J. Korean Math. Soc. **45** (2008), No. 5, pp. 1243–1253

A CONSTRUCTION OF ONE-FACTORIZATION

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Abstract. In this paper, we construct one-factorizations of given complete graphs of even order. These constructions partition the edges of the complete graph into one-factors and triples. Our new constructions of one-factors and triples can be applied to a recursive construction of Steiner triple systems for all possible orders ≥ 15.

1. Introduction

A graph $G = (V, E)$ consists of a finite set V of objects called *vertices* together with a set E of unordered pairs of vertices called *edges*. A subgraph of a graph G that contains every vertex of G is called a *factor* (or a *spanning subgraph* in [10]). A *factorization* of G is a set of factors of G which are pairwise *edge-disjoint* (that is, no two have a common edge) and whose union is all of G. Since G is a factor of itself, $\{G\}$ is a factorization of G so that every graph has a factorization. However, it is more interesting to consider factorizations in which the factors satisfy certain conditions. For a given graph G, a *one-factor* is a factor which is a regular graph of degree one. In other words, a one-factor is a set of pairwise disjoint edges of G which between them contain every vertex. A *one-factorization* of G is a partition of all the edges into one-factors each of which is, in its turn, a partition of the set of vertices.

One-factors and one-factorizations of the complete graph of even order $2n$, written as K_{2n}, have been studied by several authors in [4, 7, 8]. A *cyclic* graph of order $2n$ (see [10]) is a graph whose vertices are the integers modulo $2n$ with the property that if $\{x, y\}$ is an edge then so is $\{x + i, y + i\}$ for $1 \leq i \leq 2n - 1$ when $|x - y| \neq n$; for $1 \leq i \leq n - 1$ when $|x - y| = n$. As general backgrounds on one-factorizations of the complete graphs, we refer to [1, 5, 9, 10]. The existence of a one-factor is known for all K_{2n} and it has been known that a one-factorization exists for K_{2n} as well (see [6]). As an application, it is

Received December 5, 2006; Revised March 8, 2007.
2000 *Mathematics Subject Classification.* Primary 05C70, 51E10.
Key words and phrases. one-factorizations, Steiner triple systems.
The present Research has been conducted by the Research Grant of Kwangwoon University in 2008.

well-known that a round robin tournament can be expressed as a proper one-factorization of a complete graph of even order (see [8, 10]). In the theory of designs, one-factorization of K_{2n} has been used for constructing a Steiner triple system of order v (written as $STS(v)$), which is a 2-design with v points and blocks of size 3, called *triples.* We sometimes denote a Steiner triple system by $STS(v)=(V,T)$ for the set of vertices V and the set of triples. In a given Steiner triple system S, a Steiner triple system T is called a *subsystem of* S if every block of T is a block of S. In [3], J. Doyen and R. Wilson show that a $STS(u)$ is a subsystem of $STS(v)$ for every u and v such that $u,\ v \equiv 1,\ 3 \pmod 6$ and $v > 2u$. Similar recursive construction method of $STS(2v+s)$ for $s = 1$, or 7 based on a given $STS(v)$ can be found in [2, 8, 10], which shows the existence of Steiner triple system for all feasible orders.

In this paper, we give new constructions of one-factorizations of a cyclic complete graph K_{2n} depending on whether $2n/\gcd(\alpha, 2n)$ is even or odd, where α is a positive integer in the set of all edge differences in K_{2n}. For the odd case of α, we next give a factorization by modifying our one-factorization in order to obtain an infinite family of $STS(v)$ by applying Doyen-Wilson theorem [3] on a recursive construction scheme. Then, we obtain a family of $STS(v)$s for all possible orders $v \equiv 1,\ 3 \pmod 6$ derived from our modified one-factorizations for odd α's and the one-factorizations given in [10] for even α's. These Steiner triple systems are different from the ones given by Bose, seen in [2].

2. Constructions of one-factorization

Let K_{2n} be the cyclic complete graph whose vertex-set is the additive group of residue classes of integers modulo $2n$, that is,

$$\mathbb{Z}_{2n} = \{0, 1, \ldots, 2n-1\}.$$

For $\alpha = 1, 2, \ldots, n$, let

$$E_\alpha = \{\{i, j\} \mid i - j \equiv \pm\alpha \pmod{2n}\}.$$

Then $\{E_\alpha \mid \alpha = 1, 2, \ldots, n\}$ is a partition of the edge-set of K_{2n} and E_n is always a one-factor of K_{2n}. In this case, α is called the *difference of an edge* $\{i, j\}$ and E_α is said to *have α-difference.*

Let D be the set of all differences of K_{2n}. Then $D = D_e \cup D_o$, where

$$D_e = \{\alpha \mid 2n/\gcd(\alpha, 2n) \text{ is even, } 1 \le \alpha \le n\}$$

and

$$D_o = \{\alpha \mid 2n/\gcd(\alpha, 2n) \text{ is odd, } 1 \le \alpha \le n\}.$$

Note that if α is odd, then α must be in D_e.

We now give a construction of one-factors of K_{2n} as follows.

Construction 1. (1) *Let $\alpha \in D_e$, $\alpha \neq n$ and let $g_\alpha = \gcd(\alpha, 2n)$. Then, as a graph, E_α consists of g_α component cycles of length $\frac{2n}{g_\alpha}$ (see [10]) which means*

$$E_\alpha = \bigcup_{j=0}^{g_\alpha - 1} \left\{ \{j + \alpha i, j + \alpha + \alpha i\} \mid i = 0, 1, \ldots, \frac{2n}{g_\alpha} - 1 \right\}.$$

By taking alternate members of the cycles, we have two one-factors F_α and $F_\alpha + \alpha$ of the complete graph K_{2n} as follows:

$$\begin{aligned} F_\alpha &= \bigcup_{j=0}^{g_\alpha - 1} \left\{ \{j + \alpha i, j + \alpha i + \alpha\} \mid i = 0, 2, \ldots, \frac{2n}{g_\alpha} - 2 \right\}, \\ F_\alpha + \alpha &= \bigcup_{j=0}^{g_\alpha - 1} \left\{ \{j + \alpha i, j + \alpha i + \alpha\} \mid i = 1, 3, \ldots, \frac{2n}{g_\alpha} - 1 \right\}. \end{aligned}$$

That is,

$$\begin{aligned} F_\alpha &= \bigcup_{j=0}^{g_\alpha - 1} \left\{ \{j + 2\alpha i, j + 2\alpha i + \alpha\} \mid i = 0, 1, \ldots, \frac{n}{g_\alpha} - 1 \right\}, \\ F_\alpha + \alpha &= \bigcup_{j=0}^{g_\alpha - 1} \left\{ \{j + 2\alpha i + \alpha, j + 2\alpha i + 2\alpha\} \mid i = 0, 1, \ldots, \frac{n}{g_\alpha} - 1 \right\}. \end{aligned}$$

(2) *Let $\alpha \in D_o$ and $g_\alpha = \gcd(\alpha, 2n)$. For each $j = \frac{g_\alpha l}{2}$, $l = 0, 1, \ldots, \frac{2n}{g_\alpha} - 1$ and each $k = 1, 2, \ldots, \frac{2n}{g_\alpha} - 1$, define*

$$A_\alpha(j, k) = \bigcup_{i=0}^{\frac{g_\alpha}{2} - 1} \left\{ \left\{ \frac{g_\alpha k}{2} + i + j, 2n - \frac{g_\alpha k}{2} + i + j \right\}, \{i + j, n + i + j\} \right\}.$$

Then for each $j = \frac{g_\alpha l}{2}$, $l = 0, 1, \ldots, \frac{2n}{g_\alpha} - 1$,

$$F_\alpha + j = \bigcup_{k=1}^{\frac{2n}{g_\alpha} - 1} A_\alpha(j, k)$$

is a one-factor of K_{2n}.

The one-factors shown in (2) of Construction 1 satisfies the following proposition.

Proposition 2. *Let K_v be the cyclic complete graph. If $v = 2^r p$ for some odd number $p > 1$ and some $r \in \mathbb{N}$, then*

$$\bigcup \left\{ F_{2^r} + j \mid j = 2^{r-1} l, 0 \leq l \leq p - 1 \right\} = \bigcup \left\{ E_\alpha \mid \alpha \in D_o \cup \left\{ \tfrac{v}{2} \right\} \right\}.$$

Proof. Let $v = 2^r p$ for some odd number $p > 1$ and some $r \in \mathbb{N}$. Then $D_o = \left\{ 2^r, 2 \times 2^r, 3 \times 2^r, \ldots, \frac{p-1}{2} \times 2^r \right\}$. For a difference $\alpha = 2^r$, let $g_\alpha = \gcd(2^r, 2^r p) = 2^r$. For each $j = 2^{r-1} l$, $l = 0, 1, \ldots, p - 1$ we define $F_{2^r} + j$ as (2) in Construction 1. Then for each $j = 2^{r-1} l$ and $0 \leq l \leq p - 1$, $F_{2^r} + j$

is one-factor and the difference of each edge of $F_{2^r}+j$ is one of 2^r, 2×2^r, $3\times 2^r,\ldots,\left(\frac{p-1}{2}\right)\times 2^r$, and $2^{r-1}p$. Hence there are p one-factors which are pairwise edge-disjoint and the set of differences of all edges of $F_{2^r}+j$ is equal to $D_o\cup\left\{\frac{v}{2}\right\}$, so that

$$\bigcup\left\{F_{2^r}+j \mid j=2^{r-1}l,\ 0\le l\le p-1\right\}\subset\bigcup\left\{E_\alpha \mid \alpha\in D_o\cup\left\{\tfrac{v}{2}\right\}\right\}.$$

Note that the cardinal number of the right side is $(p-1)v/2+v/2$ from the definition of E_α. Since the cardinal number of left side is $pv/2$, we have the equality of the statement. This completes the proof. □

Note that if $v=2n$ is a power of 2, then $D_o=\emptyset$ so that $D=D_e$. The following theorem, which is a construction of one-factorization for $v=2^r$, is known in [4, 7, 8, 10].

Theorem 3. *If $v=2^r$ for some $r\in\mathbb{N}$, then the cyclic graph K_v has a one-factorization*

$$\mathfrak{F}_{2^r}=\left\{F_\alpha, F_\alpha+\alpha \mid \alpha\in D_e, \alpha\ne\tfrac{v}{2}\right\}\cup\left\{E_{\frac{v}{2}}\right\},$$

where $F_\alpha, F_\alpha+\alpha$ and $E_{\frac{v}{2}}$ are defined in (1) *of Construction 1.*

We now have a new construction of one-factorizations of the cyclic complete graph K_{2n} from Proposition 2 and Theorem 3.

Theorem 4. *If $v=2^rp$ for some $r\in\mathbb{N}$ and odd number p, then the cyclic graph K_v has a one-factorization*

$$\mathfrak{F}_{2^rp}=\left\{F_\alpha, F_\alpha+\alpha \mid \alpha\in D_e, \alpha\ne\tfrac{v}{2}\right\}\cup\left\{F_{2^r}+2^{r-1}l \mid l=0,1,\ldots,p-1\right\},$$

where $F_\alpha, F_\alpha+\alpha$ and $F_{2^r}+2^{r-1}l$ $(l=0,1,\ldots,p-1)$ are one-factors defined in Construction 1.

Proof. If $p=1$, it is obvious from Theorem 3. Now we suppose $v=2^rp$ for $p>1$ and $r\in\mathbb{N}$ so that $D=\left\{1,2,3,\ldots,2^{r-1}p\right\}$.
For $\alpha\in D$, let $g_\alpha=\gcd(\alpha,v)$. For $\alpha=2^r$, by Proposition 2, note that

$$\left\{F_{2^r}+2^{r-1}l \mid l=0,1,\ldots,p-1\right\}$$

consists of p edge-disjoint one-factors of K_v and

$$\bigcup\left\{F_{2^r}+2^{r-1}l \mid l=0,1,\ldots,p-1\right\}=\bigcup\left\{E_\alpha \mid \alpha\in D_o\cup\left\{\tfrac{v}{2}\right\}\right\}.$$

For each $\alpha\in D_e-\left\{\frac{v}{2}\right\}$, the one-factors F_α and $F_\alpha+\alpha$ have α-difference by (1) of Construction 1. Since

$$D_o\cup\left\{\frac{v}{2}\right\}=\left\{2^r, 2\times 2^r, 3\times 2^r,\ldots,\left(\frac{p-1}{2}\right)\times 2^r, 2^{r-1}p\right\},$$

the cardinal number of $D_e-\left\{\frac{v}{2}\right\}$ is

$$\left|D-\left\{2^r, 2\times 2^r, 3\times 2^r,\ldots,\left(\frac{p-1}{2}\right)\times 2^r, 2^{r-1}p\right\}\right|=2^{r-1}p-\frac{p-1}{2}-1.$$

Thus there are

$$\left(2^{r-1}p - \frac{p-1}{2} - 1\right) \times 2 = 2^r p - p - 1$$

one-factors of K_v and the set of all edges of $\{F_\alpha, F_\alpha + \alpha \mid \alpha \in D_e, \alpha \neq \frac{v}{2}\}$ is equal to the set $\bigcup \{E_\alpha \mid \alpha \in D_e, \alpha \neq \frac{v}{2}\}$. Hence the total number of one-factors of K_v is $p + 2^r p - p - 1 = 2^r p - 1 = v - 1$ and the set of all edges of $\mathfrak{F}_{2^r p}$ is equal to the set $\bigcup \{E_\alpha \mid \alpha \in D\}$, where

$$\mathfrak{F}_{2^r p} = \left\{F_{2^r} + 2^{r-1}l \mid l = 0, 1, \ldots, p-1\right\} \cup \left\{F_\alpha, F_\alpha + \alpha \mid \alpha \in D_e, \alpha \neq \tfrac{v}{2}\right\}.$$

Therefore, the cyclic complete graph K_v has a one-factorization $\mathfrak{F}_{2^r p}$ which consists of $v - 1$ one-factors. □

We now apply the previous Theorem 4 to a complete graph K_{10}.

Example 5. Consider the cyclic complete graph K_{10}. Then $D_e = \{1, 3, 5\}$, and $D_o = \{2, 4\}$.

If $\alpha = 1 \in D_e$, then $g_\alpha = \gcd(1, 10) = 1$ and $j = 0$; so, we have

$$\begin{aligned} F_1 &= \{\{2i, 2i+1\} \mid i = 0,1,2,3,4\} \\ &= \{\{0,1\},\{2,3\},\{4,5\},\{6,7\},\{8,9\}\}, \\ F_1 + 1 &= \{\{2i+1, 2i+2\} \mid i = 0,1,2,3,4\} \\ &= \{\{1,2\},\{3,4\},\{5,6\},\{7,8\},\{9,0\}\}. \end{aligned}$$

If $\alpha = 3 \in D_e$, then $g_\alpha = \gcd(3, 10) = 1$ and $j = 0$; so, we have

$$\begin{aligned} F_3 &= \{\{6i, 6i+3\} \mid i = 0,1,2,3,4\} \\ &= \{\{0,3\},\{6,9\},\{2,5\},\{8,1\},\{4,7\}\}, \\ F_3 + 3 &= \{\{6i+3, 6i+6\} \mid i = 0,1,2,3,4\} \\ &= \{\{3,6\},\{9,2\},\{5,8\},\{1,4\},\{7,0\}\}. \end{aligned}$$

Now, since $10 = 2^r p = 2^1 \times 5$ for the difference $\alpha = 2$ we have $g_2 = \gcd(2, 10) = 2$ and $j = 0, 1, 2, 3, 4$. Thus

$$\begin{aligned} F_2 + 0 &= \{\{k, (10-k)\}, \{0,5\} \mid k = 1,2,3,4\} \\ &= \{\{1,9\},\{2,8\},\{3,7\},\{4,6\},\{0,5\}\}, \\ F_2 + 1 &= \{\{k+1, (10-k)+1\}, \{0+1, 5+1\} \mid k = 1,2,3,4\} \\ &= \{\{2,10\},\{3,9\},\{4,8\},\{5,7\},\{1,6\}\}, \\ F_2 + 2 &= \{\{k+2, (10-k)+2\}, \{0+2, 5+2\} \mid k = 1,2,3,4\} \\ &= \{\{3,1\},\{4,10\},\{5,9\},\{6,8\},\{2,7\}\}, \\ F_2 + 3 &= \{\{k+3, (10-k)+3\}, \{0+3, 5+3\} \mid k = 1,2,3,4\} \\ &= \{\{4,2\},\{5,1\},\{6,10\},\{7,9\},\{3,8\}\}, \\ F_2 + 4 &= \{\{k+4, (10-k)+4\}, \{0+4, 5+4\} \mid k = 1,2,3,4\} \\ &= \{\{5,3\},\{6,2\},\{7,1\},\{8,10\},\{4,9\}\}. \end{aligned}$$

Hence $\{F_\alpha, F_\alpha + \alpha \mid \alpha = 1, 3\} \cup \{F_2 + j \mid j = 0,1,2,3,4\}$ forms a one-factorization of K_{10} which consists of 9 one-factors of K_{10}.

By modifying one-factors in Construction 1, we obtain a new factorization consisting of one-factors and triples for $v \equiv 0 \pmod 6$ and $v > 3$.

Construction 6. *Take $\mathfrak{F}_{2^r p}$ the one-factorization of K_v stated in Theorem 4.*
If $v \equiv 0 \pmod 6$, then for each $j = 2^{r-1}l$ ($l = 0, 1, \ldots, p-1$) and each $k = 1, 2, \ldots, p-1$, define

$$\begin{aligned} &A^*_{2^r}(j,k) \\ &= A_{2^r}(j,k) - \bigcup_{i=0}^{\frac{g_\alpha}{2}-1} \left\{ \left\{ \frac{v}{6}+j+i, \frac{5v}{6}+j+i \right\}, \left\{ \frac{v}{3}+j+i, \frac{2v}{3}+j+i \right\} \right\} \\ &\bigcup \bigcup_{i=0}^{\frac{g_\alpha}{2}-1} \left\{ \left\{ \frac{v}{6}+j+i, \frac{v}{3}+j+i \right\}, \left\{ \frac{5v}{6}+j+i, \frac{2v}{3}+j+i \right\} \right\}. \end{aligned}$$

For each $j = 2^{r-1}l$ ($l = 0, 1, \ldots, p-1$), let

$$F^*_{2^r} + j = \bigcup_{k=1}^{p-1} A^*_{2^r}(j,k)$$

be a one-factor of the cyclic complete graph K_v. Define a $\frac{v}{3}$-set of triples as

$$T_{\frac{v}{3}} = \left\{ \left\{ i, \frac{v}{3}+i, \frac{2v}{3}+i \right\} \mid i = 0, 1, \ldots, \frac{v}{3}-1 \right\}.$$

Let

$$\mathfrak{F}^*_1 = \left\{ F^*_{2^r} + 2^{r-1}l \mid l = 0, 1, \ldots, p-1 \right\}$$

and

$$\mathfrak{F}^*_2 = \left\{ F_\alpha, F_\alpha + \alpha \mid \alpha \in D_e - \left\{ \frac{v}{6}, \frac{v}{2} \right\} \right\},$$

where F_α and $F_\alpha + \alpha$ are one-factors defined in Construction 1. Finally, we define $\mathfrak{F}^*_v$ to be

$$\mathfrak{F}^*_v = \mathfrak{F}^*_1 \cup \mathfrak{F}^*_2 \cup T_{\frac{v}{3}}$$

which consists of one-factors and triples.

The set $\mathfrak{F}^*_v$ defined in Construction 6 satisfies the following theorem.

Theorem 7. *If $v \equiv 0 \pmod 6$, then the edges of the cyclic K_v are partitioned into $\frac{v}{3}$ triples and $v-3$ one-factors.*

Proof. By the definition in Construction 6, the set $\mathfrak{F}^*_1$ consists of p one-factors in which each edge of the set $\bigcup\{E_\alpha \mid \alpha \in D_o - \{\frac{v}{3}\} \cup \{\frac{v}{6}, \frac{v}{2}\}\}$ occurs exactly once. Note that for each $\alpha \in D_e - \{\frac{v}{6}, \frac{v}{2}\}$, there are two edge-disjoint one-factors. Thus $\mathfrak{F}^*_2$ defined in Construction 6 consists of

$$2 \times (2^{r-1}p - \tfrac{p-1}{2} - 1) - 1 = 2^r p - p - 3$$

pairwise edge-disjoint one-factors, and the set of all edges of $\mathfrak{F}^*_2$ is equal to the set $\bigcup\{E_\alpha \mid \alpha \in D_e - \{\frac{v}{6}, \frac{v}{2}\}\}$. Hence the total number of pairwise edge-disjoint one-factors is

$$p + (2^r p - p - 3) = v - 3.$$

From the definition of $T_{\frac{v}{3}}$ in Construction 6, $T_{\frac{v}{3}}$ consists of $\frac{v}{3}$ triples in which each edge of $E_{\frac{v}{3}}$ occurs exactly once. □

From Construction 6, we have the following example for K_{12}.

Example 8. Consider the cyclic complete graph K_{12}. Firstly we have an one-factorization which consists of 11 one-factors from Construction 1. Then $D_e = \{1,2,3,5,6\}$, and $D_o = \{4\}$.
If $\alpha = 1 \in D_e$, then we have

$$\begin{aligned} F_1 &= \{\{0,1\},\{2,3\},\{4,5\},\{6,7\},\{8,9\},\{10,11\}\}, \\ F_1+1 &= \{\{1,2\},\{3,4\},\{5,6\},\{7,8\},\{9,10\}\{11,0\}\}. \end{aligned}$$

If $\alpha = 2 \in D_e$, then we have

$$\begin{aligned} F_2 &= \{\{0,2\},\{4,6\},\{8,10\},\{1,3\},\{5,7\},\{9,11\}\}, \\ F_2+2 &= \{\{2,4\},\{6,8\},\{10,0\},\{3,5\},\{7,9\}\{11,1\}\}. \end{aligned}$$

If $\alpha = 3 \in D_e$, then we have

$$\begin{aligned} F_3 &= \{\{0,3\},\{6,9\},\{1,4\},\{7,10\},\{2,5\},\{8,11\}\}, \\ F_3+3 &= \{\{3,6\},\{9,0\},\{4,7\},\{10,1\},\{5,8\},\{11,2\}\}. \end{aligned}$$

If $\alpha = 5 \in D_e$, then we have

$$\begin{aligned} F_5 &= \{\{0,5\},\{10,3\},\{8,1\},\{6,11\},\{4,9\},\{2,7\}\}, \\ F_5+5 &= \{\{5,10\},\{3,8\},\{1,6\},\{11,4\},\{9,2\},\{7,0\}\}. \end{aligned}$$

Now, for the differences $\alpha = 4 \in D_o$ we have 3 one-factors

$$\begin{aligned} F_4+0 &= \{\{2,10\},\{0,6\},\{3,11\},\{1,7\},\{4,8\},\{5,9\}\}, \\ F_4+2 &= \{\{4,0\},\{2,8\},\{5,1\},\{3,9\},\{6,10\},\{7,11\}\}, \\ F_4+4 &= \{\{6,2\},\{4,10\},\{7,3\},\{5,11\},\{8,0\},\{9,1\}\} \end{aligned}$$

which consist of edges with differences $\alpha = 4, 6$. Hence

$$\{F_\alpha, F_\alpha + \alpha \mid \alpha = 1,2,3,5\} \cup \{F_4 + j \mid j = 0,2,4\}$$

forms a one-factorization of K_{12} which consists of 11 one-factors of K_{12}.

Applying Construction 6 to this one-factorization, we have 9 one-factors and 4 triples as follows.

$$\begin{aligned} &F_4^* + 0 = \{\{2,4\},\{10,8\},\{3,5\},\{11,9\},\{0,6\},\{1,7\}\}, \\ &F_4^* + 2 = \{\{4,6\},\{0,10\},\{5,7\},\{1,11\},\{2,8\},\{3,9\}\}, \\ &F_4^* + 4 = \{\{6,8\},\{2,0\},\{7,9\},\{3,1\},\{4,10\},\{5,11\}\}, \\ &F_1, F_1+1, F_3, F_3+3, F_5, F_5+5 \text{ and} \\ &T_4 = \{\{0,4,8\},\{1,5,9\},\{2,6,9\},\{3,7,11\}\}. \end{aligned}$$

We define

$$\begin{aligned} &\mathfrak{F}_1^* = \{F_4^*+0, F_4^*+2, F_4^*+4\}, \\ &\mathfrak{F}_2^* = \{F_1, F_1+1, F_3, F_3+3, F_5, F_5+5\}, \end{aligned}$$

and

$$\mathfrak{F}_{12}^* = \mathfrak{F}_1^* \cup \mathfrak{F}_2^* \cup\ T_4.$$

Then $\mathfrak{F}_{12}^*$ is a factorization which consists of 9 one-factors and 4 triples from Construction 6.

Next, by modifying one-factorization in Construction 1, we obtain a new factorization consisting of one-factors and triples for $v \equiv 0 \pmod 2$, $v > 7$ and $v \neq 10$ as follows.

Construction 9. *Take $\mathfrak{F}_{2^r p}$ the one-factorization of K_v stated in Theorem 4. If $v \equiv 0 \pmod 2$, $v > 7$ and $v \neq 10$, then for each $j = 2^{r-1}l$ $(l = 0, 1, \ldots, p-1)$ we have the following one-factor;*

$$\begin{aligned} F_2^* + j = &\left(F_2 + j - \left\{\{1+j, (v-1)+j\}, \{(\tfrac{v}{2}-1)+j, (\tfrac{v}{2}+1)+j\}\right\}\right) \\ &\bigcup\left\{\{1+j, (\tfrac{v}{2}-1)+j\}, \{(v-1)+j, (\tfrac{v}{2}+1)+j\}\right\}. \end{aligned}$$

Define a v-set of triples as

$$T_{1,2,3} = \left\{\{i, i+1, i+3\} \mid i = 0, 1, 2, \ldots, v-1\right\}.$$

Let

$$\begin{aligned} \mathfrak{F}_1 &= \left\{F_{2^r} + 2^{r-1}l \mid r > 1,\ l = 0, 1, \ldots, p-1\right\}, \\ \mathfrak{F}_1^* &= \left\{F_2^* + j \mid j = 0, 1, \ldots, \tfrac{v}{2} - 1\right\}, \\ \mathfrak{F}_2^* &= \left\{F_\alpha, F_\alpha + \alpha \mid \alpha \in D_e - \left\{1, 3, \tfrac{v}{2} - 2, \tfrac{v}{2}\right\}\right\}, \text{ and} \\ \mathfrak{F}_3^* &= \left\{F_\alpha, F_\alpha + \alpha \mid \alpha \in D_e - \left\{1, 2, 3, \tfrac{v}{2}\right\}\right\}. \end{aligned}$$

We define $\mathfrak{F}_v^*$ to be

$$\mathfrak{F}_v^* = \begin{cases} \mathfrak{F}_1^* \cup \mathfrak{F}_2^* \cup\ T_{1,2,3} & \text{if } v = 2p, v \neq 10, \\ \mathfrak{F}_1 \cup \mathfrak{F}_3^* \cup\ T_{1,2,3} & \text{if } v = 2^r p, r > 1 \end{cases}$$

which consists of one-factors and triples.

The set $\mathfrak{F}_v^*$ defined in Construction 9 satisfies the following theorem.

Theorem 10. *If $v \equiv 0 \pmod 2$ and $v > 7, v \neq 10$, then the edges of the cyclic K_v are partitioned into v triples and $v - 7$ one-factors.*

Proof. For $v = 2^r p$, we have two cases: $r = 1$ and $r > 1$. If $r = 1$ and $v \neq 10$, then $2 \in D_o$. From the definition of $\mathfrak{F}_1^*$ in Construction 9, $\mathfrak{F}_1^*$ consists of p pairwise edge-disjoint one-factors and the set of all edges of $\mathfrak{F}_1^*$ is equal to the set

$$\bigcup\left\{E_\alpha \mid \alpha \in D_o \cup \left\{\tfrac{v}{2} - 2, \tfrac{v}{2}\right\} - \{2\}\right\}.$$

Note that for each $\alpha \in D_e - \left\{1, 3, \frac{v}{2} - 2, \frac{v}{2}\right\}$, there are edge-disjoint two one-factors F_α, $F_\alpha + \alpha$. Thus $\mathfrak{F}_2^*$ defined in Construction 9 contains $2 \times \left(2^{r-1}p - \frac{p-1}{2} - 1 - 3\right)$ pairwise edge-disjoint one-factors and the set of all edges of $\mathfrak{F}_2^*$ is equal to

$$\bigcup\left\{E_\alpha \mid \alpha \in D_e - \left\{1, 3, \tfrac{v}{2} - 2, \tfrac{v}{2}\right\}\right\}.$$

Hence for the case $r = 1$, $\mathfrak{F}_1^* \cup \mathfrak{F}_2^*$ consists of

$$p + 2 \times \left(2^{r-1}p - \tfrac{p-1}{2} - 1 - 3\right) = v - 7$$

pairwise edge-disjoint one-factors in which each edge of the set

$$\bigcup \{E_\alpha \mid \alpha \in D - \{1,2,3\}\}$$

appears exactly once.

Now, we suppose $r > 1$; then, $2 \in D_e$. From the definition of $\mathfrak{F}_3^*$ in Construction 9, there are $2 \times \left(2^{r-1}p - \frac{p-1}{2} - 1 - 3\right) = 2^r p - p - 7$ one-factors in which every edges of

$$\bigcup \left\{E_\alpha \mid \alpha \in D_e - \left\{1,2,3,\tfrac{v}{2}\right\}\right\}$$

appears exactly once. From Proposition 2, $\mathfrak{F}_1$ defined in Construction 9 consists of p one-factors in which each edge of $\bigcup \left\{E_\alpha \mid \alpha \in D_o \cup \left\{\frac{v}{2}\right\}\right\}$ occurs exactly once. Hence, for $r > 1$, $\mathfrak{F}_1 \cup \mathfrak{F}_3^*$ consists of

$$p + \left(2^k p - p - 7\right) = v - 7$$

one-factors in which all edges of $\bigcup \{E_\alpha \mid \alpha \in D - \{1,2,3\}\}$ appears exactly once. Then, from the definition of $T_{1,2,3}$, we have v edge-disjoint triples and the set of all edges of them is equal to $\{E_\alpha \mid \alpha = 1,2,3\}$. In all, for $v \equiv 0 \pmod 2$, $v > 7$ and $v \neq 10$,

$$\mathfrak{F}_v^* = \begin{cases} \mathfrak{F}_1^* \cup \mathfrak{F}_2^* \cup \ T_{1,2,3} & \text{if } \ v = 2p, v \neq 10, \\ \mathfrak{F}_1 \cup \mathfrak{F}_3^* \cup \ T_{1,2,3} & \text{if } \ v = 2^r p \end{cases}$$

consists of $v - 7$ one-factors and v triples in which each edge of K_v appears exactly once. □

Note that Construction 6 and 9 directly imply the Theorem 7 and 10, respectively, which can be seen in [2, 7, 9].

Now we remark the excluded case that $v = 10$ from Construction 9. In K_{10}, we have three one-factors and ten triples as follows;

$$\begin{array}{l} \{(0,5),(1,7),(2,6),(3,9),(4,8)\}, \\ \{(1,6),(2,8),(3,7),(4,0),(5,9)\}, \\ \{(2,7),(3,8),(4,9),(0,6),(1,5)\}, \text{ and} \\ \{\{i, i+1, i+3\} \mid i = 0,1,2,\ldots,10-1\}. \end{array}$$

Including this construction for K_{10}, we finally have the following theorem.

Theorem 11. *If $v \equiv 0 \pmod 2$ and $v > 7$, then the edges of the cyclic K_v are partitioned into v triples and $v - 7$ one-factors.*

We remark that the result in Theorem 11 can be also seen in [2, 7, 9] with different approaches.

As an example of Construction 9, we have the following factorization of K_{14}.

Example 12. For the cyclic complete graph K_{14}, we have a factorization which consists of 7 one-factors and 14 triples.

From (2) in Construction 1, we first have 7 one-factors

$$\begin{aligned}
F_2+0 &= \{\{1,13\},\{2,12\},\{3,11\},\{4,10\},\{5,9\},\{6,8\},\{0,7\}\},\\
F_2+1 &= \{\{2,0\},\{3,13\},\{4,12\},\{5,11\},\{6,10\},\{7,9\},\{1,8\}\},\\
F_2+2 &= \{\{3,1\},\{4,0\},\{5,13\},\{6,12\},\{7,11\},\{8,10\},\{2,9\}\},\\
F_2+3 &= \{\{4,2\},\{5,1\},\{6,0\},\{7,13\},\{8,12\},\{9,11\},\{3,10\}\},\\
F_2+4 &= \{\{5,3\},\{6,2\},\{7,1\},\{8,0\},\{9,13\},\{10,12\},\{4,11\}\},\\
F_2+5 &= \{\{6,4\},\{7,3\},\{8,2\},\{9,1\},\{10,0\},\{11,13\},\{5,12\}\},\\
F_2+6 &= \{\{7,5\},\{8,4\},\{9,3\},\{10,2\},\{11,1\},\{12,0\},\{6,13\}\}
\end{aligned}$$

for the edge difference $\alpha \in D_o \cup \{7\} = \{2,4,6\} \cup \{7\}$ and 6 one-factors

$$F_1, F_1+1, F_3, F_3+3, F_5, F_5+5$$

for the edge difference $\alpha \in D_e - \{7\} = \{1,3,5,7\} - \{7\}$.

By modifying theses 13 one-factors, we construct new factorization consisting of 7 one-factors and 14 triples as follows.

$$\begin{aligned}
F_2^*+0 &= \{\{1,6\},\{2,12\},\{3,11\},\{4,10\},\{5,9\},\{13,8\},\{0,7\}\},\\
F_2^*+1 &= \{\{2,7\},\{3,13\},\{4,12\},\{5,11\},\{6,10\},\{0,9\},\{1,8\}\},\\
F_2^*+2 &= \{\{3,8\},\{4,0\},\{5,13\},\{6,12\},\{7,11\},\{1,10\},\{2,9\}\},\\
F_2^*+3 &= \{\{4,9\},\{5,1\},\{6,0\},\{7,13\},\{8,12\},\{2,11\},\{3,10\}\},\\
F_2^*+4 &= \{\{5,10\},\{6,2\},\{7,1\},\{8,0\},\{9,13\},\{3,12\},\{4,11\}\},\\
F_2^*+5 &= \{\{6,11\},\{7,3\},\{8,2\},\{9,1\},\{10,0\},\{4,13\},\{5,12\}\},\\
F_2^*+6 &= \{\{7,12\},\{8,4\},\{9,3\},\{10,2\},\{11,1\},\{5,0\},\{6,13\}\}, \text{ and}\\
T_{1,2,3} &= \{\{i,i+1,i+3\} \mid i=0,1,\ldots,13\}.
\end{aligned}$$

We define

$$\begin{aligned}
\mathfrak{F}_1^* &= \{F_2^*+j \mid j=0,1,\ldots,6\},\\
\mathfrak{F}_2^* &= \emptyset.
\end{aligned}$$

Then we have a factorization

$$\mathfrak{F}_v^* = \mathfrak{F}_1^* \cup \mathfrak{F}_2^* \cup T_{1,2,3}$$

which consists of 7 one-factors and 14 triples.

We now apply our one-factors and triples given from Constructions 1, 6, and 9 to the well-known recursive construction of STS suggested by J. Doyen and R. Wilson [3] described as follows.

Theorem 13. *If there is a STS(v), then there is a STS($2v+s$) with the original STS(v) as a subsystem for $s=1$, or 7. If $v \equiv 3 \pmod 6$ and $s=3$, then there is a STS($2v+s$) with the original STS(v) as a subsystem.*

Theorem 13 (also shown in [2, 8, 10]) guaranties that $STS(2v+s)$ is obtained from combining the given subsystem $STS(v)$ with our one-factorization of K_{v+s} for each $s=1,7$, and $STS(2v+s)$ is also guarantied for the case when $s=3$ and $v \equiv 3 \pmod 6$.

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