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UNIVERSAL QUADRATIC FORMS OVER POLYNOMIAL RINGS

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Abstract. The Fifteen Theorem proved by Conway and Schneeberger is a criterion for positive definite quadratic forms over the rational integer ring to be universal. In this paper, we give a proof of an analogy of the Fifteen Theorem for definite quadratic forms over polynomial rings, which is known as the *Four Conjecture* proposed by Gerstein.

1. Introduction

Conway and Schneeberger (see [12] and [3]) announced the so called the '*Fifteen Theorem*' which claims that an integral positive definite quadratic form represents every positive integer, i.e., the form is *universal*, if it represents 1, 2, 3, 5, 6, 7, 10, 14 and 15. For example, the well-known Lagrange's Four Square Theorem is an immediate consequence of the Fifteen Theorem. Bhargava [1] recently gave a simple proof of the theorem (see also [6], [7] and [5] for fascinating recent developments on universal forms). In [4], Gerstein studied the analogy of the Fifteen Theorem over $\mathbb{F}_q[x]$ and proposed the following conjecture.

Four Conjecture. *An integral definite quadratic form over $\mathbb{F}_q[x]$ represents every polynomial in $\mathbb{F}_q[x]$ if it represents 1, δ, x and δx, where $ch(\mathbb{F}_q) \neq 2$ and δ is a non-square element in $\mathbb{F}_q$.*

In this paper, we prove:

Theorem 1.1. *The Four Conjecture is true.*

Notations and terminology are standard and adopted from [11] if not explained. Particularly, $\mathbb{F}_q$ is a finite field with q elements, where q is an odd prime power, $\mathbb{F}_q[x]$ is the polynomial ring of one variable x, and $\mathbb{F}_q(x)$ is the quotient field of $\mathbb{F}_q[x]$.

We call a quadratic space V over $\mathbb{F}_q(x)$ *definite* (resp., *indefinite*) if the local completion V_∞ at $\infty = (1/x)$ is anisotropic (resp., isotropic). Let L be a free

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$\mathbb{F}_q[x]$-module in V. We call L an *integral lattice* if

$$B(L, L) \subseteq \mathbb{F}_q[x],$$

where B is a given symmetric bilinear form on V. For any $v \in V$, we define $Q(v) := B(v, v)$.

It should be pointed out that the Four Conjecture is not true for indefinite quadratic forms over $\mathbb{F}_q[x]$, just like the Fifteen Theorem is not true for indefinite quadratic forms over $\mathbb{Z}$, as the following example indicates.

Example. Let $L \cong \langle 1, -\delta, (x+a)^2, (x+a)^3, (x+a)^4 \rangle$ be a diagonal integral lattice over $\mathbb{F}_q[x]$, where $a \in \mathbb{F}_q^\times$ and $\delta \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^2$. Then since rank(L) = 5, the corresponding quadratic space spanned by L is indefinite. By strong approximation theorem for spin groups (see [10] and [8]), an element in $\mathbb{F}_q[x]$ is represented by L if and only if it is represented by L locally at every prime of $\mathbb{F}_q[x]$. (In fact, the class number of L is one.) It is clear that 1, δ, x and δx are represented by L locally at every prime. By [10], however, $(x + a)$ can not be represented by L locally at $(x + a)$.

From now on, we assume that all quadratic spaces are definite. If L is an integral lattice in a definite quadratic space V, then the Four Conjecture says

$$\{1, \delta, x, \delta x\} \subseteq Q(L) \iff Q(L) = \mathbb{F}_q[x]$$

for any given $\delta \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^2$. We choose such a δ and fix it afterwards.

2. Representations of ternary lattices

In this section, we extend the last proposition in [4] to all ternary integral lattices whose determinants are linear polynomials in $\mathbb{F}_q[x]$. This result plays a central role in our proof of the Four Conjecture.

Let K be an integral ternary lattice with $\det(K) = ax + b$ with $a, b \in \mathbb{F}_q$ and $a \neq 0$. For any $f(x) \in \mathbb{F}_q[x]$, one can write

$$f(x) = a_0 + a_1(ax + b) + \cdots + a_n(ax + b)^n,$$

where $a_0, a_1, \ldots, a_n \in \mathbb{F}_q$ with $a_n \neq 0$. Define

$$v_{ax+b}(f) := \min\{ i \, : \, a_i \neq 0\}.$$

Lemma 2.1. *There is only one class of ternary integral lattices with given determinant $ax + b$, $a \neq 0$, and $f(x)$ is not represented by this class if and only if*

(1) *n is odd and $-a_n \in (\mathbb{F}_q^\times)^2$; or*

(2) *$m = v_{ax+b}(f)$ is odd and $-a_m \in (\mathbb{F}_q^\times)^2$.*

Proof. Let K be an integral ternary lattice with $\det(K) = ax + b$. Since the degrees of diagonal polynomials of the Gram matrix of a reduced basis $\{v_1, v_2, v_3\}$ (see [4] or [2]) of K are dominant, one has $Q(v_1), Q(v_2) \in \mathbb{F}_q^\times$ and hence

$$K = \mathbb{F}_q[x]v_1 \perp \mathbb{F}_q[x]v_2 \perp \mathbb{F}_q[x]v_3.$$

From the definiteness condition it follows that $\mathbb{F}_q[x]v_1 \perp \mathbb{F}_q[x]v_2 \cong \langle 1, -\delta \rangle$ and therefore,

$$K \cong \langle 1, -\delta, -\delta(ax+b) \rangle.$$

This proves the first assertion.

Since the class number of K is one, it is a purely local problem to determine whether $f(x)$ is represented by K. It is clear that K is unimodular and hence universal at all primes except $(ax+b)$ and ∞. Therefore $f(x)$ is not represented by K if and only if $f(x)$ is not represented by K locally at $(ax+b)$ or ∞.

If $f(x)$ is not represented by K locally at $(ax+b)$, then $a_0 = 0$, for otherwise $f(x)$ is a unit at $(ax+b)$ and is represented by the sublattice $\langle 1, -\delta \rangle$ of K according to Hensel's lemma. Therefore, $m = v_{ax+b}(f(x)) \geq 1$ and $f(x)$ is not represented by K locally at $(ax+b)$ if and only if $f(x)$ is not represented by the quadratic space spanned by K at $(ax+b)$ by [10]. This is equivalent to the fact that the quadratic space

$$[1, -\delta, -\delta(ax+b), -f(x)] \cong [1, -\delta, -\delta(ax+b), -a_m(ax+b)^m]$$

is anisotropic at $(ax+b)$. It is a standard fact that the above quadratic space is anisotropic at $(ax+b)$ if and only if m is odd and $-a_m \in (\mathbb{F}_q^\times)^2$.

At ∞, $f(x)$ is not represented by K if and only if the space

$$[1, -\delta, -\delta(ax+b), -f(x)] \cong [1, -\delta, -\delta ax, -a_n a^n x^n]$$

is anisotropic. This is equivalent to the fact that n is odd and $-a_n \in (\mathbb{F}_q^\times)^2$. □

3. Proof of the theorem

In [4], it was proved that any quaternary integral lattice which represents $1, \delta, x$ and δx is isometric to

$$\langle 1, -\delta \rangle \perp \begin{pmatrix} \alpha x + \beta & \gamma \\ \gamma & -\delta\alpha x + \eta \end{pmatrix},$$

where $\alpha, \beta, \gamma, \eta \in \mathbb{F}_q$ with $\alpha \neq 0$. Therefore the theorem in §1 follows if one proves that an integral lattice

$$L \cong \langle 1, -\delta \rangle \perp \begin{pmatrix} x & \varepsilon \\ \varepsilon & -\delta x + \xi \end{pmatrix}$$

is universal for every $\xi \in \mathbb{F}_q$ and $\varepsilon \in \{0, 1\}$ (see [4]). Let $\{v_1, v_2, v_3, v_4\}$ be the corresponding basis of L.

Since L contains a ternary sublattice $\langle 1, -\delta, x \rangle$, it suffices to consider only those $f(x)$'s which cannot be represented by this ternary sublattice. Write

$$f(x) = a_0 + a_1(-\delta x) + \cdots + a_n(-\delta x)^n,$$

where $a_0, a_1, \ldots, a_n \in \mathbb{F}_q$ with $a_n \neq 0$. Since one can easily verify that L represents all linear polynomials in $\mathbb{F}_q[x]$ (see also [4]), we may assume that $n > 1$.

3.1. Diagonal case

In this section, we prove that L is universal when $\varepsilon = 0$, i.e., L is diagonal.

Proof. It is already proved that L is universal when $\xi = 0$ in [4]. So we assume that $\xi \neq 0$.

It is clear that there are $q+1$ solutions over $\mathbb{F}_q$ of the equation

$$y^2 - \delta z^2 = -a_n \delta^n. \tag{3.1}$$

So, there is a solution (y_0, z_0) satisfying $y_0 z_0 \neq 0$.

If n is odd and $-a_n \in (\mathbb{F}_q^\times)^2$, then we put

$$g(x) = f(x) - (-\delta x + \xi) \cdot \begin{cases} (z_0 x^{(n-1)/2})^2 & \text{if } a_0 \neq 0, \\ (z_0 x^{(n-1)/2} + 1)^2 & \text{otherwise.} \end{cases}$$

By the lemma in §2, $g(x)$ is represented by $\langle 1, -\delta, x \rangle$. Therefore $f(x)$ is represented by L. Otherwise, one can assume that $a_0 = 0$. Then

$$g(x) = f(x) - (-\delta x + \xi)$$

is represented by $\langle 1, -\delta, x \rangle$ and therefore, $f(x)$ is represented by L. □

3.2. Non-diagonal case, lower degree

It only remains to prove that L is universal for every $\xi \in \mathbb{F}_q$ when $\varepsilon = 1$. In this section, we prove that such L represents all polynomials $f(x) \in \mathbb{F}_q[x]$ of degree $n \leq 3$.

Proof. Let $n = 2$. One can assume that $f(x) = \alpha x(x - \xi\delta^{-1})$ for some $\alpha \in \mathbb{F}_q^\times$ by applying the lemma to the following ternary sublattices

$$\langle 1, -\delta, x \rangle \quad \text{and} \quad \langle 1, -\delta, -\delta x + \xi \rangle.$$

Furthermore, one only needs to consider the case when $-\xi\alpha \in (\mathbb{F}_q^\times)^2$ since $f(x)$ is represented by the former otherwise. But if $-\xi\alpha \in (\mathbb{F}_q^\times)^2$, then $f(x)$ is represented by the latter.

Let $n = 3$ and let

$$f(x) = a_0 + a_1(-\delta x) + a_2(-\delta x)^2 + a_3(-\delta x)^3 \in \mathbb{F}_q[x]$$

with $a_3 \neq 0$.

Case (i) $-a_3 \notin (\mathbb{F}_q^\times)^2$: One can assume that $a_0 = 0$ and $-a_1 \in (\mathbb{F}_q^\times)^2$. It is clear that there are $q+1$ solutions over $\mathbb{F}_q$ of the equation

$$s^2 - \delta t^2 = 1. \tag{3.2}$$

Let

$$S = \{\, \xi t^2 + 2st \,:\, (s,t) \text{ is a solution of (3.2)}\}.$$

It is clear that there are exactly two solutions (s,t) of (3.2) having the same ratio s/t if $t \neq 0$. For any solution (s,t) of (3.2) with $t \neq 0$ such that $\xi t^2 + 2st = \Delta$

for some $\Delta \in \mathbb{F}_q$, s/t satisfies a quadratic equation over $\mathbb{F}_q$ unless $\Delta = 0$. This implies that

$$\sharp S \geq \frac{\frac{q-1}{2} - 1}{2} + 1 = \frac{q+1}{4}.$$

If $q \geq 13$, then $\sharp S \geq 4$ and there is a solution (s_0, t_0) of (3.2) such that

$$Q(s_0 v_3 + t_0 v_4) = (x + \Delta_0) \nmid f(x)$$

over $\mathbb{F}_q[x]$ for some $\Delta_0 \in \mathbb{F}_q$. Then $f(x)$ is represented by the ternary sublattice

$$\mathbb{F}_q[x]v_1 \perp \mathbb{F}_q[x]v_2 \perp \mathbb{F}_q[x](s_0 v_3 + t_0 v_4) \cong \langle 1, -\delta, x + \Delta_0 \rangle.$$

If $q = 11$, then one can write $-\delta = \eta^2$ for some $\eta \in \mathbb{F}_q$. Thus S still contains more than three elements if $\xi \neq 0, \pm\eta$. So, it suffices to consider the exceptional cases. We have

$$S = \begin{cases} \{0, 3\eta^{-1}, -3\eta^{-1}\} & \text{if } \xi = 0, \\ \{0, \xi^{-1}, 6\xi^{-1}\} & \text{if } \xi = \pm\eta. \end{cases}$$

When $\xi = 0$, one may assume that

$$f(x) = -a_3\delta^3 x(x - 3\eta^{-1})(x + 3\eta^{-1}) = -a_3\delta^3 x(x^2 - 2\delta^{-1})$$

by applying the lemma to the following ternary sublattices

$$\mathbb{F}_q[x]v_1 \perp \mathbb{F}_q[x]v_2 \perp \mathbb{F}_q[x](3v_3 \pm 5\eta^{-1}v_4) \cong \langle 1, -\delta, x \pm 3\eta^{-1} \rangle.$$

It is easy to verify that $x(x^2 - 2\delta^{-1})$ is represented by

$$\begin{pmatrix} x & 1 \\ 1 & -\delta x \end{pmatrix}$$

for $\delta = 2, 6, 7, 8, 10$. So, we are done. When $\xi = \pm\eta$, one may assume that

$$f(x) = -a_3\delta^3 x(x + \xi^{-1})(x + 6\xi^{-1}) = -a_3\delta^3 x(x^2 - 4\xi^{-1}x + 5\delta^{-1})$$

by applying the lemma to the following ternary sublattices

$$\mathbb{F}_q[x]v_1 \perp \mathbb{F}_q[x]v_2 \perp \mathbb{F}_q[x](5v_3 \pm 3\eta^{-1}v_4).$$

One only needs to verify that $x(x^2 - 4\xi^{-1}x + 5\delta^{-1})$ is represented by L. Indeed, there are $a, c \in \mathbb{F}_q$ such that

$$a^2 - \delta c^2 = 1 \quad \text{and} \quad 4c^2 - 2\delta^{-1} \in \mathbb{F}_q^2$$

for $\delta = 2, 6, 7, 8, 10$. Let b be a solution of the equation

$$b^2 + 2bc - 5\delta^{-1} = 0.$$

Then

$$x(x^2 - 4\xi^{-1}x + 5\delta^{-1}) - Q((ax + b)v_3 + cxv_4) = -(4\xi^{-1} + c^2\xi + 2ab + 2ac)x^2,$$

which is represented by $\langle 1, -\delta \rangle$. Therefore, $x(x^2 - 4\xi^{-1}x + 5\delta^{-1})$ is represented by L.

If $q = 9$, then one can write $\theta^2 = -1$ and $\mathbb{F}_9 = \mathbb{F}_3(\theta)$. Then the solutions of (3.2) are

$$\begin{cases} \{(\pm 1, 0), (\pm\delta, \pm(1-\delta)), (\pm(1+\delta), \pm(1+\delta))\} & \text{if } \delta^2 = -\delta + 1 \text{ or } \delta = 1 \pm \theta, \\ \{(\pm 1, 0), (\pm\delta, \pm 1), (\pm(1-\delta), \pm\delta)\} & \text{if } \delta^2 = \delta + 1 \text{ or } \delta = -1 \pm \theta. \end{cases}$$

Therefore, S contains more than three elements except when

$$S = \begin{cases} \{0, \xi\delta, \xi(1-\delta)\} & \text{if } \delta = 1 \pm \theta \text{ and } \xi = \pm 1, \\ \{0, \xi, -\xi\delta\} & \text{if } \delta = 1 \pm \theta \text{ and } \xi = \pm(1+\delta), \\ \{0, -\xi, -\xi\delta\} & \text{if } \delta = -1 \pm \theta \text{ and } \xi = \pm\delta, \\ \{0, \xi\delta, -\xi(1+\delta)\} & \text{if } \delta = -1 \pm \theta \text{ and } \xi = \pm(1+\delta). \end{cases}$$

Here we only consider, for example, the case when $\delta = 1 \pm \theta$ and $\xi = \pm 1$. The other cases can be proved by the same argument. It is clear that we may assume that

$$f(x) = -a_3\delta^3 x(x + \xi\delta)(x + \xi(1-\delta)).$$

Since the coefficient of $-\delta(x + \xi\delta)$ in the expansion of $f(x)$ with respect to $-\delta(x + \xi\delta)$ is

$$\xi^2(2\delta^2 - \delta) = -a_3\delta^2,$$

$f(x)$ is represented by the sublattice

$$\mathbb{F}_q[x]v_1 \perp \mathbb{F}_q[x]v_2 \perp \mathbb{F}_q[x](s_0v_3 + t_0v_4) \cong \langle 1, -\delta, x + \xi\delta\rangle$$

for some solution (s_0, t_0) of (3.2).

If $q = 3, 7$, then one can assume (because $-a_3\delta^3$ is a square) that $f(x) = x^3 + \rho x^2 + \delta\gamma^2 x$, where $\rho \in \mathbb{F}_q$ and $\gamma \in \mathbb{F}_q^\times$. There are $a, c \in \mathbb{F}_q$ satisfying

$$a^2 - \delta c^2 = 1 \quad \text{and} \quad c^2 + \delta\gamma^2 \in \mathbb{F}_q^2.$$

Let b be a solution of the equation

$$b^2 + 2cb - \delta\gamma^2 = 0.$$

Then

$$f(x) - Q((ax + b)v_3 + cxv_4) = (\rho - 2ab - 2ac - \xi c^2)x^2$$

is represented by $\langle 1, -\delta\rangle$ and therefore, $f(x)$ is represented by L.

If $q = 5$, then one has $\delta^2 = -1$ and

$$S = \{0, 2\delta\xi \pm 2(\delta - 1)\}.$$

When $\xi \neq \pm(1 + \delta)$, one can assume that

$$f(x) = -a_3\delta^3 x(x + \sigma)(x + \tau),$$

where

$$\sigma = 2\delta\xi + 2(\delta - 1) \quad \text{and} \quad \tau = 2\delta\xi - 2(\delta - 1).$$

It is clear that $\sigma \neq \tau$. By assumption, one has $\sigma\tau \notin (\mathbb{F}_q^\times)^2$. By applying the lemma to the following ternary sublattices

$$\langle 1, -\delta, x + \sigma\rangle \quad \text{and} \quad \langle 1, -\delta, x + \tau\rangle,$$

one only needs to consider the case when

$$-\sigma(\tau-\sigma)\notin(\mathbb{F}_q^\times)^2 \quad\text{and}\quad -\tau(\sigma-\tau)\notin(\mathbb{F}_q^\times)^2.$$

But this is impossible. When $\xi=\pm(1+\delta)$, one has $S=\{0,-\delta\xi\}$. By the same argument as in the case of $q=3,7$, one may assume (because $-a_3\delta^3$ is a square) that $f(x)=x^3+\rho x^2-\delta x$, where $\rho\in\mathbb{F}_q$. By applying the lemma to the ternary sublattice $\langle 1,-\delta,x-\delta\xi\rangle$, one may further assume that

$$f(x)=x(x-\delta\xi)(x+\xi^{-1}).$$

Since $\langle 1,-\delta\rangle\cong\langle\xi,3\xi^{-1}\rangle$,

$$f(x)-Q(xv_3-\delta v_4)=\xi^{-1}(1-\delta\xi^2)x^2+\xi=\xi^{-1}(1-2\delta^2)x^2+\xi=3\xi^{-1}x^2+\xi$$

is represented by $\langle 1,-\delta\rangle$ and therefore, $f(x)$ is represented by L.

Case (ii) $-a_3\in(\mathbb{F}_q^\times)^2$: Observe that the orthogonal complement of the ternary lattice $\langle 1,-\delta,x\rangle$ in L is

$$\langle -\delta x^3+\xi x^2-x\rangle.$$

Suppose that $a_0\neq 0$. There are $q+1$ solutions of the equation

$$y^2-\delta z^2=-a_3\delta^3$$

over $\mathbb{F}_q$ and at least one of them, say (y_0,z_0), satisfies $y_0z_0\neq 0$. Then

$$f(x)-z_0^2(-\delta x^3+\xi x^2-x)=y_0^2x^3+\cdots+a_0$$

is represented by $\langle 1,-\delta,x\rangle$ and hence $f(x)$ is represented by L. So, we may assume that $a_0=0$.

It is clear that there are $q+1$ solutions over $\mathbb{F}_q$ of the equation

$$s^2-\delta t^2=\delta. \tag{3.3}$$

Let

$$T=\{\,\xi t^2+2st\ :(s,t)\text{ is a solution of (3.3)}\}.$$

Then, as before, one has

$$\sharp T\geq\frac{q+1}{4}.$$

If $q\geq 13$, then $\sharp T\geq 4$ and there is a solution (s_0,t_0) of (3.3) such that

$$Q(s_0v_3+t_0v_4)=(\delta x+\Delta_0)\nmid f(x)$$

over $\mathbb{F}_q[x]$ for some $\Delta_0\in\mathbb{F}_q$. Then $f(x)$ is represented by the ternary sublattice

$$\mathbb{F}_q[x]v_1\perp\mathbb{F}_q[x]v_2\perp\mathbb{F}_q[x](s_0v_3+t_0v_4)\cong\langle 1,-\delta,\delta x+\Delta_0\rangle.$$

If $q=11$, then one can write $-\delta=\eta^2$ for some $\eta\in\mathbb{F}_q$. Thus T still contains more than three elements if $\xi\neq\pm 4\eta$. So, it suffices to consider the case:

$$T=\{2\xi,5\xi,8\xi\}\quad\text{when}\quad\xi=\pm 4\eta.$$

By the lemma, one may assume that

$$f(x)=-a_3(\delta x+2\xi)(\delta x+5\xi)(\delta x+8\xi)=-a_3\delta^3x^3-4a_3\xi\delta^2x^2+4a_3\xi\delta.$$

But this contradicts the assumption that $a_0=0$.

If $q = 9$, then one can write $\theta^2 = -1$ and $\mathbb{F}_9 = \mathbb{F}_3(\theta)$. Then T contains more than three elements except when

$$T = \begin{cases} \{-\xi, \xi\delta, \xi(1+\delta)\} & \text{if } \delta = 1 \pm \theta \text{ and } \xi = \pm 1, \\ \{-\xi, -\xi\delta, -\xi(1+\delta),\} & \text{if } \delta = 1 \pm \theta \text{ and } \xi = \pm(1+\delta), \\ \{-\xi, \xi\delta, -\xi(1-\delta)\} & \text{if } \delta = -1 \pm \theta \text{ and } \xi = \pm\delta, \\ \{-\xi, -\xi\delta, \xi(1-\delta)\} & \text{if } \delta = -1 \pm \theta \text{ and } \xi = \pm(1+\delta). \end{cases}$$

Here we only consider, for example, the case when $\delta = 1 \pm \theta$ and $\xi = \pm 1$. The other cases can be proved by the same argument. But then, by the lemma, one may assume that

$$f(x) = -a_3(\delta x - \xi)(\delta x + \xi\delta)(\delta x + \xi(1+\delta)),$$

which again contradicts the assumption that $a_0 = 0$.

If $q = 7$, then $\delta^3 = -1$ and

$$T = \{4\xi \pm 2\delta^{-1}, 2\xi \pm 2\delta^{-1}\}.$$

So, T contains more than three elements if $\xi \neq 0, \pm 2\delta^{-1}$. When $\xi = \pm 2\delta^{-1}$, one has

$$T = \{\xi, 3\xi, 5\xi\}.$$

By the lemma, one may assume that

$$f(x) = -a_3(\delta x + \xi)(\delta x + 3\xi)(\delta x + 5\xi).$$

This contradicts the assumption $a_0 = 0$. When $\xi = 0$, one has $T = \{\pm 2\delta^{-1}\}$. Then by the lemma, one may assume that

$$f(x) = -a_3\delta x(\delta x - 2\delta^{-1})(\delta x + 2\delta^{-1}) = -a_3\delta x(\delta^2 x^2 + 4\delta).$$

Then

$$f(x) - (-4a_3)\delta^2(-\delta x^3 + \xi x^2 - x) = -5a_3\delta^3 x^3 - a_3\delta^2 x = 5a_3(-\delta x)^3 + a_3\delta(-\delta x)$$

is represented by $\langle 1, -\delta, x\rangle$ and therefore, $f(x)$ is represented by L.

If $q = 5$, then

$$T = \{-\xi, \xi \pm 2(1+\delta)\}.$$

When $\xi \neq \pm(1+\delta)$, T contains three elements

$$\sigma = -\xi, \quad \varsigma = \xi + 2(1+\delta) \quad \text{and} \quad \tau = \xi - 2(1+\delta).$$

By applying the lemma to the following ternary sublattices

$$\langle 1, -\delta, \delta x + \sigma\rangle, \quad \langle 1, -\delta, \delta x + \varsigma\rangle \quad \text{and} \quad \langle 1, -\delta, \delta + \tau\rangle,$$

one may assume that

$$f(x) = -a_3(\delta x + \sigma)(\delta x + \varsigma)(\delta x + \tau)$$

and from this it follows that

$$(\varsigma - \sigma)(\tau - \sigma) \notin (\mathbb{F}_q^\times)^2, \ (\sigma - \varsigma)(\tau - \varsigma) \notin (\mathbb{F}_q^\times)^2, \ \text{ and } \ (\varsigma - \tau)(\sigma - \tau) \notin (\mathbb{F}_q^\times)^2.$$

This, however, implies that

$$-(\varsigma-\sigma)^2(\tau-\sigma)^2(\varsigma-\tau)^2 \notin (\mathbb{F}_q^\times)^2,$$

which is a contradiction. When $\xi = \pm(1+\delta)$, one has $T = \{-\xi, 3\xi\}$ and may assume that

$$f(x) = -a_3\delta x(\delta x - \xi)(\delta x + 3\xi).$$

Since the coefficient of $-\delta(\delta x - \xi)$ and $(-\delta(\delta x - \xi))^3$ in the expansion of $f(x)$ are $-4a_3\delta^{-1}\xi^2$ and $a_3\delta^{-3}$, respectively, whose negatives are non-squares, $f(x)$ is represented by $\langle 1, -\delta, \delta x - \xi\rangle$.

If $q = 3$, then $T = \{\xi \pm 1\}$. One can assume that

$$f(x) = -a_3\delta^3 x(-x+\xi+1)(-x+\xi-1) = -a_3\delta^3(x^3 + \xi x^2 + (\xi^2-1)x).$$

When $\xi = 0$, one has

$$f(x) - a_3\delta^3 Q((x+1)v_3 + (x-1)v_4) = a_3\delta^3(x^2 - x - 1),$$

which is irreducible over $\mathbb{F}_q$ and is represented by $\langle 1, -\delta\rangle = \langle 1, 1\rangle$ (see [9]). Therefore, $f(x)$ is represented by L. When $\xi = \pm 1$, consider

$$f(x) - a_3\delta^3 x^2 Q(v_3 \pm v_4) = -a_4\delta^3 x^2.$$

Since this is represented by $\langle 1, -\delta\rangle$, one may conclude that $f(x)$ is represented by L. □

3.3. Non-diagonal case, higher degree

In this section, we show that L, when $\varepsilon = 1$ and $\xi \in \mathbb{F}_q$ is arbitrary, represents all polynomials $f(x) \in \mathbb{F}_q[x]$ of degree $n \geq 4$. Let

$$f(x) = a_0 + a_1(-\delta x) + \cdots + a_n(-\delta x)^n \in \mathbb{F}_q[x]$$

with $a_n \neq 0$ and $n \geq 4$.

Proof. **Case (i)** n is even or n is odd with $-a_n \notin (\mathbb{F}_q^\times)^2$: By applying the lemma to the ternary sublattice $\langle 1, -\delta, x\rangle$, one can assume that $a_0 = 0$. Suppose that $a_1 = 0$. Then

$$f(x) - (-\delta x^3 + \xi x^2 - x)$$

is represented by $\langle 1, -\delta, x\rangle$ and therefore, $f(x)$ is represented by L. Assume that $a_1 \neq 0$. One may further assume that $-a_1 \in (\mathbb{F}_q^\times)^2$.

If $q > 3$, then there are $q - 1 \geq 4$ solutions over $\mathbb{F}_q$ of the equation

$$u^2 - v^2 = -a_1\delta. \tag{3.4}$$

Then there is a solution (u_0, v_0) of (3.4) satisfying $u_0 v_0 \neq 0$. Therefore

$$f(x) - v_0^2(-\delta x^3 + \xi x^2 - x)$$

is represented by $\langle 1, -\delta, x\rangle$ and therefore, $f(x)$ is represented by L.

If $q = 3$ and $n \geq 5$, then consider a_2. When $a_2 \neq 0$, one can choose a proper sign such that $a_3 \pm a_2 = 0$ or 1. Then

$$f(x) - (x^3 + \xi x^2 - x)h(x)$$

is represented by $\langle 1, -\delta, x\rangle$ by the lemma, where

$$h(x) = \begin{cases} (x+1)^2 & \text{if } \xi = a_2, \\ 1 & \text{if } \xi \neq a_2. \end{cases}$$

This proves that $f(x)$ is represented by L. When $a_2 = 0$,

$$f(x) - (x^3 + \xi x^2 - x)k(x)$$

is represented by $\langle 1, -\delta, x\rangle$, where

$$k(x) = \begin{cases} (x+1)^2 & \text{if } \xi = a_3 = 0, \\ 1 & \text{otherwise,} \end{cases}$$

which proves that $f(x)$ is represented by L.

If $q = 3$ and $n = 4$, then one only needs to consider

$$f(x) = \begin{cases} a_4x^4 + a_3x^3 + \xi x^2 - x & \text{if } \xi \neq 0, \\ a_4x^4 - x & \text{if } \xi = 0. \end{cases}$$

When $\xi \neq 0$, one can further assume that $a_3 = 0$ by the same argument as above. Since $-x$ is represented by $\mathbb{F}_q[x]v_3 + \mathbb{F}_q[x]v_4$, one may assume that $a_4 = -\xi$. It is clear that $-x - \xi$ is also represented by $\mathbb{F}_q[x]v_3 + \mathbb{F}_q[x]v_4$. Then

$$f(x) - (-x - \xi) = -\xi x^4 + \xi x^2 + \xi$$

has no roots over $\mathbb{F}_q$ and is represented by $\langle 1, -\delta\rangle = \langle 1, 1\rangle$ by [9]. When $\xi = 0$, $-x - a_4$ is represented by $\mathbb{F}_q[x]v_3 + \mathbb{F}_q[x]v_4$. Then

$$f(x) - (-x - a_4) = a_4(x^4 + 1)$$

has no roots over $\mathbb{F}_q$ and is represented by $\langle 1, -\delta\rangle$ by [9]. In any case $f(x)$ is represented by L.

Case (ii) n is odd and $-a_n \in (\mathbb{F}_q^\times)^2$: Suppose $a_0 \neq 0$ or $-a_1 \notin \mathbb{F}_q^2$. Then there are $q+1$ solutions of (3.1) and at least one of them, say (y_0, z_0), satisfies $y_0 z_0 \neq 0$. Then

$$f(x) - (-\delta x^3 + \xi x^2 - x)(z_0 x^{(n-3)/2})^2$$

is represented by $\langle 1, -\delta, x\rangle$ and therefore, $f(x)$ is represented by L. Suppose $a_0 = a_1 = 0$. We also take a solution (y_0, z_0) of (3.1) satisfying $y_0 z_0 \neq 0$. Then

$$f(x) - (-\delta x^3 + \xi x^2 - x)(z_0 x^{(n-3)/2} + 1)^2$$

is represented by $\langle 1, -\delta, x\rangle$, which implies that $f(x)$ is represented by L. So, we may assume that $a_0 = 0$ and $-a_1 \in (\mathbb{F}_q^\times)^2$. Let (y_0, z_0) be a solution of (3.1) satisfying $y_0 z_0 \neq 0$.

If $q > 3$, then there is a solution of (3.4) such that $u_0 v_0 \neq 0$. Then

$$f(x) - (-\delta x^3 + \xi x^2 - x)(z_0 x^{\frac{n-3}{2}} + v_0)^2$$

is represented by $\langle 1, -\delta, x\rangle$ and therefore, $f(x)$ is represented by L.

If $q = 3$ and $n \geq 7$, then

$$f(x) - (x^3 + \xi x^2 - x)\ell(x)$$

is represented by $\langle 1, 1, x\rangle$, where

$$\ell(x) = \begin{cases} (x^{(n-3)/2} + 1)^2 & \text{if } \xi \neq a_2, \\ (x^{(n-3)/2} + x + 1)^2 & \text{if } \xi = a_2. \end{cases}$$

Therefore $f(x)$ is represented by L.

If $q = 3$ and $n = 5$, then one can obtain $a_2 - \xi \pm 1 \neq 0$ by choosing a proper sign. Then

$$f(x) - (x^3 + \xi x^2 - x)(x \mp 1)^2$$

is represented by $\langle 1, 1, x\rangle$, which implies that $f(x)$ is again represented by L. □

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