

THE BFK-GLUING FORMULA FOR ZETA-DETERMINANTS AND THE VALUE OF RELATIVE ZETA FUNCTIONS AT ZERO

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ABSTRACT. The purpose of this paper is to discuss the constant term appearing in the BFK-gluing formula for the zeta-determinants of Laplacians on a complete Riemannian manifold when the warped product metric is given on a collar neighborhood of a cutting compact hypersurface. If the dimension of a hypersurface is odd, generally this constant is known to be zero. In this paper we describe this constant by using the heat kernel asymptotics and compute it explicitly when the dimension of a hypersurface is 2 and 4. As a byproduct we obtain some results for the value of relative zeta functions at $s = 0$.

1. Introduction

The gluing formula for zeta-determinants of Laplacians on a compact Riemannian manifold with boundary had been given by Burghelea, Friedlander, and Kappeler in [2] and later was extended by Carron in [3]. Their formula, however, contains a constant term which is expressed by the zero coefficient of some asymptotic expansions ([2], [8]). If the dimension of a cutting compact hypersurface is odd, this constant is known to be zero ([7]). If the product metric is given on a collar neighborhood of a cutting hypersurface, this constant was computed explicitly in [3] and [9]. The BFK-gluing formula also contains some informations about the value of relative zeta functions at $s = 0$. In this paper we discuss this constant term when the warped product metric is given on a collar neighborhood of a cutting compact hypersurface. More precisely, we describe this constant in terms of heat kernel asymptotics and compute it explicitly when the dimension of a cutting compact hypersurface is 2 and 4. As a byproduct we obtain some informations about the value of relative zeta functions at $s = 0$, which we discuss in the last section.

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Let (M, g) be either a complete oriented Riemannian manifold or a compact oriented Riemannian manifold with boundary W (W is possibly empty) with dimension $m+1$. Suppose that Y is a compact hypersurface of M with $Y \cap W = \emptyset$. Choose a collar neighborhood $N_{-1,1}$ of Y such that $N_{-1,1}$ is diffeomorphic to $([-1, 1] \times Y)$ with $N_{-1,1} \cap W = \emptyset$ and Y is identified with $\{0\} \times Y$. We assume that g is a warped product metric on $N_{-1,1}$, i.e.,

$$(1.1) \quad g|_{N_{-1,1}} = (du^2 + e^{h(u)}g_Y),$$

where g_Y is a Riemannian metric on Y , u is the normal direction to Y and $h : [-1, 1] \rightarrow \mathbb{R}$ is a smooth function with $h(0) = 0$. We denote by $\frac{\partial}{\partial u}$ the unit vector field on $N_{-1,1}$ which is normal to $Y_u := \{u\} \times Y$. Let $E \rightarrow M$ be a complex vector bundle on M having the product structure on $N_{-1,1}$, which means that $E|_{N_{-1,1}} = p^*E|_Y$, where $p : [-1, 1] \times Y \rightarrow [-1, 1]$ is the projection on the first component. Then the Laplacian Δ_M corresponding to the metric g is described on $N_{-1,1}$ as follows.

$$(1.2) \quad \Delta_M|_{N_{-1,1}} = -\frac{\partial^2}{\partial u^2} - \frac{m}{2}h'(u)\frac{\partial}{\partial u} + e^{-h(u)}\Delta_Y,$$

where $m = \dim Y$ and Δ_Y is a Laplacian on Y . We impose a boundary condition P_W on W so that Δ_M can be extended to a non-negative self-adjoint operator. We denote by M_{cut} the manifold with boundary $W \cup Y \cup Y$ obtained by cutting M along $Y (= Y_0)$ and by $\Delta_{M_{cut}}$ the natural extension of Δ_M to M_{cut} . We also denote by $Y_{0,1}$ ($Y_{0,2}$) the component of the boundary of M_{cut} which is the copy of Y and $\frac{\partial}{\partial u}$ points outward (inward). We impose the Dirichlet boundary condition on $Y_{0,1} \cup Y_{0,2}$ and denote by $\Delta_{M_{cut}, \gamma_0}$ the realization of $\Delta_{M_{cut}}$ with respect to P_W on W and the Dirichlet boundary condition on $Y_{0,1} \cup Y_{0,2}$. Then for $\lambda \in \mathbb{R}^+$ $(e^{-t(\Delta_M + \lambda)} - e^{-t(\Delta_{M_{cut}, \gamma_0} + \lambda)})$ is a trace class operator (see [3]) and we define the relative zeta function and relative zeta-determinant for $(\Delta_M + \lambda, \Delta_{M_{cut}, \gamma_0} + \lambda)$ (cf. [13]) by

$$(1.3) \quad \begin{aligned} \zeta(s; \Delta_M + \lambda, \Delta_{M_{cut}, \gamma_0} + \lambda) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr} \left(e^{-t(\Delta_M + \lambda)} - e^{-t(\Delta_{M_{cut}, \gamma_0} + \lambda)} \right) dt, \\ \log \text{Det} (\Delta_M + \lambda, \Delta_{M_{cut}, \gamma_0} + \lambda) &= -\zeta'(0; \Delta_M + \lambda, \Delta_{M_{cut}, \gamma_0} + \lambda). \end{aligned}$$

Throughout this paper we assume that for $\lambda \in \mathbb{R}^+$ the Dirichlet boundary value problem for $\Delta_{M_{cut}} + \lambda$ has a unique solution, i.e., for $(f, g) \in C^\infty(Y_{0,1} \cup Y_{0,2})$ there exists a unique solution ϕ such that

$$(\Delta_{M_{cut}} + \lambda)\phi = 0, \quad \phi|_{Y_{0,1}} = f, \quad \phi|_{Y_{0,2}} = g, \quad P_W(\phi)|_W = 0.$$

Then Burghelera, Friedlander, and Kappeler ([2]) had studied

$$\log \text{Det} (\Delta_M + \lambda, \Delta_{M_{cut}, \gamma_0} + \lambda)$$

for $\lambda \in \mathbb{R}^+$ on a compact manifold and later Carron ([3]) extended their result to the case of a complete non-compact manifold.

To state their results, we introduce elliptic Ψ DO's $Q_1(\lambda)$, $Q_2(\lambda)$, and $R(\lambda)$ acting on $C^\infty(Y)$ as follows. For $f \in C^\infty(Y)$, we choose unique sections $\phi, \psi \in C^\infty(M_{cut})$ satisfying

$$\begin{aligned} (\Delta_{M_{cut}} + \lambda)\phi &= 0, & \phi|_{Y_{0,1}} &= f, & \phi|_{Y_{0,2}} &= 0, & P_W(\phi)|_W &= 0, \\ (\Delta_{M_{cut}} + \lambda)\psi &= 0, & \psi|_{Y_{0,1}} &= 0, & \psi|_{Y_{0,2}} &= f, & P_W(\psi)|_W &= 0. \end{aligned}$$

Then we define

$$\begin{aligned} (1.4) \quad Q_1(\lambda)(f) &= \left(\frac{\partial}{\partial u}\phi\right)|_{Y_{0,1}} - \left(\frac{\partial}{\partial u}\phi\right)|_{Y_{0,2}}, \\ Q_2(t)(\lambda) &= \left(\frac{\partial}{\partial u}\psi\right)|_{Y_{0,1}} - \left(\frac{\partial}{\partial u}\psi\right)|_{Y_{0,2}} \end{aligned}$$

and define the Dirichlet-to-Neumann operator $R(\lambda) : C^\infty(Y) \rightarrow C^\infty(Y)$ by

$$(1.5) \quad R(\lambda) = Q_1(\lambda) + Q_2(\lambda).$$

It is a well-known fact that for $\lambda \in \mathbb{R}^+$, $R(\lambda)$ is a positive elliptic Ψ DO of order 1. Then they proved the following equality. For some real constants π_j ($0 \leq j \leq [\frac{m}{2}]$),

$$(1.6) \quad \log \text{Det} (\Delta_M + \lambda, \Delta_{M_{cut}, \gamma_0} + \lambda) = \sum_{j=0}^{[\frac{m}{2}]} \pi_j \lambda^j + \log \text{Det} R(\lambda).$$

It is known ([2], [3]) that $\log \text{Det} (\Delta_M + \lambda, \Delta_{M_{cut}, \gamma_0} + \lambda)$ and $\log \text{Det} R(\lambda)$ have asymptotic expansions as $\lambda \rightarrow \infty$, whose coefficients can be computed by integral of some densities determined by the symbols of operators. Moreover, the zero coefficient in the asymptotic expansion of $\log \text{Det} (\Delta_M + \lambda, \Delta_{M_{cut}, \gamma_0} + \lambda)$ is zero (Lemma 3.2 or [19]). Hence $-\pi_0$ in (1.6) is, in fact, the zero coefficient in the asymptotic expansion of $\log \text{Det} R(\lambda)$. This fact enables us to compute π_0 in some cases. It is known that $\pi_0 = 0$ when $\dim M$ is even ([7]). If the metric g is a product one on $N_{-1,1}$ so that $\Delta_M = -\partial_u^2 + \Delta_Y$ on $N_{-1,1}$, it is known that $\pi_0 = \log 2 \cdot (\zeta_{\Delta_Y}(0) + \dim \ker \Delta_Y)$ ([3], [9]).

On the other hand, if M is compact or the essential spectrum $\sigma_{ess}(\Delta_M)$ of Δ_M has a positive lower bound, the coefficient of $\log \lambda$ in the asymptotic expansion of $\log \text{Det} (\Delta_M + \lambda, \Delta_{M_{cut}, \gamma_0} + \lambda)$ is $\zeta(0; \Delta_M, \Delta_{M_{cut}, \gamma_0}) + \dim \ker \Delta_M$ (Lemma 3.2). Hence, the comparison of the $\log \lambda$ -coefficients in the asymptotic expansions of (1.6) gives some informations about the value of the relative zeta functions at zero.

In this paper we are going to discuss the constant π_0 in (1.6), or equivalently the polynomial $\sum_{j=0}^{[\frac{m}{2}]} \pi_j \lambda^j$, and the value of relative zeta functions at $s = 0$ when the Laplacian is given by (1.2) on $N_{-1,1}$. We next discuss, using the result in [10], the value of the zeta function at $s = 0$ for a compatible Dirac Laplacian with the Atiyah-Patodi-Singer (APS) boundary condition on a compact manifold with boundary, which extends the result given by the author in the Appendix of [15].

2. Description of the Dirichlet-to-Neumann operator $R(\lambda)$

In this section we are going to describe the Dirichlet-to-Neumann operator $R(\lambda)$ by using the variations of $Q_1(\lambda), Q_2(\lambda)$. To do this we extend M_{cut} so that we put $\widetilde{M}_{cut} := ([-1, 0] \times Y_{0,2}) \cup_{Y_{0,2}} M_{cut} \cup_{Y_{0,1}} ([0, 1] \times Y_{0,1})$ and denote by $Y_{u,1} (Y_{u,2})$ for $-1 \leq u \leq 1$ the copy of $Y_u := \{u\} \times Y$ belonging to the same part with $Y_{0,1} (Y_{0,2})$ in \widetilde{M}_{cut} . Then for $f \in C^\infty(Y_{u,1}) = C^\infty(Y_{u,2})$ choose $\phi_u, \psi_u \in C^\infty(\widetilde{M}_{cut})$ satisfying

$$\begin{aligned} (\Delta_{\widetilde{M}_{cut}} + \lambda) \phi_u &= 0, & \phi_u|_{Y_{u,1}} &= f, & \phi_u|_{Y_{0,2}} &= 0, & P_W(\phi_u)|_W &= 0, \\ (\Delta_{\widetilde{M}_{cut}} + \lambda) \psi_u &= 0, & \phi_u|_{Y_{0,1}} &= 0, & \psi_u|_{Y_{u,2}} &= f, & P_W(\psi_u)|_W &= 0. \end{aligned}$$

Then we define

(2.1)

$$\begin{aligned} Q_{1,u}(\lambda)(f) &= \left(\frac{\partial}{\partial u} \phi_u\right)|_{Y_{u,1}} - \left(\frac{\partial}{\partial u} \phi_u\right)|_{Y_{0,2}} =: \widetilde{Q}_{1,u}(\lambda)(f) + \Psi_{1,u}(\lambda)(f), \\ Q_{2,u}(\lambda)(f) &= \left(\frac{\partial}{\partial u} \psi_u\right)|_{Y_{0,1}} - \left(\frac{\partial}{\partial u} \psi_u\right)|_{Y_{u,2}} =: \Psi_{2,u}(\lambda)(f) + \widetilde{Q}_{2,u}(\lambda)(f), \end{aligned}$$

where $\widetilde{Q}_{1,u}(\lambda)(f) := (\frac{\partial}{\partial u} \phi_u)|_{Y_{u,1}}$ and $\Psi_{1,u}(\lambda)(f) := -(\frac{\partial}{\partial u} \phi_u)|_{Y_{0,2}}$. $\widetilde{Q}_{2,u}(\lambda)(f)$ and $\Psi_{2,u}(\lambda)(f)$ are defined similarly.

Now for $f \in C^\infty(Y_0)$, choose $\phi(u, y) \in C^\infty(M_{cut})$ such that

$$(\Delta_{M_{cut}} + \lambda) \phi(u, y) = 0, \quad \phi|_{Y_{0,1}} = f, \quad \phi|_{Y_{0,2}} = 0, \quad P_W(\phi)|_W = 0.$$

Then for each $u, -1 \leq u \leq 1$, we have

$$\left(\frac{\partial}{\partial u} \phi(u, y)\right)|_{Y_{u,1}} = \widetilde{Q}_{1,u}(\lambda) (\phi(u, y)|_{Y_{u,1}}).$$

Taking the derivative with respect to u again,

(2.2)

$$\begin{aligned} \left(\frac{\partial^2}{\partial u^2} \phi(u, y)\right)|_{Y_{u,1}} &= \frac{\partial \widetilde{Q}_{1,u}(\lambda)}{\partial u} (\phi(u, y)|_{Y_{u,1}}) + \widetilde{Q}_{1,u}(\lambda) \left(\frac{\partial \phi(u, y)}{\partial u}\right)|_{Y_{u,1}} \\ &= \left(\frac{\partial}{\partial u} \widetilde{Q}_{1,u}(\lambda) + \widetilde{Q}_{1,u}(\lambda)^2\right) (\phi(u, y)|_{Y_{u,1}}). \end{aligned}$$

Since ϕ satisfies $(\Delta_{M_{cut}} + \lambda) \phi = 0$, (1.2) leads to

$$\begin{aligned} &\left(-\frac{m}{2} h'(u) \widetilde{Q}_{1,u}(\lambda) + e^{-h(u)} \Delta_Y + \lambda\right) (\phi(u, y)|_{Y_{u,1}}) \\ &= \left(\frac{\partial}{\partial u} \widetilde{Q}_{1,u}(\lambda) + \widetilde{Q}_{1,u}(\lambda)^2\right) (\phi(u, y)|_{Y_{u,1}}), \end{aligned}$$

which shows that

$$(2.3) \quad \left(\widetilde{Q}_{1,u}(\lambda) + \frac{m}{4} h'(u)\right)^2 = e^{-h(u)} \Delta_Y + \lambda + \frac{m^2}{16} h'(u)^2 - \frac{\partial}{\partial u} \widetilde{Q}_{1,u}(\lambda).$$

Similarly, we have

$$(2.4) \quad \left(\tilde{Q}_{2,u}(\lambda) - \frac{m}{4}h'(u) \right)^2 = e^{-h(u)}\Delta_Y + \lambda + \frac{m^2}{16}h'(u)^2 + \frac{\partial}{\partial u}\tilde{Q}_{2,u}(\lambda).$$

We next note that $\Psi_{1,u}(\lambda) (\phi(u, y)|_{Y_{u,1}}) = -(\frac{\partial}{\partial u}\phi(u, y))|_{Y_{0,2}}$. Taking the derivative, we have

$$(2.5) \quad \left(\frac{\partial}{\partial u}\Psi_{1,u}(\lambda) \right) \phi(u, y)|_{Y_{u,1}} + \Psi_{1,u}(\lambda) \left(\tilde{Q}_{1,u}(\lambda)\phi(u, y)|_{Y_{u,1}} \right) = 0,$$

which shows that $\frac{\partial}{\partial u}\Psi_{1,u}(\lambda) + \Psi_{1,u}(\lambda)\tilde{Q}_{1,u}(\lambda) = 0$. Since $\frac{\partial}{\partial u}\Psi_{1,u}(\lambda)$ and $\Psi_{1,u}(\lambda)$ are operators of the same order and $\tilde{Q}_{1,u}(\lambda)$ is an elliptic operator of order 1, this equality implies that $\Psi_{1,u}(\lambda)$ is a smoothing operator. As the same way, $\Psi_{2,u}(\lambda)$ is also a smoothing operator. Setting $R_u(\lambda) = Q_{1,u}(\lambda) + Q_{2,u}(\lambda)$, we have the following lemma.

Lemma 2.1. *Let $\Omega_u(\lambda) = Q_{1,u}(\lambda) - Q_{2,u}(\lambda) + \frac{m}{2}h'(u)$. Then $R_u(\lambda)^2$ is expressed as follows.*

$$R_u(\lambda)^2 = 4 \left(e^{-h(u)}\Delta_Y + \lambda + \frac{m^2}{16}h'(u)^2 + \frac{m}{4}h''(u) \right) - 2\frac{\partial}{\partial u}\Omega_u(\lambda) - \Omega_u(\lambda)^2 + \text{a smoothing operator.}$$

Remark. It is well-known ([2], [3]) that $R(\lambda)^{-1} = \gamma_0 (\Delta_M + \lambda)^{-1} (\cdot \otimes \delta_Y)$ and this fact implies that $R(\lambda)$ is positive definite for $\lambda \in \mathbb{R}^+$, where γ_0 is the restriction map to Y .

We now discuss the asymptotic symbols of $\frac{\partial}{\partial u}\Omega_u(\lambda)$ and $\Omega_u(\lambda)^2$. The equations (2.3), (2.4), and (2.5) show that

$$(2.6) \quad \begin{aligned} Q_{1,u}(\lambda)^2 &= e^{-h(u)}\Delta_Y + \lambda - \frac{m}{2}h'(u)Q_{1,u}(\lambda) - \frac{\partial}{\partial u}Q_{1,u}(\lambda) \\ &\quad + \text{a smoothing operator,} \\ Q_{2,u}(\lambda)^2 &= e^{-h(u)}\Delta_Y + \lambda + \frac{m}{2}h'(u)Q_{2,u}(\lambda) + \frac{\partial}{\partial u}Q_{2,u}(\lambda) \\ &\quad + \text{a smoothing operator.} \end{aligned}$$

We denote the asymptotic symbols of $Q_{1,u}(\lambda)$, $Q_{2,u}(\lambda)$, and $e^{-h(u)}\Delta_Y + \lambda$ as follows.

$$\begin{aligned} \sigma(Q_{1,u}(\lambda))(y, \xi, \lambda) &\sim \alpha_{1,u}(y, \xi, \lambda) + \alpha_{0,u}(y, \xi, \lambda) + \alpha_{-1,u}(y, \xi, \lambda) + \dots, \\ \sigma(Q_{2,u}(\lambda))(y, \xi, \lambda) &\sim \beta_{1,u}(y, \xi, \lambda) + \beta_{0,u}(y, \xi, \lambda) + \beta_{-1,u}(y, \xi, \lambda) + \dots, \\ \sigma(e^{-h(u)}\Delta_Y + \lambda) &= (e^{-h(u)}|\xi|^2 + \lambda) + e^{-h(u)}p_1(y, \xi) + e^{-h(u)}p_0(y, \xi). \end{aligned}$$

Then the asymptotic symbols of $Q_{1,u}(\lambda)^2$ and $Q_{2,u}(\lambda)^2$ are given by
(2.7)

$$\begin{aligned}\sigma(Q_{1,u}(\lambda)^2)(y, \xi, \lambda) &\sim \sum_{k=0}^{\infty} \sum_{\substack{|\omega|+i+j=k \\ i,j \geq 0}} \frac{1}{\omega!} d_{\xi}^{\omega} \alpha_{1-i,u}(y, \xi, \lambda) \cdot D_y^{\omega} \alpha_{1-j,u}(y, \xi, \lambda), \\ \sigma(Q_{2,u}(\lambda)^2)(y, \xi, \lambda) &\sim \sum_{k=0}^{\infty} \sum_{\substack{|\omega|+i+j=k \\ i,j \geq 0}} \frac{1}{\omega!} d_{\xi}^{\omega} \beta_{1-i,u}(y, \xi, \lambda) \cdot D_y^{\omega} \beta_{1-j,u}(y, \xi, \lambda),\end{aligned}$$

which lead to

$$\begin{aligned}(2.8) \quad \alpha_{1,u} &= \beta_{1,u} = \sqrt{e^{-h(u)}|\xi|^2 + \lambda}, \\ \alpha_{0,u} &= \frac{1}{2}\alpha_{1,u}^{-1} \left(-d_{\xi}\alpha_{1,u} \cdot D_y\alpha_{1,u} + e^{-h(u)}p_1 - \frac{m}{2}h'(u)\alpha_{1,u} + \frac{h'(u)e^{-h(u)}|\xi|^2}{2\sqrt{e^{-h(u)}|\xi|^2 + \lambda}} \right), \\ \beta_{0,u} &= \frac{1}{2}\beta_{1,u}^{-1} \left(-d_{\xi}\beta_{1,u} \cdot D_y\beta_{1,u} + e^{-h(u)}p_1 + \frac{m}{2}h'(u)\beta_{1,u} - \frac{h'(u)e^{-h(u)}|\xi|^2}{2\sqrt{e^{-h(u)}|\xi|^2 + \lambda}} \right).\end{aligned}$$

Using the relation (2.7) we can compute the homogeneous components $\alpha_{1-k,u}$ and $\beta_{1-k,u}$ for any $k \geq 1$, and hence the asymptotic symbols of $\frac{\partial}{\partial u}\Omega_u(\lambda)$ and $\Omega_u(\lambda)^2$. For instance, the principal symbols of $\frac{\partial}{\partial u}\Omega_u(\lambda)$ and $\Omega_u(\lambda)^2$ are given as follows.

$$\begin{aligned}(2.9) \quad \sigma_L \left(2 \frac{\partial}{\partial u} \Omega_u(\lambda) \right) &= \frac{(h''(u) - h'(u)^2) e^{-h(u)} |\xi|^2}{e^{-h(u)} |\xi|^2 + \lambda} + \frac{h'(u)^2 e^{-2h(u)} |\xi|^4}{(e^{-h(u)} |\xi|^2 + \lambda)^2}, \\ \sigma_L (\Omega_u(\lambda)^2) &= \frac{h'(u)^2 e^{-2h(u)} |\xi|^4}{4 (e^{-h(u)} |\xi|^2 + \lambda)^2}.\end{aligned}$$

Corollary 2.2. $\frac{\partial}{\partial u}\Omega_u(\lambda)$ and $\Omega_u(\lambda)^2$ are ΨDO 's of order 0 and each homogeneous part of the asymptotic symbol tends to 0 as $\lambda \rightarrow \infty$.

We next discuss the heat kernel asymptotics of $R_u(\lambda)^2$ at $u = 0$. We assume that $h(0) = 0$ and denote by

$$(2.10) \quad c_0 = \frac{m^2}{16} h'(0)^2 + \frac{m}{4} h''(0), \quad \mathfrak{S}_0(\lambda) = -2 \frac{\partial}{\partial u} \Omega_u(\lambda)|_{u=0} - \Omega_0(\lambda)^2.$$

We suppose that as $t \rightarrow 0^+$,

$$\begin{aligned}(2.11) \quad \text{Tr } e^{-t(\Delta_Y + c_0)} &\sim \sum_{j=0}^{\infty} \mathbf{a}_j t^{-\frac{m}{2} + j}, \\ \text{Tr } e^{-t(\Delta_Y + c_0 + \frac{1}{4}\mathfrak{S}_0(\lambda))} &\sim \sum_{j=0}^{\infty} \mathbf{a}_j(\lambda) t^{-\frac{m}{2} + j} + \sum_{k=1}^{\infty} \mathbf{b}_k(\lambda) t^k \log t.\end{aligned}$$

Then each coefficient $\mathfrak{a}_j(\lambda)$ and \mathfrak{a}_j in (2.11) can be computed by the following formula (cf. [5], [18]). Let Γ be a contour in the complex plane \mathbb{C} defined by

$$(2.12) \quad \Gamma = \{re^{i\pi} | \infty > r \geq \epsilon\} \cup \{\epsilon e^{i\phi} | \pi \geq \phi \geq -\pi\} \cup \{re^{-i\pi} | \epsilon \leq r < \infty\}$$

for sufficiently small $\epsilon > 0$ and oriented counterclockwise. Let us fix a finite number of local coordinate charts covering Y , local trivializations for E , and a partition of unity subordinate to this covering. In each coordinate chart we denote the asymptotic symbol of $(\Delta_Y + \mathfrak{c}_0 + \frac{1}{4}\mathfrak{S}_0(\lambda))$ by

$$(2.13) \quad \begin{aligned} &\sigma\left(\Delta_Y + \mathfrak{c}_0 + \frac{1}{4}\mathfrak{S}_0(\lambda)\right) \\ &\sim p_2(y, \xi) + p_1(y, \xi) + (p_0(y, \xi) + q_0(y, \xi, \lambda)) + q_{-1}(y, \xi, \lambda) + \dots, \end{aligned}$$

where $p_2(y, \xi) = |\xi|^2$ and $\sum_{j=0}^2 p_j(y, \xi)$, $\sum_{j=0}^\infty q_{-j}(y, \xi, \lambda)$ are the (asymptotic) symbols of $\Delta_Y + \mathfrak{c}_0$ and $\frac{1}{4}\mathfrak{S}_0(\lambda)$, respectively. Then the asymptotic symbol $\sum_{j=0}^\infty r_{-2-j}(y, \xi, \lambda, \mu)$ of the resolvent $(\mu - (\Delta_Y + \mathfrak{c}_0 + \frac{1}{4}\mathfrak{S}_0(\lambda)))^{-1}$ is given recursively as follows.

$$(2.14) \quad \begin{aligned} r_{-2}(y, \xi, \lambda, \mu) &= (\mu - |\xi|^2)^{-1}, \\ r_{-2-j}(y, \xi, \lambda, \mu) &= -(\mu - |\xi|^2)^{-1} \sum_{k=0}^{j-1} \sum_{|\omega|+l+k=j} \frac{1}{\omega!} \partial_\xi^\omega \tilde{p}_{2-l}(y, \xi, \lambda) \cdot D_y^\omega r_{-2-k}(y, \xi, \lambda, \mu), \end{aligned}$$

where

$$\tilde{p}_j(y, \xi, \lambda) = \begin{cases} p_j(y, \xi) & \text{for } j = 1, 2, \\ p_0(y, \xi) + q_0(y, \xi, \lambda) & \text{for } j = 0, \\ q_j(y, \xi, \lambda) & \text{for } j < 0. \end{cases}$$

Now $\mathfrak{a}_j(\lambda)$ ($0 \leq j \leq [\frac{m}{2}]$) can be computed by the following integral (cf. Formula (12) in [5] or Theorem 13.1 in [18]).

$$(2.15) \quad \mathfrak{a}_j(\lambda) = \begin{cases} \frac{\Gamma(\frac{m}{2}-j)}{2 \cdot 2\pi i} \int_Y d\text{vol}(Y) \int_{|\xi|=1} \frac{1}{(2\pi)^m} d\xi' \int_\Gamma \mu^{-\frac{m}{2}+j} r_{-2-2j}(y, \xi, \lambda, \mu) d\mu & \text{for } 0 \leq j < \frac{m}{2}, \\ \frac{1}{2} \int_Y d\text{vol}(Y) \int_{|\xi|=1} \frac{1}{(2\pi)^m} d\xi' \int_0^\infty r_{-2-m}(y, \xi, \lambda, -\nu) d\nu & \text{for } j = \frac{m}{2} \in \mathbb{Z}^+, \end{cases}$$

where $d\xi'$ is the usual surface element of the sphere $|\xi| = 1$. Similarly, let us denote the asymptotic symbol of $(\mu - (\Delta_Y + \mathfrak{c}_0))^{-1}$ by

$$(2.16) \quad \sigma\left((\mu - (\Delta_Y + \mathfrak{c}_0))^{-1}\right) \sim \sum_{j=0}^\infty \delta_{-2-j}(y, \xi, \mu),$$

where $\delta_{-2-j}(y, \xi, \mu)$ is defined recursively by the same formula as (2.14). Then \mathfrak{a}_j can be computed by the integral (2.15) with the integrand $\delta_{-2-j}(y, \xi, \mu)$.

We here note that

$$(2.17) \quad r_{-2-j}(y, \xi, \lambda, \mu) = \delta_{-2-j}(y, \xi, \mu) + \tilde{r}_{-2-j}(y, \xi, \lambda, \mu),$$

where $\tilde{r}_{-2-j}(y, \xi, \lambda, \mu)$ tends to 0 as $\lambda \rightarrow \infty$. In particular, $r_{-2}(y, \xi, \lambda, \mu) = \delta_{-2}(y, \xi, \mu)$. Then we have the following result.

Lemma 2.3. *Each $\mathbf{a}_j(\lambda)$, $0 \leq j \leq [\frac{m}{2}]$, has an asymptotic expansion for $\lambda \rightarrow \infty$ of the following form:*

$$\mathbf{a}_j(\lambda) \sim \mathbf{a}_j + \sum_{k=1}^{\infty} \mathbf{a}_{j, \frac{k}{2}} \lambda^{-\frac{k}{2}}.$$

In particular,

$$\mathbf{a}_0(\lambda) = \mathbf{a}_0 \quad \text{and} \quad \lim_{\lambda \rightarrow \infty} \mathbf{a}_j(\lambda) = \mathbf{a}_j, \quad (1 < j \leq [\frac{m}{2}]).$$

Proof. In view of (2.14), $r_{-2-j}(y, \xi, \lambda, \mu)$ can be expressed by

$$r_{-2-j}(y, \xi, \lambda, \mu) = \sum_{k \geq 0} \frac{f_k(y, \xi, \mu)}{(|\xi|^2 + \lambda)^{\frac{k}{2}}},$$

where $f_k(y, \xi, \mu)$ is a homogeneous polynomial of degree $k - 2 - j$ with respect to ξ and μ . Hence, (2.15) shows that for $0 \leq j < \frac{m}{2}$,

$$\mathbf{a}_j(\lambda) = \sum_k \frac{1}{(\lambda + 1)^{\frac{k}{2}}} \frac{1}{2 \cdot 2\pi i} \int_Y d\text{vol}(Y) \int_{|\xi|=1} \frac{1}{(2\pi)^m} d\xi' \int_{\Gamma} \mu^{-\frac{m}{2}-j} f_k(y, \xi, \mu) d\mu,$$

from which the result follows. The case of $j = \frac{m}{2}$ can be treated in the same way. □

3. The main results and their proofs

We begin this section with the following lemma, which is straightforward (cf. [8] or [19]).

Lemma 3.1. *Let P be an elliptic Ψ DO of order > 0 on a compact manifold and $\{\alpha_j\}, \{\beta_j\}$ be increasing sequences with $\beta_0 > 0$, and tending to ∞ . Suppose that*

$$\text{Tr } e^{-tP} \sim \sum_{j=0}^{\infty} a_j t^{\alpha_j} + \sum_{j=0}^{\infty} b_j t^{\beta_j} \log t \quad \text{for } t \rightarrow 0^+.$$

Then, as $\lambda \rightarrow \infty$,

$$\begin{aligned} & \log \text{Det}(P + \lambda) \\ & \sim - \sum_{j=0}^{\infty} a_j \frac{d}{ds} \left(\frac{\Gamma(s + \alpha_j)}{\Gamma(s)} \right)_{s=0} \cdot \lambda^{-\alpha_j} + \sum_{j=0}^{\infty} a_j \left(\frac{\Gamma(s + \alpha_j)}{\Gamma(s)} \right)_{s=0} \cdot \lambda^{-\alpha_j} \log \lambda \\ & \quad + \sum_{j=0}^{\infty} b_j \Gamma(\beta_j) \cdot \lambda^{-\beta_j} \log \lambda - \sum_{j=0}^{\infty} b_j \int_0^{\infty} x^{\beta_j-1} e^{-x} \log x dx \cdot \lambda^{-\beta_j} + O(e^{-c\lambda}). \end{aligned}$$

In particular, the constant term does not appear and the coefficient of $\log \lambda$ is $(\zeta_P(0) + \dim \ker P)$.

We next consider the asymptotic expansions of

$$\log \text{Det} (\Delta_M + \lambda, \Delta_{M_{cut}, \gamma_0} + \lambda)$$

and $\log \text{Det} R(\lambda)$ as $\lambda \rightarrow \infty$. Let K be a collar neighborhood of Y whose closure is a compact subset of M . Then it is shown in [1] (see also [3]) that for some positive constant C ,

$$| \text{Tr} (1_{M-K} (e^{-t\Delta_M} - e^{-t\Delta_{M_{cut}, \gamma_0}}) 1_{M-K}) | \leq e^{-\frac{C}{t}}.$$

It is a well-known fact that for some real constants ω_j ,

$$\text{Tr} (1_K (e^{-t\Delta_M} - e^{-t\Delta_{M_{cut}, \gamma_0}}) 1_K) \sim \sum_{j=0}^{\infty} \omega_j t^{\frac{j-m}{2}}, \quad \text{as } t \rightarrow 0^+.$$

Hence, for $t \rightarrow 0$,

$$(3.1) \quad \text{Tr} (e^{-t\Delta_M} - e^{-t\Delta_{M_{cut}, \gamma_0}}) \sim \sum_{j=0}^{\infty} \omega_j t^{\frac{j-m}{2}},$$

where $\dim M = m + 1$. This fact together with Lemma 3.1 yields the following result.

Lemma 3.2. *As $\lambda \rightarrow \infty$, we have the following asymptotic expansion.*

$$\begin{aligned} & \log \text{Det} (\Delta_M + \lambda, \Delta_{M_{cut}, \gamma_0} + \lambda) \\ & \sim - \sum_{j=0}^{\infty} \omega_j \left(\frac{d}{ds} \left(\frac{\Gamma(s - \frac{m-j}{2})}{\Gamma(s)} \right) \right)_{s=0} \lambda^{\frac{m-j}{2}} \\ & \quad + \sum_{j=0}^m \omega_j \left(\frac{\Gamma(s - \frac{m-j}{2})}{\Gamma(s)} \right)_{s=0} \lambda^{\frac{m-j}{2}} \cdot \log \lambda + O(e^{-c\lambda}). \end{aligned}$$

In particular, the constant term does not appear. If M is compact or complete with $\sigma_{ess}(\Delta_M)$ having a positive lower bound, the coefficient of $\log \lambda$ is $\zeta(0; \Delta_M, \Delta_{M_{cut}, \gamma_0}) + \dim \ker \Delta_M$.

The above lemma shows that $-\pi_0$ in (1.6) is, in fact, the zero coefficient in the asymptotic expansion of $\log \text{Det} R(\lambda)$ as $\lambda \rightarrow \infty$. If $\dim Y$ is odd, this zero coefficient is known to be zero ([7]) and hence we need to discuss the case of $\dim Y$ even. By Lemma 2.1 and Lemma 3.1 with (2.11) we have the following result.

Lemma 3.3. *Let m be the dimension of Y . Then*

$$\begin{aligned} & \log \text{Det}R(\lambda) \\ &= \log 2 \cdot \zeta_{(\Delta_Y + c_0 + \frac{1}{4}\mathfrak{S}_0(\lambda) + \lambda)}(0) - \frac{1}{2} \sum_{j=0}^{\infty} \mathfrak{a}_j(\lambda) \frac{d}{ds} \left(\frac{\Gamma(s - \frac{m}{2} + j)}{\Gamma(s)} \right)_{s=0} \lambda^{\frac{m}{2}-j} \\ & \quad + \frac{1}{2} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \mathfrak{a}_j(\lambda) \left(\frac{\Gamma(s - \frac{m}{2} + j)}{\Gamma(s)} \right)_{s=0} \lambda^{\frac{m}{2}-j} \log \lambda + O(\lambda^{-c}), \quad (c > 0), \end{aligned}$$

where

$$\zeta_{(\Delta_Y + c_0 + \frac{1}{4}\mathfrak{S}_0(\lambda) + \lambda)}(0) = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ \sum_{j=0}^{\frac{m}{2}} \frac{(-1)^{\frac{m}{2}-j}}{(\frac{m}{2}-j)!} \mathfrak{a}_j(\lambda) \lambda^{\frac{m}{2}-j} & \text{if } m \text{ is even.} \end{cases}$$

Specially, if $\dim Y$ is even, Lemma 3.3 together with Lemma 2.3 leads to the following corollary.

Corollary 3.4. *Let $m = \dim Y$ be even. Then $\log \text{Det}R(\lambda)$ has the following asymptotic expansion for $\lambda \rightarrow \infty$.*

$$\log \text{Det}R(\lambda) \sim \sum_{j=0}^{\infty} \mathfrak{p}_j \lambda^{\frac{m-j}{2}} + \sum_{j=0}^{\infty} \mathfrak{q}_j \lambda^{\frac{m-j}{2}} \log \lambda,$$

where

$$\begin{aligned} \mathfrak{p}_m &= \log 2 \sum_{j=1}^{\frac{m}{2}} \frac{(-1)^{\frac{m}{2}-j}}{(\frac{m}{2}-j)!} \mathfrak{a}_{j, \frac{m}{2}-j} - \frac{1}{2} \sum_{j=1}^{\frac{m}{2}-1} \frac{(-1)^{\frac{m}{2}-j}}{(\frac{m}{2}-j)!} \left(1 + \frac{1}{2} + \dots + \frac{1}{\frac{m}{2}-j} \right) \mathfrak{a}_{j, \frac{m}{2}-j}, \\ \mathfrak{q}_m &= \frac{1}{2} \sum_{j=1}^{\frac{m}{2}} \frac{(-1)^{\frac{m}{2}-j}}{(\frac{m}{2}-j)!} \mathfrak{a}_{j, \frac{m}{2}-j}. \end{aligned}$$

On the other hand, let $P(\eta)$ be a non-negative classical elliptic Ψ DO of order k with parameter η of weight χ ($\chi > 0$) on a d -dimensional compact closed manifold X . We refer to [2] or [18] for the definitions. Suppose that the asymptotic symbol of $P(\eta)$ is given as follows.

$$\sigma(P(\eta)) = \sum_{j=0}^{\infty} p_{k-j}(\eta, x, \xi),$$

where for $\tau > 0$, $p_{k-j}(\tau^\chi \eta, x, \tau \xi) = \tau^{k-j} p_{k-j}(\eta, x, \xi)$. Then it was shown in the Appendix of [2] that $\log \text{Det}P(\eta)$ has the following asymptotic expansion as $|\eta| \rightarrow \infty$.

$$(3.2) \quad \log \text{Det}P(\eta) \sim \sum_{j=0}^{\infty} \kappa_j |\eta|^{\frac{d-j}{\chi}} + \sum_{j=0}^d \theta_j |\eta|^{\frac{j}{\chi}} \log |\eta|,$$

where each κ_j and θ_j can be computed in terms of the symbol of $P(\frac{\eta}{|\eta|})$. Specially, κ_d and θ_0 can be computed as follows. Let us denote the symbol of $(\mu - P(\eta))^{-1}$ by

$$\sigma\left((\mu - P(\eta))^{-1}\right)(\mu, \eta, x, \xi) \sim \sum_{j=0}^{\infty} r_{-k-j}(\mu, \eta, x, \xi).$$

We define $J_d(s, \frac{\eta}{|\eta|}, x)$ by

$$(3.3) \quad J_d(s, \frac{\eta}{|\eta|}, x) = \frac{1}{2\pi i} \int_{\mathbb{R}^d} d\xi \int_{\Gamma} \mu^{-s} r_{-k-d}(\mu, \frac{\eta}{|\eta|}, x, \xi) d\mu,$$

where Γ is a contour in \mathbb{C} defined in (2.12). Then,

$$(3.4) \quad \begin{aligned} \kappa_d &= \frac{d}{ds} \left(\frac{1}{(2\pi)^d} \int_M J_d(s, \frac{\eta}{|\eta|}, x) \, d\text{vol}(x) \right) \Big|_{s=0}, \\ \theta_0 &= \left(\frac{1}{(2\pi)^d} \frac{k}{\chi} \int_M J_d(s, \frac{\eta}{|\eta|}, x) \, d\text{vol}(x) \right) \Big|_{s=0}. \end{aligned}$$

If the symbol of $P(\eta)$ satisfies the following property

$$(3.5) \quad p_{k-j}(\eta, x, -\xi) = (-1)^j p_{k-j}(\eta, x, \xi),$$

and $d = \dim X$ is odd, then $J_d(s, \frac{\eta}{|\eta|}, x) = 0$ and hence $\kappa_d = \theta_0 = 0$. It is known that the Dirichlet-to-Neumann operator $R(\lambda)$ is a classical elliptic Ψ DO of order 1 with parameter λ of weight 2 and satisfies (3.5) (cf. [7]). This fact with Lemma 3.2 leads to the following result.

Corollary 3.5. *Suppose that either M is compact or $\sigma_{\text{ess}}(\Delta_M)$ has a positive lower bound. If $\dim M$ is even, then*

$$\zeta(0; \Delta_M, \Delta_{M_{\text{cut}}, \gamma_0}) + \dim \ker \Delta_M = 0.$$

Since $-\pi_0$ in (1.6) is the zero coefficient in the asymptotic expansion of

$$\log \text{Det}(R(\lambda)),$$

Corollary 3.4 leads to the following result, which is the main result of this paper.

Theorem 3.6. *Let (M, g) , M_{cut} be as above and g , Δ_M be given by (1.1), (1.2) on $N_{-1,1}$. Suppose that $\dim M = m + 1$, $\mathfrak{c}_0 = \frac{m^2}{16} h'(0)^2 + \frac{m}{4} h''(0)$ and $l = \dim \ker(\Delta_Y + \mathfrak{c}_0)$. Then the constant π_0 in (1.6) is the following.*

- (1) *If m is odd, then $\pi_0 = 0$.*
- (2) *If $m = 2$, then $\pi_0 = -\log 2 \cdot (\zeta_{(\Delta_Y + \mathfrak{c}_0)}(0) + l)$.*
- (3) *If $m = 4$, $\pi_0 = -\log 2 \cdot (\zeta_{(\Delta_Y + \mathfrak{c}_0)}(0) + l) - (\log 2 - \frac{1}{2}) \frac{\text{vol}(Y)}{64\pi^2} (h''(0) - h'(0)^2)$, where $\text{vol}(Y)$ is the volume of Y .*

(4) Generally, if $m \geq 4$ and even, then we have

$$\begin{aligned} \pi_0 &= -\mathfrak{p}_m \\ &= -\log 2 \cdot (\zeta_{(\Delta_Y+c_0)}(0) + l) \\ &\quad - \sum_{j=1}^{\frac{m}{2}-1} \frac{(-1)^{\frac{m}{2}-j}}{(\frac{m}{2}-j)!} \mathfrak{a}_{j, \frac{m}{2}-j} \left(\log 2 - \frac{1}{2} \left(1 + \frac{1}{2} + \dots + \frac{1}{\frac{m}{2}-j} \right) \right). \end{aligned}$$

Proof. The assertion (4) follows from Corollary 3.4. It's enough to prove the assertion (3). If $m = 4$, the assertion (4) shows that

$$\pi_0 = -\log 2 \cdot (\zeta_{(\Delta_Y+c_0)}(0) + l) + \mathfrak{a}_{1,1} \left(\log 2 - \frac{1}{2} \right).$$

By (2.9), (2.15), and Lemma 2.3, $\mathfrak{a}_{1,1}$ can be computed as follows. We note that

$$\begin{aligned} r_{-2-1}(y, \xi, \lambda, \mu) &= \delta_{-2-1}(y, \xi, \mu), \\ r_{-2-2}(y, \xi, \lambda, \mu) &= \delta_{-2-2}(y, \xi, \mu) - (\mu - |\xi|^2)^{-2} \cdot q_0(y, \xi, \lambda) \\ &= \delta_{-2-2}(y, \xi, \mu) - (\mu - |\xi|^2)^{-2} \left(-\frac{1}{4} \frac{(h''(0) - h'(0)^2) |\xi|^2}{|\xi|^2 + \lambda} - \frac{5}{16} \frac{h'(0)^2 |\xi|^4}{(|\xi|^2 + \lambda)^2} \right). \end{aligned}$$

Then by (2.15) we have

$$\begin{aligned} \mathfrak{a}_1(\lambda) &= \mathfrak{a}_1 - \frac{1}{2} \frac{\text{vol}(Y) \cdot \text{vol}(S^3)}{16\pi^4} \left(\frac{1}{4} \frac{h''(0) - h'(0)^2}{(\lambda + 1)} + \frac{5}{16} \frac{h'(0)^2}{(\lambda + 1)^2} \right) \\ &\sim \mathfrak{a}_1 - \frac{\text{vol}(Y)}{16\pi^2} \left(\frac{1}{4} (h''(0) - h'(0)^2) \frac{1}{\lambda} + \frac{1}{16} (-4h''(0) + 9h'(0)^2) \frac{1}{\lambda^2} + \dots \right), \end{aligned}$$

which shows that $\mathfrak{a}_{1,1} = -\frac{\text{vol}(Y)}{64\pi^2} (h''(0) - h'(0)^2)$. This completes the proof of Theorem 3.6. \square

Remark. Replacing Δ_M and $\Delta_{M_{cut}, \gamma_0}$ by $\Delta_M + \nu$ and $\Delta_{M_{cut}, \gamma_0} + \nu$ for $\nu \in \mathbb{R}^+$, we can rewrite (1.6) as follows.

$$(3.6) \quad \log \text{Det} (\Delta_M + \nu + \lambda, \Delta_{M_{cut}, \gamma_0} + \nu + \lambda) = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \pi_j(\nu) \lambda^j + \log \text{Det} R(\nu + \lambda).$$

Since $\Delta_M + \nu$, $\Delta_{M_{cut}, \gamma_0} + \nu$, and $R(\nu)$ are invertible operators, we have

$$(3.7) \quad \log \text{Det} (\Delta_M + \nu, \Delta_{M_{cut}, \gamma_0} + \nu) = \pi_0(\nu) + \log \text{Det} R(\nu),$$

where $\pi_0(\nu)$ is the polynomial part in (1.6). Similarly, we can express $\pi_0(\nu)$ in terms of coefficients in Lemma 2.3. For instance, $\pi_0(\nu) = 0$ when $\dim M$ is even.

We next discuss what data determines π_0 . We consider the collar neighborhood $N_{-1,1}$ of Y and denote by $\Delta_{N_{-1,1}}, \Delta_{N_{-1,0}}, \Delta_{N_{0,1}}$ the restrictions of Δ_M to $N_{-1,1}, N_{-1,0}, N_{0,1}$. We impose the Dirichlet boundary condition on each

boundary and denote the realizations by $\Delta_{N_{-1,1},\gamma_0}$, $\Delta_{N_{-1,0},\gamma_0}$, $\Delta_{N_{0,1},\gamma_0}$. In this case (1.6) can be written by

$$(3.8) \quad \begin{aligned} & \log \text{Det} (\Delta_{N_{-1,1},\gamma_0} + \lambda) - \log \text{Det} (\Delta_{N_{-1,0},\gamma_0} + \lambda) - \log \text{Det} (\Delta_{N_{0,1},\gamma_0} + \lambda) \\ &= \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \tilde{\pi}_j \lambda^j + \log \text{Det} R_{N_{-1,1}}(\lambda), \end{aligned}$$

where $R_{N_{-1,1}}(\lambda) = Q_{N_{-1,0}}(\lambda) + Q_{N_{0,1}}(\lambda)$ is defined as follows. For $f \in C^\infty(Y)$, choose $\phi \in C^\infty(N_{-1,0})$, $\psi \in C^\infty(N_{0,1})$ such that

$$(\Delta_{N_{-1,0}} + \lambda) \phi = 0, \quad (\Delta_{N_{0,1}} + \lambda) \psi = 0, \quad \phi|_{Y_{-1}} = \psi|_{Y_1} = 0, \quad \phi|_{Y_0} = \psi|_{Y_0} = f.$$

We define

$$(3.9) \quad Q_{N_{-1,0}}(\lambda)f = (\partial_u \phi)|_{Y_0}, \quad Q_{N_{0,1}}(\lambda)f = -(\partial_u \psi)|_{Y_0}.$$

Then $Q_{N_{-1,0}}(\lambda)$ and $Q_{N_{0,1}}(\lambda)$ satisfy the equations (2.3) and (2.4), respectively, which shows that $Q_{N_{-1,0}}(\lambda)$ ($Q_{N_{0,1}}(\lambda)$) has the same asymptotic symbol as $Q_1(\lambda)$ ($Q_2(\lambda)$) and hence $R(\lambda) - R_{N_{-1,1}}(\lambda)$ is a smoothing operator. This fact shows that $\log \text{Det} R(\lambda)$ and $\log \text{Det} R_{N_{-1,1}}(\lambda)$ have the same asymptotic expansions as $\lambda \rightarrow \infty$. In particular, they have the same zero- and log λ -coefficients, which leads to the following result.

Corollary 3.7. *The coefficients \mathfrak{p}_m and \mathfrak{q}_m in Corollary 3.4 are determined by the data on a small collar neighborhood of Y .*

4. The value of relative zeta functions at $s = 0$

Theorem 3.6 gives some informations about the value of relative zeta functions at $s = 0$, which we discuss in this section. Corollary 3.4, Theorem 3.6, and Corollary 3.7 lead to the following result.

Theorem 4.1. *Let (M, g) , M_{cut} be as above and g , Δ_M be given by (1.1), (1.2) on $N_{-1,1}$. Suppose that $\dim M = m + 1$, $\mathfrak{c}_0 = \frac{m^2}{16}h'(0)^2 + \frac{m}{4}h''(0)$ and $l = \dim \ker(\Delta_Y + \mathfrak{c}_0)$. We assume that either M is compact or complete with $\sigma_{ess}(\Delta_M)$ having a positive lower bound. Then*

- (1) *If $\dim M$ is even, $\zeta(0; \Delta_M, \Delta_{M_{cut}, \gamma_0}) + \dim \ker \Delta_M = 0$.*
- (2) *If $\dim M$ is odd,*

$$\zeta(0; \Delta_M, \Delta_{M_{cut}, \gamma_0}) + \dim \ker \Delta_M = \frac{1}{2} \sum_{j=1}^{\frac{m}{2}} \frac{(-1)^{\frac{m}{2}-j}}{(\frac{m}{2}-j)!} \mathfrak{a}_{j, \frac{m}{2}-j},$$

which is determined by the data on a collar neighborhood of Y .

- (3) *If $m = 2$, $\zeta(0; \Delta_M, \Delta_{M_{cut}, \gamma_0}) + \dim \ker \Delta_M = \frac{1}{2}(\zeta_{\Delta_Y + \mathfrak{c}_0}(0) + l)$.*

(4) If $m = 4$,

$$\begin{aligned} & \zeta(0; \Delta_M, \Delta_{M_{cut, \gamma_0}}) + \dim \ker \Delta_M \\ &= \frac{1}{2} \left(\zeta_{\Delta_Y + \epsilon_0}(0) + l + \frac{\text{vol}(Y)}{64\pi^2} (h''(0) - h'(0)^2) \right). \end{aligned}$$

Remark. Since the coefficient θ_0 in (3.2) vanishes when $\dim Y$ is odd, the assertion (1) holds generally.

As an application of Theorem 4.1 we consider the case of $h(u) \equiv 0$, i.e., the product metric on $N_{-1,1}$ and $\Delta_M|_{N_{-1,1}} = -\partial_u^2 + \Delta_Y$. Then it is known that $\pi_0 = -\log 2 \cdot (\zeta_{\Delta_Y}(0) + \dim \ker \Delta_Y)$ and it is not difficult to check that $\mathfrak{a}_{j, \frac{m}{2}-j} = 0$ for $1 \leq j \leq \frac{m}{2} - 1$ (cf. Lemma 2.3), which gives the following result.

Corollary 4.2. *Suppose that M is a compact closed manifold and the metric g is the product one on $N_{-1,1}$ so that $\Delta_M = -\partial_u^2 + \Delta_Y$ on $N_{-1,1}$. Then*

- (1) $\zeta(0; \Delta_M, \Delta_{M_{cut, \gamma_0}}) + \dim \ker \Delta_M = \frac{1}{2} (\zeta_{\Delta_Y}(0) + \dim \ker \Delta_Y)$.
- (2) If $\dim M$ is odd, $\zeta_{\Delta_{M_1, \gamma_0}}(0) = -\frac{1}{4} (\zeta_{\Delta_Y}(0) + \dim \ker \Delta_Y)$.
- (3) If $\dim M$ is even, $\zeta_{\Delta_{M_1, \gamma_0}}(0) = \frac{1}{2} (\zeta_{\Delta_{\widetilde{M}_1}}(0) + \dim \ker \Delta_{\widetilde{M}_1})$, where \widetilde{M}_1 is the double of M_1 and $\Delta_{\widetilde{M}_1}$ is the natural extension of Δ_{M_1} to \widetilde{M}_1 .

Let M_1 be a compact manifold with boundary Y and $M_{1, \infty} = M_1 \cup_Y [0, \infty) \times Y$. We give the product metric on the cylinder part of $M_{1, \infty}$ so that on the cylinder part $\Delta_{M_{1, \infty}} = -\partial_u^2 + \Delta_Y$. Suppose that $\mu_1 > 0$ is the smallest positive eigenvalue of Δ_Y and we denote the scattering matrix by

$$S(s) : \ker \Delta_Y \rightarrow \ker \Delta_Y, \quad |s| < \sqrt{\mu_1}.$$

It was shown in [14] that for $t \rightarrow \infty$

$$(4.1) \quad \text{Tr} \left(e^{-t\Delta_{M_{1, \infty}}} - e^{-t(-\partial_u^2 + \Delta_Y)|_{[0, \infty) \times Y, \gamma_0}} \right) = b_0 + O(t^{-\rho}),$$

where $b_0 = \dim \ker \Delta_{M_{1, \infty}} + \frac{1}{4} (\text{Tr } S(0) + \dim \ker \Delta_Y)$ and $\rho > 0$. Then we have the following result.

Corollary 4.3.

$$\begin{aligned} & \zeta(0; \Delta_{M_{1, \infty}}, (-\partial_u^2 + \Delta_Y)|_{[0, \infty) \times Y, \gamma_0}) \\ &= \zeta_{\Delta_{M_1, \gamma_0}}(0) - \dim \ker \Delta_{M_{1, \infty}} + \frac{1}{2} (\zeta_{\Delta_Y}(0) + \dim \ker \Delta_Y) - b_0 \\ &= \zeta_{\Delta_{M_1, \gamma_0}}(0) - 2 \dim \ker \Delta_{M_{1, \infty}} + \frac{1}{2} \zeta_{\Delta_Y}(0) + \frac{1}{2} \dim \ker (Id + S(0)). \end{aligned}$$

We next discuss the case of the Neumann boundary condition. We still assume the product metric and product structure near the boundary. Let \mathfrak{N} be the Neumann boundary condition imposed on the boundary of M_1 and $\Delta_{M_1, \mathfrak{N}}$

be the realization. Then it can be shown by the method presented in [10] (cf. [16]) that

$$(4.2) \quad \log \text{Det} (\Delta_{M_1, \mathfrak{N}} + \lambda) - \log \text{Det} (\Delta_{M_1, \gamma_0} + \lambda) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} a_k \lambda^k + \log \text{Det} Q_1(\lambda),$$

where $Q_1(\lambda)$ differs from $\sqrt{\Delta_Y + \lambda}$ by a smoothing operator and hence $\log \text{Det}$ of these two operators have the same asymptotic expansions. Since the coefficient of $\log \lambda$ in the asymptotic expansion of $\log \text{Det} (\sqrt{\Delta_Y + \lambda})$ is

$$\frac{1}{2} (\zeta_{\Delta_Y}(0) + \dim \ker \Delta_Y),$$

this fact together with Corollary 4.2 leads to the following result.

Theorem 4.4. *Let (M_1, g) be a compact manifold with boundary. We assume that the metric g is the product one and $\Delta_M = -\partial_u^2 + \Delta_Y$ near the boundary. Then*

- (1) $\zeta_{\Delta_{M_1, \mathfrak{N}}}(0) + \dim \ker \Delta_{M_1, \mathfrak{N}} - \zeta_{\Delta_{M_1, \gamma_0}}(0) = \frac{1}{2} (\zeta_{\Delta_Y}(0) + \dim \ker \Delta_Y)$.
- (2) *If $\dim M_1$ is odd,*

$$\zeta_{\Delta_{M_1, \mathfrak{N}}}(0) + \dim \ker \Delta_{M_1, \mathfrak{N}} = \frac{1}{4} (\zeta_{\Delta_Y}(0) + \dim \ker \Delta_Y).$$

- (3) *If $\dim M_1$ is even, $\zeta_{\Delta_{M_1, \mathfrak{N}}}(0) + \dim \ker \Delta_{M_1, \mathfrak{N}} = \zeta_{\Delta_{M_1, \gamma_0}}(0)$.*

Finally, we discuss the value of the zeta function associated with a compatible Dirac Laplacian with the Atiyah-Patodi-Singer (APS) boundary condition on a compact manifold with boundary. Let (M, g) be a compact oriented $(m + 1)$ -dimensional Riemannian manifold and $E \rightarrow M$ be a Clifford module bundle. Suppose that Y is a hypersurface of M such that $M - Y$ has two components whose closures are denoted by M_1, M_2 . We denote by M_{cut} the compact manifold with boundary obtained by cutting M along Y as before, i.e., $M_{cut} = M_1 \cup_Y M_2$. We choose a collar neighborhood $N_{-1,1}$ of Y which is diffeomorphic to $[-1, 1] \times Y$ and assume that the metric g is the product one on $N_{-1,1}$ and the bundle E has the product structure on $N_{-1,1}$. Suppose that D_M is a compatible Dirac operator acting on smooth sections of E , having the following form on $N_{-1,1}$

$$D_M = G(\partial_u + B),$$

where $G : E|_Y \rightarrow E|_Y$ is a bundle automorphism, ∂_u is the outward normal derivative to M_1 on $N_{-1,1}$ and B is a Dirac operator on Y . We further assume that G and B are independent of the normal coordinate u and satisfy

$$(4.3) \quad \begin{aligned} G^* &= -G, & G^2 &= -I, & B^* &= B, & GB &= -BG, \\ \dim (\ker(G - i) \cap \ker B) &= \dim (\ker(G + i) \cap \ker B). \end{aligned}$$

Then we have, on $N_{-1,1}$, the Dirac Laplacian

$$D_M^2 = -\partial_u^2 + B^2.$$

We also denote by $D_{M_{cut}}, D_{M_i}$ ($i = 1, 2$) the extension and restriction of D_M to M_{cut} and M_i . We denote by $\Pi_{<} (\Pi_{>})$ the orthogonal projection onto the space spanned by negative (positive) eigensections of B and by $\sigma : \ker B \rightarrow \ker B$ a unitary operator satisfying

$$\sigma G = -G\sigma, \quad \sigma^2 = Id_{\ker B}.$$

We define the generalized APS boundary condition $\Pi_{<,\sigma^-}, \Pi_{>,\sigma^+}$ by

$$\Pi_{<,\sigma^-} = \Pi_{<} + \frac{1}{2}(I - \sigma)|_{\ker B}, \quad \Pi_{>,\sigma^+} = \Pi_{>} + \frac{1}{2}(I + \sigma)|_{\ker B}$$

and denote by $D_{M_1,\Pi_{<,\sigma^-}}, D_{M_1,\Pi_{<,\sigma^-}}^2$ ($D_{M_2,\Pi_{>,\sigma^+}}, D_{M_2,\Pi_{>,\sigma^+}}^2$) the realizations of $D_{M_i}, D_{M_i}^2$ with respect to the boundary condition $\Pi_{<,\sigma^-} (\Pi_{>,\sigma^+})$. Then it was shown in [10] that

$$\begin{aligned} & \log \text{Det} \left(D_{M_1,\Pi_{<,\sigma^-}}^2 + \lambda \right) - \log \text{Det} \left(D_{M_1,\gamma_0}^2 + \lambda \right) \\ (4.4) \quad &= \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} a_{1,j} \lambda^j + \log \text{Det} \left(\Pi_{>,\sigma^+} (Q_1(\lambda) + |B|) \Pi_{>,\sigma^+} \right), \\ & \log \text{Det} \left(D_{M_2,\Pi_{>,\sigma^+}}^2 + \lambda \right) - \log \text{Det} \left(D_{M_2,\gamma_0}^2 + \lambda \right) \\ &= \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} a_{2,j} \lambda^j + \log \text{Det} \left(\Pi_{<,\sigma^-} (Q_2(\lambda) + |B|) \Pi_{<,\sigma^-} \right), \end{aligned}$$

where $Q_1(\lambda)$ and $Q_2(\lambda)$ are defined by the same way as in (1.4). We now consider the first equation of (4.4) and we can treat the second equation in the same way. As before, $\log \text{Det} \left(D_{M_1,\Pi_{<,\sigma^-}}^2 + \lambda \right)$, $\log \text{Det} \left(D_{M_1,\gamma_0}^2 + \lambda \right)$, and $\log \text{Det} \left(\Pi_{>,\sigma^+} (Q_1(\lambda) + |B|) \Pi_{>,\sigma^+} \right)$ have asymptotic expansions for $\lambda \rightarrow \infty$. Moreover, the coefficients of $\log \lambda$ in the asymptotic expansions of

$$\log \text{Det} \left(D_{M_1,\Pi_{<,\sigma^-}}^2 + \lambda \right)$$

and $\log \text{Det} \left(D_{M_1,\gamma_0}^2 + \lambda \right)$ are $\left(\zeta_{D_{M_1,\Pi_{<,\sigma^-}}^2}(0) + \dim \ker D_{M_1,\Pi_{<,\sigma^-}}^2 \right)$ and $\zeta_{D_{M_1,\gamma_0}^2}(0)$.

Now let us consider the asymptotic expansion of

$$\log \text{Det} \left(\Pi_{>,\sigma^+} (Q_1(\lambda) + |B|) \Pi_{>,\sigma^+} \right).$$

We note that

$$\begin{aligned} & \log \text{Det} \left(\Pi_{>,\sigma^+} (Q_1(\lambda) + |B|) \Pi_{>,\sigma^+} \right) \\ &= -\log \text{Det} \left(\Pi_{<,\sigma^-} \left(\sqrt{B^2 + \lambda} + |B| \right) \Pi_{<,\sigma^-} \right) \\ & \quad + \log \text{Det} \left(\left(\sqrt{B^2 + \lambda} + |B| \right) + \Pi_{>,\sigma^+} \left(Q_1(\lambda) - \sqrt{B^2 + \lambda} \right) \Pi_{>,\sigma^+} \right). \end{aligned}$$

It was shown in [9] that

$$Q_i(\lambda) = \sqrt{B^2 + \lambda} + \text{a smoothing operator}, \quad i = 1, 2,$$

which shows that $\log \lambda$ -coefficient in the asymptotic expansion of

$$\log \text{Det} \left(\Pi_{>,\sigma+} (Q_1(\lambda) + |B|) \Pi_{>,\sigma+} \right)$$

is the same as that in the asymptotic expansion of $\frac{1}{2} \log \text{Det} (\sqrt{B^2 + \lambda} + |B|)$. We compute the $\log \lambda$ -coefficient in the asymptotic expansion of

$$\log \text{Det} \left(\sqrt{B^2 + \lambda} + |B| \right)$$

as follows.

Lemma 4.5. *For $1 \leq k \in \mathbb{Z}$ let us denote $f_k(s, \lambda)$ by*

$$f_k(s, \lambda) = \frac{1}{\Gamma(s)} \int_0^\infty t^{\frac{s+k}{2}-1} \text{Tr} \left(|B|^k e^{-t(B^2+\lambda)} \right) dt.$$

Then the coefficients of $\log \lambda$ in the asymptotic expansions of $f_k(0, \lambda)$ and $-f'_k(0, \lambda)$, as $\lambda \rightarrow \infty$, are zero.

Proof. We first note that $\text{Tr} \left(|B|^k e^{-tB^2} \right)$ has the following asymptotic expansion for $t \rightarrow 0^+$ (Theorem 2.7 in [4]).

$$(4.5) \quad \text{Tr} \left(|B|^k e^{-tB^2} \right) \sim \sum_{j=0}^\infty b_j^{(k)} t^{\frac{j-m-k}{2}} + \sum_{j=0}^\infty \left(c_j^{(k)} \log t + d_j^{(k)} \right) t^j.$$

Then direct computation shows that the coefficients of $\log \lambda$ in the asymptotic expansions of $f_k(0, \lambda)$ and $-\zeta'_k(0, \lambda)$ for $\lambda \rightarrow \infty$ are zero and $b_m^{(k)}$, respectively. On the other hand, we note that

$$(4.6) \quad \zeta_{|B|}(s) = \frac{1}{\Gamma(\frac{s+k}{2})} \int_0^\infty t^{\frac{s+k}{2}-1} \text{Tr} \left(|B|^k e^{-tB^2} \right) dt.$$

The equation (4.5) shows that the RHS of (4.6) has a simple pole at $s = 0$ with residue $\frac{2b_m^{(k)}}{\Gamma(\frac{k}{2})}$. Since $\zeta_{|B|}(s)$ has a regular value at $s = 0$, this fact implies that each $b_m^{(k)} = 0$ for $k \geq 1$, which completes the proof of the lemma. \square

Lemma 4.6. *The coefficient of $\log \lambda$ in the asymptotic expansion of*

$$\log \text{Det} \left(\sqrt{B^2 + \lambda} + |B| \right)$$

for $\lambda \rightarrow \infty$ is $\frac{1}{2} (\zeta_{B^2}(0) + \dim \ker B)$.

Proof. We first note that

$$\begin{aligned} & \zeta_{(\sqrt{B^2+\lambda+|B|})}(s) \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr} e^{-t(\sqrt{B^2+\lambda+|B|})} dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{q=0}^\infty \frac{t^q}{q!} \text{Tr} \left(\left(\sqrt{B^2+\lambda} - |B| \right)^q e^{-2t\sqrt{B^2+\lambda}} \right) dt \\ &= \sum_{q=0}^\infty \frac{1}{q!} \frac{1}{\Gamma(s)} \int_0^\infty t^{s+q-1} \text{Tr} \left(\left(\sqrt{B^2+\lambda} - |B| \right)^q e^{-2t\sqrt{B^2+\lambda}} \right) dt. \end{aligned}$$

We now set

$$\zeta_q(s, \lambda) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s+q-1} \text{Tr} \left(\left(\sqrt{B^2+\lambda} - |B| \right)^q e^{-2t\sqrt{B^2+\lambda}} \right) dt.$$

In case of $q = 0$, the coefficient of $\log \lambda$ in the asymptotic expansion of $-\zeta'_0(0, \lambda)$ for $\lambda \rightarrow \infty$ is $\frac{1}{2}(\zeta_{B^2}(0) + \dim \ker B)$ (cf. Lemma 3.1). For $q \geq 1$,

$$\begin{aligned} \zeta_q(s, \lambda) &= \sum_{k=0}^q (-1)^k \binom{q}{k} \frac{1}{\Gamma(s)} \int_0^\infty t^{s+q-1} \text{Tr} \left(\left(\sqrt{B^2+\lambda} \right)^{q-k} |B|^k e^{-2t\sqrt{B^2+\lambda}} \right) dt \\ &= 2^{-s-q} \sum_{k=0}^q (-1)^k \binom{q}{k} \frac{\Gamma(s+q)}{\Gamma(\frac{s+k}{2})} \frac{1}{\Gamma(s)} \int_0^\infty t^{\frac{s+k}{2}-1} \text{Tr} \left(|B|^k e^{-t(B^2+\lambda)} \right) dt. \end{aligned}$$

Then Lemma 4.5 shows that each $\log \lambda$ -coefficient of $-\zeta'_q(0, \lambda)$ is zero, which completes the proof of the lemma. \square

We can say the similar assertions for $D_{M_2, \Pi_{>, \sigma^+}}^2$. The comparison of $\log \lambda$ -coefficients in the asymptotic expansions of (4.4) and Corollary 4.2 leads to the following result.

Theorem 4.7. *Let M, M_1 and M_2 be as above. We denote $\dim \ker D_M^2, \dim \ker D_{M_1, \Pi_{<, \sigma^-}}^2, \dim \ker D_{M_2, \Pi_{>, \sigma^+}}^2$ by $l_M, l_{M_1, \Pi_{<, \sigma^-}}, l_{M_2, \Pi_{>, \sigma^+}}$, respectively.*

Then

- (1) $\zeta_{D_{M_1, \Pi_{<, \sigma^-}}^2}(0) + l_{M_1, \Pi_{<, \sigma^-}} - \zeta_{D_{M_1, \gamma_0}^2}(0) = \frac{1}{4}(\zeta_{B^2}(0) + \dim \ker B)$.
- (2) *If $\dim M_1$ is odd, $\zeta_{D_{M_1, \Pi_{<, \sigma^-}}^2}(0) + l_{M_1, \Pi_{<, \sigma^-}} = 0$.*
- (3) *If $\dim M_1$ is even, $\zeta_{D_{M_1, \Pi_{<, \sigma^-}}^2}(0) + l_{M_1, \Pi_{<, \sigma^-}} = \zeta_{D_{M_1, \gamma_0}^2}(0)$.*
- (4) $(\zeta_{D_M^2}(0) + l_M) - (\zeta_{D_{M_1, \Pi_{<, \sigma^-}}^2}(0) + l_{M_1, \Pi_{<, \sigma^-}}) - (\zeta_{D_{M_2, \Pi_{>, \sigma^+}}^2}(0) + l_{M_2, \Pi_{>, \sigma^+}}) = 0$.

Remark. (1) The second assertion was proved earlier in the appendix of [15] by the author.

(2) The zeta-determinant of a compatible Dirac operator is defined by (4.7)

$$\text{Det} D_M = e^{-\frac{1}{2}\zeta'_{D_M^2}(0)} e^{\frac{\pi i}{2}(\zeta_{D_M^2}(0) - \eta_{D_M}(0))} = \sqrt{\text{Det} D_M^2} e^{\frac{\pi i}{2}(\zeta_{D_M^2}(0) - \eta_{D_M}(0))},$$

where $\eta_{D_M}(0)$ is the eta-invariant for D_M (cf. [17]). The gluing formula for the eta-invariant of a Dirac operator with respect to the APS boundary condition is given in [6] (or [11]) and the gluing formula for the zeta-determinant of a Dirac Laplacian with respect to the APS boundary condition is given in [10] (or [11]). Hence, the assertion (4) together with these results completes the gluing formula for the zeta-determinant of a compatible Dirac operator.

In view of Corollary 4.3 we conclude this section with the computation of the value of relative zeta function for Dirac Laplacian with the APS boundary condition on a manifold with cylindrical end. As before, we denote by $M_{1,\infty} := M_1 \cup_Y [0, \infty) \times Y$ and by $D_{M_{1,\infty}}$ the natural extension of D_{M_1} to $M_{1,\infty}$. Then,

$$\begin{aligned}
 & \zeta(s, D_{M_{1,\infty}}^2, (-\partial_u^2 + B^2)_{[0,\infty) \times Y, \Pi_{>,\sigma^+}}) \\
 &= \zeta(s, D_{M_{1,\infty}}^2, (-\partial_u^2 + B^2)_{[0,\infty) \times Y, \gamma_0}) \\
 (4.8) \quad &+ \zeta(s, (-\partial_u^2 + B^2)_{[0,\infty) \times Y, \gamma_0}, (-\partial_u^2 + B^2)_{[0,\infty) \times Y, \Pi_{>,\sigma^+}}) \\
 &= \zeta(s, D_{M_{1,\infty}}^2, (-\partial_u^2 + B^2)_{[0,\infty) \times Y, \gamma_0}) - \frac{1}{4\sqrt{\pi}} \frac{\Gamma(s + \frac{1}{2})}{\Gamma(s + 1)} \zeta_{B^2}(s).
 \end{aligned}$$

Here we used the fact (cf. [10]) that

$$\zeta(s, (-\partial_u^2 + B^2)_{[0,\infty) \times Y, \gamma_0}, (-\partial_u^2 + B^2)_{[0,\infty) \times Y, \Pi_{>,\sigma^+}}) = -\frac{1}{4\sqrt{\pi}} \frac{\Gamma(s + \frac{1}{2})}{\Gamma(s + 1)} \zeta_{B^2}(s).$$

Since $\dim \ker (Id + S(0)) = \frac{1}{2} \dim \ker B$ on a manifold with cylindrical end ([12]), Corollary 4.3 together with (4.8) yields the following result.

Corollary 4.8.

$$\begin{aligned}
 & \zeta(0, D_{M_{1,\infty}}^2, (-\partial_u^2 + B^2)_{[0,\infty) \times Y, \Pi_{>,\sigma^+}}) \\
 &= \zeta_{D_{M_1, \gamma_0}^2}(0) - 2 \dim \ker D_{M_{1,\infty}}^2 + \frac{1}{4} (\zeta_{B^2}(0) + \dim \ker B).
 \end{aligned}$$

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