \langle \nabla' \nabla' \frac{ds}{dt} e_1, e_2 \rangle + (\nabla' \frac{ds}{dt} e_1, \nabla' e_2)\]

\[= \langle \nabla' \nabla' \frac{ds}{dt} e_1, e_2 \rangle - k_2 + (k_1 e_2 + k_2 e_3, (k_2)_s e_1 - \frac{k_1^2 + k_2^2}{2} e_3)\]

\[= \langle \nabla' \nabla' \frac{ds}{dt} (- (k_2)_s e_2 + (k_1)_s e_3), e_2 \rangle - k_2 - k_2 \frac{k_1^2 + k_2^2}{2}\]

\[= -(k_2)_s - k_2 - k_2 \frac{k_1^2 + k_2^2}{2}\]

similarly

\[(2.34) \quad (k_2)_t = (k_1)_s + k_1 + k_1 \frac{k_1^2 + k_2^2}{2}.\]

Therefore

\[(2.35) \quad q_t = (k_1)_t + i(k_2)_t\]

\[= -(k_2)_s - k_2 - k_2 \frac{k_1^2 + k_2^2}{2} + i(k_1)_s + ik_1 + ik_1 \frac{k_1^2 + k_2^2}{2}\]

\[= i[q_{ss} + \left(\frac{1}{2} |q|^2 + 1\right)q].\]

\[\square\]

References