Entire Functions That Share One Value With Their Derivatives

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Abstract. In the paper, we use the theory of normal family to study the problem on entire function that share a finite non-zero value with their derivatives and prove a uniqueness theorem which improve the result of J.P. Wang and H.X. Yi.

1. Introduction and main results

Let \( f \) and \( g \) be some non-constant meromorphic functions. We say \( f \) and \( g \) share a value \( b \) IM(CM) iff \( f - b = 0 \Leftrightarrow g - b = 0 \) (\( f - b = 0 \Leftrightarrow g - b = 0 \)), ignoring multiplicities (counting multiplicities). We assume that the reader is familiar with fundamental results and the standard notations of the Nevanlinna theory([5],[9],[10]).

In 1986, Jank, Mues and Volkmann proved the following result.

Theorem A. Let \( f \) be a nonconstant entire function. If \( f \) and \( f' \) share a finite, nonzero value \( a \) IM, and if \( f''(z) = a \) whenever \( f(z) = a \), then \( f \equiv f' \).

Remark 1. From the hypothesis of Theorem A, it can be easily seen that the value \( a \) is shared by \( f \) and \( f' \) CM. Theorem A suggests the following Question of Yi and Yang.

Question(see [9], [10]). Let \( f \) be a nonconstant meromorphic function, let \( a \) be a finite, nonzero constant, and let \( n \) and \( m \) be positive integers. If \( f \), \( f'(n) \) and \( f'(m) \) share \( a \) CM, where \( n \) and \( m \) are not both even or both odd, must \( f \equiv f'(n) \)?

An example ([7]) given by Yang shows that the answer to the above Question is,
in general, negative. Recently, related to the Question, Li and Yang ([4]) obtained the following theorem.

**Theorem B.** Let $f$ be an entire function, let $a$ be a finite nonzero value, and let $n(\geq 2)$ be a positive integer. If $f, f', f^{(k)}$ share the value $a$ CM, then $f$ assumes the form $f(z) = b e^{cz} + a - \frac{a}{c}$, where $b, c$ are nonzero constants and $e^{n-1} = 1$.

In 2003, J. P. Wang and H. X. Yi ([6]) proved the next result.

**Theorem C.** Let $f$ be a nonconstant entire function, let $a(\neq 0)$ be a constant, and $k(\geq 2)$ be a positive integer. If $f$ and $f'$ share a CM, and if $f^{(k)}(z) = a$ whenever $f(z) = a$, then $f$ assumes the form $f(z) = Ae^{\lambda z} + a - \frac{a}{\lambda}$, where $A(\neq 0)$ and $\lambda$ are constants satisfying $\lambda^{k-1} = 1$.

**Remark 2.** Under the hypothesis of Theorem C, we must have $f' \equiv f^{(k)}$. In Theorem C, if $k = 2$, then we have $\lambda = 1$ which implies $f \equiv f'$. So Theorem C contains Theorem A. Obviously, Theorem C has improved Theorem B.

It is natural to ask the following question: what can we say if CM is replaced by IM in Theorem C? In this paper, we use the theory of normal families to prove the following results.

**Theorem 1.** Let $f$ be a nonconstant entire function, let $a(\neq 0)$ be a constant, and $k(\geq 2)$ be a positive integer. If $f$ and $f'$ share a IM, and $f^{(k)}(z) = a$ whenever $f(z) = a$, and if there exist $z_0 \in C$ satisfying $f^{(k)}(z_0) = f'(z_0) = b$, where $b \neq a$ is a constant, then $f$ assumes the form $f(z) = Ae^{\lambda z} + a - \frac{a}{\lambda}$, where $A(\neq 0)$ and $\lambda$ are constants satisfying $\lambda^{k-1} = 1$.

**Corollary 1.** Let $f$ be a nonconstant entire function, let $a(\neq 0)$ be a constant, let $k \geq 2$ be a positive integer. If $f$ and $f'$ share a IM and $f'(z) = a \Rightarrow f^{(k)}(z) = a$, then $f$ assumes the form $f(z) = Ae^{\lambda z} + a - \frac{a}{\lambda}$, where $A(\neq 0)$ and $\lambda$ are constants satisfying $\lambda^{k-1} = 1$.

**Corollary 2.** Let $f$ be a nonconstant entire function, let $a(\neq 0)$ be a constant, let $k \geq 2$ be a positive integer. If $f(z) = a \Rightarrow f'(z) = a \Rightarrow |f^{(k)}(z)| \leq M$, $M$ is a positive number, then $f' - a \overline{f - a} = c$, where $c$ is a nonzero constant.

2. Some lemmas

**Lemma 1([1]).** Let $\zeta$ be a family of holomorphic functions in a domain $D$, let $k \geq 2$ be a positive integer, and let $\alpha$ be a function holomorphic in $D$, such that $\alpha(z) \neq 0$ for $z \in D$. If for every $f \in \zeta$, $f(z) = 0 \Rightarrow f'(z) = \alpha(z)$ and $f^{(k)}(z) = \alpha(z) \Rightarrow |f^{(k)}(z)| \leq h$, where $h$ is a positive number, then $\zeta$ is normal in $D$.

**Lemma 2([2]).** Let $f$ be an entire function and $M$ be a positive number. If $f^{3}(z) \leq
$M$ for any $z \in C$, then $f$ is of exponential type.

Here, as usual, $f^2(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$ is the spherical derivative.

**Lemma 3** ([3]). Let $\zeta$ be a family of meromorphic functions in a domain $D$, then $\zeta$ is normal in $D$ if and only if the spherical derivatives of functions $f \in \zeta$ are uniformly bounded on compact subsets of $D$.

**Lemma 4** ([8]). Let $Q(z)$ be a nonconstant polynomial. Then every solution $F$ of the differential equation $F^{(k)} - e^{Q(z)}F = 1$ is an entire function of infinite order.

Using the same argument as in the proof of Lemma 4, we can prove the following lemma. We omit the details here.

**Lemma 5.** Let $P(z) \not\equiv 0$ be a polynomial and $Q(z)$ be a nonconstant polynomial. Then every solution $F$ of the differential equation $F^{(k)} - P(z)e^{Q(z)}F = 1$ is an entire function of infinite order.

**Lemma 6.** Let $f$ be a transcendental entire function with $\rho(f) \leq 1$. Let $k \geq 2$ be a positive integer. Let $h$ be a positive number and $a$ be a nonzero constant. If $f(z) = 0 \Rightarrow f'(z) = a$, $f'(z) = a \Rightarrow |f^{(k)}(z)| \leq h$ and $N(r, \frac{f}{f' - a}) = S(r, f)$, then $\frac{f'}{f} - a = c$, where $c$ is a nonzero constant.

**Proof.** From $f(z) = 0 \Rightarrow f'(z) = a$, we get $f(z)$ only has simple zeros. Let

$$
\mu = \frac{f' - a}{f},
$$

then $\mu$ is a entire function. Since $f$ is a transcendental function, we get $\mu \not\equiv 0$, then

$$
T(r, \mu) = m(r, \mu) \leq m(r, \frac{a}{f}) + S(r, f) \leq T(r, f) + S(r, f).
$$

From this we can get $\rho(\mu) \leq \rho(f) \leq 1$, where $\rho(f)$ denote the order of $f$.

$$
N(r, \frac{1}{\mu}) = N(r, \frac{f}{f' - a}) = S(r, f) = O(\log r) \quad (r \not\in E).
$$

Hence $\mu$ has finite zeros. We set $\mu = P(z)e^{bz}$, where $P(z)$ is a polynomial and $b$ is a constant. Form (2.1), we have

$$
(2.2) \quad f' - P(z)e^{bz}f = a.
$$

Let $F = \frac{f}{a}$. Then

$$
(2.3) \quad F' - P(z)e^{bz}F = 1.
$$
If $b \neq 0$, by Lemma 5 we have the order of $f$ is infinite, which is a contradiction. Thus we get $b = 0$ and

\begin{equation}
(2.4) \quad f' = P(z) f + a,
\end{equation}

it follows from (2.4) that

\begin{equation}
(2.5) \quad f^{(k)}(z) = P_1(z) f + P_2(z),
\end{equation}

where $P_1(z)$ and $P_2(z)$ are polynomials, $\deg(P_1) = k \deg(P)$, $\deg(P_2) = (k - 1) \deg(P)$.

**Case 1:** If $f$ has finite zeros, we can get $f' - a$ also has finite zeros, therefore $f$ is a polynomial, which is a contradiction.

**Case 2:** If $f$ has infinite zeros $z_1, z_2, \ldots, z_n, \ldots$, and

\[ |z_1| \leq |z_2| \leq \cdots \leq |z_n| \leq \cdots, |z_n| \rightarrow \infty (n \rightarrow \infty). \]

From (2.5), we have $f^{(k)}(z_n) = P_2(z_n)$. By $|f^{(k)}(z_n)| \leq h$, we see that $P_2(z)$ is a constant, thus $P(z)$ is a constant. Let $P(z) = c$, $c$ is a nonzero constant. From (2.4), we obtain

\[ \frac{f' - a}{f} = c. \]

This completes the proof of Lemma 6.

**Lemma 7**([1]). Let $g$ be a nonconstant entire function with $\rho(g) \leq 1$; let $k \geq 2$ be an integer, and let $a$ be a nonzero finite value. If $g(z) = 0 \Rightarrow g'(z) = a$, and $g'(z) = a \Rightarrow g^{(k)}(z) = 0$, then $g(z) = a(z - z_0)$, where $z_0$ is a constant.

**Lemma 8**([1]). There does not exist entire function $f$ satisfying that

\[ f(z) = \sum_{j=0}^{s} C_j \exp(w^j z), \]

where $w = \exp(2\pi i/k)$ and $C_j$ are constants, and

\[ f(z) = 0 \Leftrightarrow f'(z) = a. \]

**Proof.** From the proof of Lemma 7 in [1], we can get the conclusion.

**3. Proof of theorem 1**

From the assumption, we see that $f$ is a transcendental entire function. Let us now show that $f$ is of exponential type. Let $F = f - a$, then

\[ F = 0 \Leftrightarrow F' = a \Rightarrow F^{(k)} = a. \]
Set $\zeta = \{ F(z + w) : w \in \mathbb{C} \}$, then $\zeta$ is a family of holomorphic functions on the unit disc $\Delta$. By the assumption, for any function $g(z) = F(z + w)$, we have

$$g(z) = 0 \Leftrightarrow g'(z) = a \Rightarrow |g^{(k)}(z)| = |a|,$$

hence by Lemma 1, $\zeta$ is normal in $\Delta$. Thus by Lemma 3, there exist $M > 0$ satisfying $f^2(z) \leq M$ for all $z \in \mathbb{C}$. By Lemma 2, $f$ is of exponential type. Then

$$\rho(f) = \rho(F) \leq 1,$$

(3.1) $f(z) = a \Leftrightarrow f'(z) = a \Rightarrow f^{(k)}(z) = a.$

We distinguish the following two cases.

**Case 1.** If $f' - a$ has finite multiple zeros. We know that $f$ and $f'$ share a IM, so $\frac{f' - a}{f - a}$ have finite zeros, and $f$ is a transcendental entire function, we derive that

$$N(r, \frac{F}{F' - a}) = N(r, \frac{f - a}{f' - a}) = S(r, f) = S(r, F).$$

Therefore by lemma 6, we get

$$\frac{f' - a}{f - a} = \frac{F' - a}{F} = c,$$

where $c$ is a nonzero constant. Consequently, $f$ and $f'$ share a CM, we can get the conclusion by Theorem A.

**Case 2.** If $f' - a$ has infinite multiple zeros. Then there exists

(3.2) $|a_1| \leq |a_2| \leq \cdots \leq |a_n| \leq \cdots, |a_n| \to \infty \quad (n \to \infty),$

where $a_n$ is the multiple $a$-point of $f'$. We claim:

(3.3) $|f^{(k+1)}(a_n)| \leq M_1 \quad (n = 1, 2, 3, \cdots).$

If the inequality (3.3) is not right, we suppose

(3.4) $|f^{(k+1)}(a_n)| = b_n \to \infty \quad (n \to \infty).$

Let $g_n(z) = f(z + a_n)$, we know that $\zeta$ is normal in $\Delta$, we have $\{ f(z + w) : w \in \mathbb{C} \}$ is normal in $\Delta$. We see that

$$\{g_n\} \subset \{ f(z + w) : w \in \mathbb{C} \},$$

thus we get $\{g_n\}$ is normal in $\Delta$, $\forall g_n \in \{g_n\}$ we have

$$g_n(0) = f(a_n) = a.$$
hence \( \{g_n\} \) is uniformly bounded on compact subsets of \( \triangle \). We can get \( \{g_n^{(k)}\} \) is uniformly bounded in \( |z| \leq \frac{1}{2} \). From this we get \( \{g_n^{(k)}\} \) is normal in \( |z| \leq \frac{1}{2} \), but by (3.3) and (3.4), we have

\[
|g_n^{(k)}(0)| = \frac{|g_n^{(k+1)}(0)|}{1 + |g_n^{(k)}(0)|^2} = \frac{b_n}{1 + |a|^2} \to \infty,
\]

which is a contradiction. Thus we prove the claim.

Let

(3.5) \[ f(z) = a + a(z - a_n) + A_3(z - a_n)^3 + \cdots \quad (n = 1, 2, 3 \ldots). \]

Then

(3.6) \[ f'(z) = a + 3A_3(z - a_n)^2 + \cdots \quad (n = 1, 2, 3 \ldots), \]

(3.7) \[ f^{(k)}(z) = a + f^{(k+1)}(a_n)(z - a_n) + \cdots \quad (n = 1, 2, 3 \ldots). \]

Let

(3.8) \[ \varphi = \frac{f^{(k)} - f'}{f - a}. \]

We also distinguish the following two cases.

**Subcase 2.1.** \( \varphi \neq 0 \). From the assumption and (3.8), we get \( \varphi \) is a entire function and

\[
T(r, \varphi) = m(r, \varphi) = S(r, f) = O(\log r) \quad (r \notin \mathbb{E}).
\]

Hence we can get \( \varphi \) is a polynomial.

From (3.5),(3.6),(3.7) and (3.8), we have

\[
\varphi(a_n) = \left. \frac{f^{(k)} - f'}{f - a} \right|_{z = a_n} = \frac{1}{a} f^{(k+1)}(a_n),
\]

hence

(3.9) \[ |\varphi(a_n)| = \left| \frac{1}{a} f^{(k+1)}(a_n) \right| \leq M_1. \]

We know \( \varphi(z) \) is a polynomial and \( |a_n| \to \infty \) \( (n \to \infty) \), from (3.9) we get \( \varphi \) is a nonzero constant. Let \( \varphi = c \), thus we obtain

(3.10) \[ f^{(k)} = f' + c(f - a) \quad (c \neq 0). \]

By the assumption, we substitute \( z_0 \) into (3.10) and get a contradiction.
Subcase 2.2. $\phi \equiv 0$, then we get

\[(3.11) \quad f^{(k)} = f'.\]

In the following we deal with the equation (3.11) in the similar way of Lemma 7.

By (3.11), we have

\[(3.12) \quad f(z) = \sum_{j=0}^{k-2} C_j \exp(w^j z) + D,\]

where $w = \exp(2\pi i/k - 1)$ and $C_j$ and $D$ are constants.

Since $f$ is transcendental, there exists $C_j$ such that $C_j \neq 0$. We denote the nonzero constants in $C_j$ by $C_{jm}(0 \leq j_m \leq k - 2, m = 0, 1, \ldots, s, s \leq k - 2)$. Thus we have

\[(3.13) \quad f(z) = \sum_{m=0}^{s} C_{jm} \exp(w^{jm} z) + D,\]

Let $z_n = r_n e^{i\theta_n}$, where $0 \leq \theta_n < 2\pi$. Without loss of generality, we may assume that $\theta_n \to \theta_0$ as $n \to \infty$. Let

\[(3.14) \quad L = \max_{0 \leq m \leq s} \cos(\theta_0 + \frac{2j_m \pi}{k - 1}).\]

Then, either there exists an index $m_0$ such that $\cos(\theta_0 + \frac{2j_{m_0} \pi}{k - 1}) = L > \cos(\theta_0 + \frac{2j_m \pi}{k - 1})$, for $m \neq m_0$. Then there exists $\delta > 0$ such that for $n$ sufficiently large,

\[(3.15) \quad \cos(\theta_0 + \frac{2j_{m_0} \pi}{k - 1}) - \cos(\theta_0 + \frac{2j_m \pi}{k - 1}) \geq \delta, \quad \text{for} \quad m \neq m_0.\]

We consider these cases separately.

**Case 2.2.1.** There exists an index $m_0$ such that

$\cos(\theta_0 + \frac{2j_{m_0} \pi}{k - 1}) = L > \cos(\theta_0 + \frac{2j_m \pi}{k - 1})$,

for $m \neq m_0$. Then there exists $\delta > 0$ such that for $n$ sufficiently large,

\[(3.15) \quad \cos(\theta_0 + \frac{2j_{m_0} \pi}{k - 1}) - \cos(\theta_0 + \frac{2j_m \pi}{k - 1}) \geq \delta, \quad \text{for} \quad m \neq m_0.\]

We differentiate (3.13) twice and get

\[(3.16) \quad f'' = \sum_{m=0}^{s} C_{jm} (w^{jm})^2 \exp(w^{jm} z).\]

Since $f''(z_n) = 0$, we have

\[(3.17) \quad (w^{jm_0})^2 C_{jm_0} + \sum_{m \neq m_0} C_{jm} (w^{jm})^2 \exp(w^{jm} z_n - w^{jm_0} z_n) = 0.\]
By (3.15), we have

\[
|\exp(w^j m_n - w^{jm_0} z_n)| = \exp \left\{ r_n \left( \cos(\theta_0 + \frac{2j m \pi}{k-1}) - \cos(\theta_0 + \frac{2j m_0 \pi}{k-1}) \right) \right\}
\leq e^{-\delta r_n} \to 0 \quad (n \to \infty).
\]

Thus we obtain \( C_{jm_0} = 0 \), which contradicts our assumption.

**Case 2.2.2.** There exist two indices \( m_1, m_2 (m_1 \neq m_2) \) such that

(3.18) \( \cos(\theta_0 + \frac{2j m_1 \pi}{k-1}) = \cos(\theta_0 + \frac{2j m_2 \pi}{k-1}) = L > \cos(\theta_0 + \frac{2j m \pi}{k-1}) \),

for \( m \neq m_1, m_2 \). Then there exists a \( \delta > 0 \) such that for \( n \) sufficiently large,

(3.19) \( \cos(\theta_0 + \frac{2j m_i \pi}{k-1}) - \cos(\theta_0 + \frac{2j m_j \pi}{k-1}) \geq \delta \), \quad \text{for } (m \neq m_1, m_2) \quad (i = 1, 2).

Since \( f(z_n) = a, f'(z_n) = a \) and \( f''(z_n) = 0 \), we have

(3.20) \( C_{jm_1} \exp(w^{jm_1} z_n) + C_{jm_2} \exp(w^{jm_2} z_n) + \sum_{m \neq m_1, m_2} C_{jm} \exp(w^{jm} z_n) + D = a, \)

and

(3.21) \[
C_{jm_1} w^{jm_1} \exp(w^{jm_1} z_n) + C_{jm_2} w^{jm_2} \exp(w^{jm_2} z_n) + \sum_{m \neq m_1, m_2} C_{jm} w^{jm} \exp(w^{jm} z_n) = a,
\]

(3.22) \[
C_{jm_1} (w^{jm_1})^2 \exp(w^{jm_1} z_n) + C_{jm_2} (w^{jm_2})^2 \exp(w^{jm_2} z_n) + \sum_{m \neq m_1, m_2} C_{jm} (w^{jm})^2 \exp(w^{jm} z_n) = 0.
\]

Thus we get

(3.23) \[
C_{jm_1} w^{jm_1} (w^{jm_1} - w^{jm_2}) \exp(w^{jm_1} z_n) + \sum_{m \neq m_1, m_2} C_{jm} w^{jm_1} (w^{jm} - w^{jm_2}) \exp(w^{jm} z_n) = aw^{jm_2}.
\]

Using the same argument as that used in proving \( C_{jm_0} = 0 \) above and the fact that \( w^j \neq w^l (j \neq l, 0 \leq j, l \leq k - 2) \), we obtain

(3.24) \( \exp(w^{jm_1} z_n) \to c_0, \quad (n \to \infty), \)

where \( c_0 \neq 0 \) is a constant.
It follows that
\begin{equation}
\cos(\theta_0 + \frac{2jm_1 \pi}{k-1}) = \lim_{n \to \infty} \cos(\theta_n + \frac{2jm_1 \pi}{k-1}) = 0.
\end{equation}

Similarly, we get
\[
\cos(\theta_0 + \frac{2jm_2 \pi}{k-1}) = 0.
\]
Thus we have
\begin{equation}
\left| \frac{2jm_1 \pi}{k-1} - \frac{2jm_2 \pi}{k-1} \right| = \pi \quad \text{and} \quad w^{jm_2} = -w^{jm_1}.
\end{equation}

From (3.20), (3.22), (3.25) and (3.26), we can get \( D = a \).

Let
\begin{equation}
g = \sum_{j=0}^{s} C_j \exp(w^j z),
\end{equation}
then
\begin{equation}
g = 0 \iff g' = a.
\end{equation}

From Lemma 8 and (3.27)-(3.28), we can get a contradiction.

Thus we complete the proof of Theorem 1.

In the similar way, we can prove the Corollary 1 and Corollary 2.

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References

