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ON GENERALIZED NONLINEAR QUASI-VARIATIONAL-LIKE INCLUSIONS DEALING WITH (h,η)-PROXIMAL MAPPING

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ABSTRACT. In this paper, a new class of (h,η)-proximal mappings for proper functionals in Hilbert spaces is introduced. The existence and Lipschitz continuity of the (h,η)-proximal mappings for proper functionals are proved. A class of generalized nonlinear quasi-variational-like inclusions in Hilbert spaces is introduced. A perturbed three-step iterative algorithm with errors for the generalized nonlinear quasi-variational-like inclusion is suggested. The existence and uniqueness theorems of solution for the generalized nonlinear quasi-variational-like inclusion are established. The convergence and stability results of iterative sequence generated by the perturbed three-step iterative algorithm with errors are discussed.

1. Introduction

Variational inequality theory has become a very effective and power tool in pure and applied mathematics and has been used in a large variety of problems arising in differential equations, mechanics, contact problems in elasticity and general equilibrium, see [1]-[10], [12]-[14], [16]-[27].

In 1991, Yao [25] investigated a generalized quasi-variational inequality and gave its applications. In 1994, Hassouni-Moudafi [13] studied a class of variational inclusions and suggested a perturbed algorithm for the variational inclusion. In 2000, Ding-Luo [6] introduced a class of quasi-variational-like inclusions in Hilbert spaces and developed perturbed η-proximal point algorithms of Mann and Ishikawa type for finding approximate solutions of the quasi-variational-like inclusions. In 2004, Liu-Kang [16] developed a perturbed three-step iterative algorithm for a generalized quasivariational inequality involving strongly

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monotone and generalized pseudocontractive mappings and proved the stability results of iterative sequence generated by the perturbed three-step iterative algorithm.

Inspired and motivated by the recent research works [3]-[6], [13]-[14], [16]-[20], [23]-[25], in this paper, we introduce and study a new class of generalized nonlinear quasi-variational-like inclusions, which includes as special cases, the classes of variational inclusions and quasivariational inclusions studied in [1], [3]-[6], [13]-[14], [16], [23]. Applying η-subdifferential and (h, η)-proximal mappings, we establish the equivalence between the generalized nonlinear quasi-variational-like inclusion and the fixed point problem. By using the equivalence, a new perturbed three-step iterative algorithm with errors is suggested and analyzed. The convergence criteria of the algorithm is also discussed. Our results extend and improve the recent results due to Cho-Kim-Huang-Kang [3], Ding [4] and [5], Ding-Luo [6], Hassouni-Moudafi [13], Kazmi [14], Liu-Kang [16] and Verma [23].

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space endowed with the norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, respectively. Let a, b, c, d, g: $H \to H$ and $\eta, M, N : H \times H \to H$ be mappings and $\phi : H \times H \to \mathbb{R} \cup \{+\infty\}$ be a proper functional such that for each fixed $y \in H$, $\phi(\cdot, y) : H \to H$ is lower semicontinuous and η-subdifferentiable on H and $g(H) \cap \mathrm{dom}\Delta\phi(\cdot, y) \neq \emptyset$. Given $f \in H$, we consider the following generalized nonlinear quasi-variational-like inclusion problem:

Find $x \in H$ such that $gx \in \mathrm{dom}\Delta\phi(\cdot, x)$ and

$$\begin{aligned} &\langle N(ax, bx) - M(cx, dx) - f, \eta(y, gx)\rangle \\ &\geq \phi(gx, x) - \phi(y, x), \quad \forall y \in H. \end{aligned} \tag{2.1}$$

Special cases:

If $f = 0$, $N(ax, bx) = ax - bx$ and $M(cx, dx) = 0$ for all $x, y \in H$, then the problem (2.1) reduces to the following variational inclusion problem:

Find $x \in H$ such that $gx \in \mathrm{dom}\Delta\phi$ and

$$\langle ax - bx, \eta(y, gx)\rangle \geq \phi(gx) - \phi(y), \quad \forall y \in H,$$

which was introduced and studied by Ding-Luo [6].

If $f = 0$, $\phi(x, y) = \phi(x)$, $N(ax, bx) = ax - bx$, $M(cx, dx) = 0$ and $\eta(y, x) = y - x$ for all $x, y \in H$, then the problem (2.1) reduces to the following inclusion problem:

Find $x \in H$ such that $gx \in \mathrm{dom}\Delta\phi$ and

$$\langle ax - bx, y - gx\rangle \geq \phi(gx) - \phi(y), \quad \forall y \in H,$$

which was studied by Hassouni-Moudafi [13].

If $f = 0$, $\eta(x, y) = x - y$, $N(ax, bx) = ax - bx$, $M(cx, dx) = 0$ and $K(x) = mx + K$ for all $x, y \in H$, where $m : H \to H$ is a mapping and K is a closed convex subset of H, $\phi : H \times H \to H$ is defined by

$$\phi(x, y) = I_{K(y)}(x), \quad \forall x, y \in H,$$

and $I_{K(y)}(x)$ is the indicator function of $K(y)$, that is,

$$I_{K(y)}(x) = \begin{cases} 0, & \text{if } x \in K(y), \\ +\infty, & \text{otherwise,} \end{cases}$$

then the problem (2.1) reduces to the following strongly nonlinear quasi-variational inequality problem:

Find $x \in H$ such that $gx \in K(x)$ and

$$\langle ax - bx, y - gx \rangle \geq 0, \quad \forall y \in K(x),$$

which includes a number of variational inequalities, quasi-variational inequalities, complementarity and quasi-complementarity problems as special cases, studied previously by many authors.

Now we recall the following definitions and results.

Definition 2.1. Let $h : H \to H$, $\eta : H \times H \to H$ be mappings.

(1) h is said to be *δ-η-strongly monotone* if there exists a constant $\delta > 0$ such that

$$\langle hx - hy, \eta(x, y) \rangle \geq \delta \|x - y\|^2, \quad \forall x, y \in H;$$

(2) h is said to be *σ-Lipschitz continuous* if there exists a constant $\sigma > 0$ such that

$$\|hx - hy\| \leq \sigma \|x - y\|, \quad \forall x, y \in H;$$

(3) η is said to be *α-Lipschitz continuous* if there exists a constant $\alpha > 0$ such that

$$\|\eta(x, y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in H.$$

Definition 2.2. Let $a, b, c : H \to H$ be mappings. A mapping $N : H \times H \to H$ is called

(1) *strongly monotone* with respect to a in the first argument if there exists a constant $r > 0$ such that

$$\langle N(ax, u) - N(ay, u), x - y \rangle \geq r\|x - y\|^2, \quad \forall x, y, u \in H;$$

(2) *relaxed coercive* with respect to a in the first argument if there exist constants $\gamma > 0$ and $r > 0$ such that

$$\langle N(ax, u) - N(ay, u), x - y \rangle \geq r\|x - y\|^2 - \gamma\|ax - ay\|^2, \quad \forall x, y, u \in H;$$

(3) *Lipschitz continuous* in the first argument if there exists a constant $t > 0$ satisfying

$$\|N(x, u) - N(y, u)\| \leq t\|x - y\|, \quad \forall x, y, u \in H;$$

(4) *generalized pseudocontractive* with respect to a in the first argument if there exists a constant $s > 0$ such that

$$\langle N(ax,u) - N(ay,u), x - y\rangle \leq s\|x-y\|^2, \quad \forall x, y, u \in H;$$

(5) *mixed Lipschitz continuous* with respect to a and b in the first and second arguments if there exists a constant $\beta > 0$ such that

$$\|N(ax,bx) - N(ay,by)\| \leq \beta\|x-y\|, \quad \forall x, y \in H.$$

In a similar way, we can define the Lipschitz continuity of the mapping N in the second argument.

Definition 2.3 ([24]). A functional $f : H \times H \to \mathbb{R} \cup \{+\infty\}$ is said to be 0-*diagonally quasi-concave* in the first argument if for any finite set $\{x_1, \ldots, x_n\} \subset H$ and for any $y = \sum_{i=1}^n \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$,

$$\min_{1\leq i\leq n} f(x_i, y) \leq 0.$$

Definition 2.4. Let $\eta : H \times H \to H$ be a mapping. A proper functional $\phi : H \to \mathbb{R} \cup \{+\infty\}$ is said to be η-*subdifferentiable* at a point $x \in H$ if there exists a point $f^* \in H$ such that

$$\phi(y) \geq \phi(x) + \langle f^*, \eta(y,x)\rangle, \quad \forall y \in H,$$

where f^* is called a η-subgradient of ϕ at x. The set of all η-subgradient of ϕ at x is denoted by $\Delta\phi(x)$. The mapping $\Delta\phi : H \to 2^H$ denoted by

$$\Delta\phi(x) = \{f^* \in H : \phi(y) \geq \phi(x) + \langle f^*, \eta(y,x)\rangle, \ \forall y \in H\}, \quad \forall x \in H,$$

is said to be η-*subdifferential* of ϕ.

Definition 2.5. Let $\phi : H \to \mathbb{R}\cup\{+\infty\}$ be a proper functional, $\eta : H\times H \to H$ and $h : H \to H$ be mappings, and $\rho > 0$ be a constant. Assume that for any given $x \in H$ there exists a unique point $u \in H$ such that

$$\langle hu - x, \eta(y,u)\rangle \geq \rho\phi(u) - \rho\phi(y), \quad \forall y \in H.$$

Then the mapping $x \mapsto u$, denoted by $J_{\rho,h}^{\Delta\phi}(x)$, is said to be (h,η)-*proximal mapping* of ϕ.

It follows from the definition of $J_{\rho,h}^{\Delta\phi}(x)$ that $x - hu \in \rho\Delta\phi(u)$, that is, $J_{\rho,h}^{\Delta\phi}(x) = (h + \rho\Delta\phi)^{-1}(x)$.

Definition 2.6 ([11]). Let $T : H \to H$ be a mapping and $x_0 \in H$. Assume that $x_{n+1} = f(T, x_n)$ define an iterative procedure which yields a sequence of points $\{x_n\}_{n\geq 0} \subseteq H$. Suppose that $F(T) = \{x \in H : x = Tx\} \neq \emptyset$ and $\{x_n\}_{n\geq 0}$ converges to some $u \in F(T)$. Let $\{z_n\}_{n\geq 0} \subset H$ and $\varepsilon_n = \|z_{n+1} - f(T, z_n)\|$ for $n \geq 0$. If $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} z_n = u$, then the iterative procedure defined by $x_{n+1} = f(T, x_n)$ is said to be T-*stable* or *stable* with respect to T.

Lemma 2.1 ([15]). *Let $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$, $\{c_n\}_{n\geq 0}$ and $\{t_n\}_{n\geq 0}$ be four sequences of nonnegative numbers satisfying*

$$a_{n+1} \leq (1-t_n)a_n + t_n b_n + c_n, \quad \forall n \geq 0,$$

where $\{t_n\}_{n\geq 0} \subset [0,1]$, $\sum_{n=0}^{\infty} t_n = +\infty$, $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=0}^{\infty} c_n < +\infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 ([7]). *Let D be a nonempty convex subset of a topological vector space and $f : D \times D \to \mathbb{R} \cup \{+\infty\}$ be such that*

(a) *for each $x \in D$, $y \mapsto f(x,y)$ is lower semicontinuous on each compact subset of D,*

(b) *for each finite set $\{x_1, \dots, x_m\} \subset D$ and for each $y = \sum_{i=1}^m \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$, $\min_{0\leq i\leq 1} f(x_i, y) \leq 0$,*

(c) *there exist a nonempty compact convex subset D_0 of D and a nonempty compact subset K of D such that for each $y \in D \setminus K$, there is an $x \in co(D_0 \cup \{y\})$ satisfying $f(x,y) > 0$.*

Then there exists $\hat{y} \in D$ such that $f(x,\hat{y}) \leq 0$.

Now we show the existence and Lipschitz continuity of the (h,η)-proximal mapping for a proper functional ϕ in Hilbert spaces.

Theorem 2.1. *Let $\eta : H \times H \to H$ be τ-Lipschitz continuous such that $\eta(x,y) = -\eta(y,x)$ for all $x,y \in H$, $\phi : H \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous η-subdifferentiable proper functional and $h : H \to H$ be γ-η-strongly monotone and λ-Lipschitz continuous. Assume that for any given $x \in H$, the functional $k : H \times H \to \mathbb{R}$ defined by $k(y,u) = \langle x - hu, \eta(y,u)\rangle$ is 0-DQCV in the first argument. Let $\rho > 0$ be a constant. Then for any given $x \in H$, there exists a unique $u \in H$ such that*

$$\langle hu - x, \eta(y,u)\rangle \geq \rho\phi(u) - \rho\phi(y), \quad \forall y \in H. \tag{2.2}$$

That is, $u = J_{\rho,h}^{\Delta\phi}(x)$.

Proof. For any given $x \in H$, define a functional $f : H \times H \to \mathbb{R} \cup \{+\infty\}$ by

$$f(y,u) = \langle x - hu, \eta(y,u)\rangle + \rho\phi(u) - \rho\phi(y), \quad \forall y \in H.$$

It follows from the continuity of $\eta(y,u)$ and lower semicontinuity of ϕ that for each $y \in H$, $u \to f(y,u)$ satisfies the condition (b) of Lemma 2.2. Otherwise, there exists a finite set $\{y_1, \dots, y_m\} \subset H$ and some $u_0 = \sum_{i=1}^m \lambda_i y_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$ such that

$$\langle x - hu_0, \eta(y_i,u_0)\rangle + \rho\phi(u_0) - \rho\phi(y_i) > 0, \quad 1 \leq i \leq m. \tag{2.3}$$

Since ϕ is η-subdifferentiable at u_0, there exists $f_{u_0}^* \in H$ such that

$$\phi(y) \geq \phi(u_0) + \langle f_{u_0}^*, \eta(y,u_0)\rangle, \quad \forall y \in H. \tag{2.4}$$

By virtue of (2.3), (2.4), we infer that

$$\begin{aligned}\langle x - hu_0, \eta(y_i,u_0)\rangle &> \rho\phi(y_i) - \rho\phi(u_0)\\ &\geq \rho\langle f_{u_0}^*, \eta(y_i,u_0)\rangle, \quad 1 \leq i \leq m,\end{aligned}$$

which implies that

$$(2.5) \qquad \langle x-\rho f^*_{u_0}-hu_0,\eta(y_i,u_0)\rangle > 0, \quad 1\le i\le m.$$

Since $k(y_i,u_0)=\langle x-\rho f^*_{u_0}-hu_0,\eta(y_i,u_0)\rangle$ is 0-DQCV in the first argument, it follows that

$$\min_{1\le i\le m}\langle x-\rho f^*_{u_0}-hu_0,\eta(y_i,u_0)\rangle \le 0 < \min_{1\le i\le m}\langle x-\rho f^*_{u_0}-hu_0,\eta(y_i,u_0)\rangle,$$

which is a contradiction. Hence f satisfies the condition (b) of Lemma 2.2. Let $\overline{y}\in \mathrm{dom}\phi$. Since ϕ is η-subdifferentiable at $\overline{y}$, there exists $f^*_{\overline{y}}\in H$ such that

$$\phi(u)-\phi(\overline{y})\ge \langle f^*_{\overline{y}},\eta(u,\overline{y})\rangle, \quad \forall u\in H,$$

which implies that

$$\begin{aligned} f(\overline{y},u) &= \langle x-hu,\eta(\overline{y},u)\rangle+\rho\phi(u)-\rho\phi(\overline{y})\\ &\ge \langle h\overline{y}-hu,\eta(\overline{y},u)\rangle+\langle x-h\overline{y},\eta(\overline{y},u)\rangle+\rho\langle f^*_{\overline{y}},\eta(u,\overline{y})\rangle\\ &\ge \gamma\|\overline{y}-u\|^2-\tau\|x-h\overline{y}\|\|\overline{y}-u\|-\tau\rho\|f^*_{\overline{y}}\|\|\overline{y}-u\|\\ &= \|\overline{y}-u\|[\gamma\|\overline{y}-u\|-\tau\|x-h\overline{y}\|-\tau\rho\|f^*_{\overline{y}}\|]. \end{aligned}$$

Let $r=(1/\gamma)\tau[\|x-h\overline{y}\|+\rho\|f^*_{\overline{y}}\|]$, $K=\{u\in H:\|\overline{y}-u\|\le r\}$ and $D_0=\{\overline{y}\}$. Clearly D_0 and K are both weakly compact convex subsets of H and for each $u\in H\backslash K$, there exists $\overline{y}\in co(D_0\cup\{\overline{y}\})$ such that $f(\overline{y},u)>0$. Hence all the conditions of Lemma 2.2 are satisfied. By Lemma 2.2, there exists $\overline{u}\in H$ such that $f(y,\overline{u})\le 0$, $\forall y\in H$, that is,

$$\langle h\overline{u}-x,\eta(y,\overline{u})\rangle\ge\rho\phi(\overline{u})-\rho\phi(y), \quad \forall y\in H.$$

Now we show that $\overline{u}$ is a unique solution of the problem (2.2). Suppose that $u_1,u_2\in H$ are arbitrary two solutions of the problem (2.2). It follows that

$$(2.6) \qquad \langle hu_1-x,\eta(y,u_1)\rangle\ge\rho\phi(u_1)-\rho\phi(y), \quad \forall y\in H,$$

$$(2.7) \qquad \langle hu_2-x,\eta(y,u_2)\rangle\ge\rho\phi(u_2)-\rho\phi(y), \quad \forall y\in H.$$

Taking $y=u_2$ in (2.6) and $y=u_1$ in (2.7), and adding these inequalities, we have

$$(2.8) \qquad \langle hu_1-x,\eta(u_2,u_1)\rangle+\langle hu_2-x,\eta(u_1,u_2)\rangle\ge 0.$$

Since $\eta(x,y)=-\eta(y,x)$, $\forall x,y\in H$ and h is γ-η-strongly monotone, by (2.8) we deduce that

$$\gamma\|u_1-u_2\|^2\le\langle hu_1-hu_2,\eta(u_1,u_2)\rangle\le 0,$$

which implies that $u_1=u_2$. This completes the proof. □

Theorem 2.2. *Let $\eta: H\times H\to H$ be τ-Lipschitz continuous such that $\eta(x,y)=-\eta(y,x)$ for all $x,y\in H$, $\phi: H\to\mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous η-subdifferentiable proper functional and $h: H\to H$ be γ-η-strongly monotone and λ-Lipschitz continuous. Let $\rho>0$ be a constant. Assume that for any given $x\in H$, the functional $k: H\times H\to\mathbb{R}$ defined by $k(y,u)=\langle x-hu,\eta(y,u)\rangle$*

is 0-DQCV in the first argument. Then the (h,η)-proximal mapping $J^{\Delta\phi}_{\rho,h}$ of ϕ is τ/γ-Lipschitz continuous.

Proof. Let $x_1, x_2 \in H$, $u_1 = J^{\Delta\phi}_{\rho,h}(x_1)$ and $u_2 = J^{\Delta\phi}_{\rho,h}(x_2)$. It follows from Theorem 2.1 that

$$\langle hu_1 - x_1, \eta(y,u_1)\rangle \geq \rho\phi(u_1) - \rho\phi(y), \quad \forall y \in H, \tag{2.9}$$

$$\langle hu_2 - x_2, \eta(y,u_2)\rangle \geq \rho\phi(u_2) - \rho\phi(y), \quad \forall y \in H. \tag{2.10}$$

Taking $y = u_2$ in (2.9) and $y = u_1$ in (2.10), and adding these inequalities, we conclude that

$$\langle hu_1 - x_1, \eta(u_2,u_1)\rangle + \langle hu_2 - x_2, \eta(u_1,u_2)\rangle \geq 0.$$

Since h is γ-η-strongly monotone and η is τ-Lipschitz continuous with $\eta(x,y) = -\eta(y,x)$ for all $x, y \in H$, we infer that

$$\begin{aligned}\gamma\|u_1 - u_2\|^2 &\leq \langle hu_1 - hu_2, \eta(u_1,u_2)\rangle \leq \langle x_1 - x_2, \eta(u_1,u_2)\rangle \\ &\leq \|x_1 - x_2\|\|\eta(u_1,u_2)\| \leq \tau\|x_1 - x_2\|\|u_1 - u_2\|,\end{aligned}$$

which implies that

$$\left\|J^{\Delta\phi}_{\rho,h}(x_1) - J^{\Delta\phi}_{\rho,h}(x_2)\right\| = \|u_1 - u_2\| \leq \frac{\tau}{\gamma}\|x_1 - x_2\|,$$

that is, $J^{\Delta\phi}_{\rho,h}$ is τ/γ-Lipschitz continuous. This completes the proof. □

Remark 2.1. If $\eta(x,y) = x - y$ for all $x, y \in H$, where h is the identity mapping in H, then Theorem 2.2 reduces to Lemma 2.2 in [13].

3. A perturbed three-step iterative algorithm with errors

We first transfer the problem (2.1) into a fixed point problem.

Lemma 3.1. *$x^* \in H$ is a solution of the problem (2.1) if and only if x^* satisfies the following relation*

$$gx^* = J^{\Delta\phi(\cdot,x^*)}_{\rho,h}[hgx^* - \rho(N(ax^*,bx^*) - M(cx^*,dx^*) - f)], \tag{3.1}$$

where $J^{\Delta\phi}_{\rho,h}(\cdot,x^*) = (h + \rho\Delta\phi(\cdot,x^*))^{-1}$ is the (h,η)-proximal mapping of $\phi(\cdot,x^*)$ and $\rho > 0$ is a constant.

Proof. The fact directly follows from Definitions 2.4 and 2.5. □

Based on Lemma 3.1, we suggest the following perturbed three-step iterative algorithm with errors.

Algorithm 3.1. Let $N, M, \eta : H \times H \to H$, $a, b, c, d, g, h : H \to H$ be mappings, and $\{\phi_n\}_{n\geq 0} : H \times H \to \mathbb{R} \cup \{+\infty\}$ be a sequence of proper functionals such that for each given $x \in H, n \geq 0$, the (h,η)-proximal mapping of $\phi_n(\cdot,x)$

exists. Let $Ex = hgx - \rho N(ax, bx) + \rho M(cx, dx) + \rho f$ for all $x \in H$. For any given $u_0 \in H$, compute sequence $\{u_n\}_{n\geq 0}$ by the following iterative schemes:

$$\begin{aligned} u_{n+1} &= (1 - a_n - b_n)u_n \\ &\quad + a_n\big[v_n - gv_n + J_{\rho,h}^{\Delta\phi_n(\cdot,v_n)}(Ev_n)\big] + b_n p_n, \\ v_n &= (1 - a_n' - b_n')u_n \\ &\quad + a_n'\big[w_n - gw_n + J_{\rho,h}^{\Delta\phi_n(\cdot,w_n)}(Ew_n)\big] + b_n' q_n, \\ w_n &= (1 - a_n'' - b_n'')u_n \\ &\quad + a_n''\big[u_n - gu_n + J_{\rho,h}^{\Delta\phi_n(\cdot,u_n)}(Eu_n)\big] + b_n'' r_n,\ n \geq 0, \end{aligned} \tag{3.2}$$

where $\rho > 0$, $\{p_n\}_{n\geq 0}$, $\{q_n\}_{n\geq 0}$ and $\{r_n\}_{n\geq 0}$ are bounded sequences in H introduced to take into account possible in exact computation and sequences

$$\{a_n\}_{n\geq 0},\ \{b_n\}_{n\geq 0},\ \{a_n'\}_{n\geq 0},\ \{b_n'\}_{n\geq 0},\ \{a_n''\}_{n\geq 0}$$

and $\{b_n''\}_{n\geq 0}$ are in $[0,1]$ satisfying

$$\max\{a_n + b_n, a_n' + b_n', a_n'' + b_n''\} \leq 1, \quad n \geq 0, \tag{3.3}$$

$$\sum_{n=0}^{\infty} a_n = +\infty, \quad \lim_{n\to\infty} a_n' b_n'' = \lim_{n\to\infty} b_n' = 0 \tag{3.4}$$

and one of the following conditions:

$$\sum_{n=0}^{\infty} b_n < +\infty; \tag{3.5}$$

(3.6) there exists a nonnegative sequence $\{h_n\}_{n\geq 0}$ with $\lim_{n\to\infty} h_n = 0$ and $b_n = a_n h_n$, $\forall n \geq 0$.

Remark 3.1. In case $a_n'' = b_n'' = 0$ for all $n \geq 0$, then Algorithm 3.1 reduces to the Ishikawa type perturbed iterative algorithm in [3]. If $a_n' = b_n' = 0$, for all $n \geq 0$, then the Ishikawa type perturbed iterative algorithm reduces to the Mann type perturbed iterative algorithm in [23]. If h is the identity mapping in H and $M(x,y) = 0, N(x,y) = x - y = \eta(x,y)$ for all $x, y \in H$, and $b_n = b_n' = b_n'' = a_n'' = 0$ for all $n \geq 0$, Algorithm 3.1 reduces to the algorithm in [13].

4. Existence, convergence and stability

In this section, we give the existence and uniqueness theorems of solution for the generalized nonlinear quasi-variational-like inclusion, and show the convergence and stability results of iterative sequence generated by the perturbed three-step iterative algorithm with errors.

Theorem 4.1. *Let $a, b, c, d, g : H \to H$ be mappings, a, b, g be Lipschitz continuous with constants $\lambda_a, \lambda_b, \lambda_g$, respectively, and g be σ-strongly monotone. Let $\eta : H \times H \to H$ be δ-strongly monotone and τ-Lipschitz continuous such that $\eta(x, y) = -\eta(y, x)$, $\forall x, y \in H$. Let $h : H \to H$ be γ-η-strongly monotone and λ-Lipschitz continuous. For each given $x \in H$, the functional $k : H \times H \to \mathbb{R}$ defined by $k(y, u) = \langle x - hu, \eta(y, u)\rangle$ is 0-DQCV in the first argument. Let $N : H \times H \to H$ be Lipschitz continuous in the first and second arguments with constants ξ and ζ, respectively, and be strongly monotone with respect to a in the first argument with constant α. Assume that $M : H \times H \to H$ is mixed Lipschitz continuous with respect to c and d in the first and second arguments with constant β. Let ϕ and $\phi_n : H \times H \to \mathbb{R} \cup \{+\infty\}$ be such that for each fixed $y \in H$ and $n \geq 0$, $\phi(\cdot, y)$ and $\phi_n(\cdot, y)$ are lower semicontinuous η-subdifferentiable proper functionals satisfying $g(H) \cap \mathrm{dom}\Delta\phi(\cdot, y) \neq \emptyset$, $g(H) \cap \mathrm{dom}\Delta\phi_n(\cdot, y) \neq \emptyset$ and*

$$\begin{aligned}&\sup\left\{\left\|J_{\rho,h}^{\Delta\phi(\cdot,x)}(z) - J_{\rho,h}^{\Delta\phi(\cdot,y)}(z)\right\|, \left\|J_{\rho,h}^{\Delta\phi_n(\cdot,x)}(z) - J_{\rho,h}^{\Delta\phi_n(\cdot,y)}(z)\right\| : n \geq 0\right\}\\ &\leq \mu\|x - y\|, \quad \forall x, y, z \in H,\end{aligned}$$

$$\lim_{n\to\infty}\left\|J_{\rho,h}^{\Delta\phi_n(\cdot,y)}(z) - J_{\rho,h}^{\Delta\phi(\cdot,y)}(z)\right\| = 0, \quad \forall y, z \in H, \tag{4.1}$$

where $\mu > 0$ is a constant. Let $\{x_n\}_{n\geq 0}$ be any sequence in H and define $\{\varepsilon_n\}_{n\geq 0} \subset [0, \infty)$ by

$$\begin{aligned}\varepsilon_n &= \big\|x_{n+1} - \big[(1 - a_n - b_n)x_n\\ &\qquad + a_n(y_n - gy_n + J_{\rho,h}^{\Delta\phi_n(\cdot,y_n)}(Ey_n) + b_n p_n)\big]\big\|,\\ y_n &= (1 - a_n' - b_n')x_n\\ &\qquad + a_n'\big[z_n - gz_n + J_{\rho,h}^{\Delta\phi_n(\cdot,z_n)}(Ez_n)\big] + b_n' q_n,\\ z_n &= (1 - a_n'' - b_n'')x_n\\ &\qquad + a_n''\big[x_n - gx_n + J_{\rho,h}^{\Delta\phi_n(\cdot,x_n)}(Ex_n)\big] + b_n'' r_n, \quad n \geq 0.\end{aligned} \tag{4.2}$$

Let $K = (1+\frac{\tau}{\delta})\sqrt{1 - 2\sigma + \lambda_g^2} + \frac{\tau}{\delta}\lambda_g\sqrt{1 - 2\gamma + \lambda^2} + \mu$, $P = \lambda_a^2\xi^2\tau^2 - \tau^2(\zeta\lambda_b + \beta)^2$, $Q = \alpha\tau^2$. If there exists a constant $\rho > 0$ satisfying one of the following conditions:

$$\begin{aligned}&P > 0, \quad [Q - (1 - K)(\zeta\lambda_b + \beta)\tau\delta]^2 > (\tau^2 - \delta^2(1 - K)^2)P,\\ &\left|\rho - \frac{Q - (1 - K)(\zeta\lambda_b + \beta)\tau\delta}{P}\right|\\ &< \frac{\sqrt{[Q - (1 - K)(\zeta\lambda_b + \beta)\tau\delta]^2 - P(\tau^2 - (1 - K)^2\delta^2)}}{P};\end{aligned} \tag{4.3}$$

$$P < 0,$$

$$\left|\rho - \frac{Q - (1-K)(\zeta\lambda_b + \beta)\tau\delta}{P}\right| \tag{4.4}$$
$$> \frac{\sqrt{[Q - (1-K)(\zeta\lambda_b + \beta)\tau\delta]^2 - P(\tau^2 - (1-K)^2\delta^2)}}{-P},$$

then the problem (2.1) *has a unique solution* $x^* \in H$ *and the iterative sequence* $\{x_n\}_{n\geq 0}$ *defined by Algorithm 3.1 converges strongly to* x^*. *Moreover, if there exists a constant* $A > 0$ *satisfying*

$$a_n \geq A, \quad n \geq 0, \tag{4.5}$$

then $\lim_{n\to\infty} x_n = x^*$ *if and only if* $\lim_{n\to\infty} \varepsilon_n = 0$.

Proof. First, we define a mapping $F : H \to H$ by

$$Fx = x - gx + J_{\rho,h}^{\Delta\phi(\cdot,x)}(Ex), \quad \forall x \in H.$$

For any $x, y \in H$, we conclude that

$$\begin{aligned}
&\|Fx - Fy\| \\
&\leq \|x - y - (gx - gy)\| + \left\|J_{\rho,h}^{\Delta\phi(\cdot,x)}(Ex) - J_{\rho,h}^{\Delta\phi(\cdot,x)}(Ey)\right\| \\
&\quad + \left\|J_{\rho,h}^{\Delta\phi(\cdot,x)}(Ey) - J_{\rho,h}^{\Delta\phi(\cdot,y)}(Ey)\right\| \\
&\leq \left(1 + \frac{\tau}{\delta}\right)\|x - y - (gx - gy)\| \\
&\quad + \frac{\tau}{\delta}\|gx - gy - (hgx - hgy)\| \\
&\quad + \frac{\tau}{\delta}\|x - y - \rho(N(ax, bx) - N(ay, by))\| \\
&\quad + \frac{\rho\tau}{\delta}\|M(cx, dx) - M(cy, dy)\| + \mu\|x - y\| \\
&\leq \left(1 + \frac{\tau}{\delta}\right)\sqrt{1 - 2\sigma + \lambda_g^2}\|x - y\| + \frac{\tau}{\delta}\lambda_g\sqrt{1 - 2\gamma + \lambda^2}\|x - y\| \\
&\quad + \frac{\tau}{\delta}\left(\sqrt{1 - 2\rho\alpha + \rho^2\xi^2\lambda_a^2} + \rho\zeta\lambda_b\right)\|x - y\| \\
&\quad + \frac{\tau\rho\beta}{\delta}\|x - y\| + \mu\|x - y\| \\
&= \theta\|x - y\|,
\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}
\theta = \left(1 + \frac{\tau}{\delta}\right)\sqrt{1 - 2\sigma + \lambda_g^2} + \frac{\tau}{\delta}\big[\lambda_g\sqrt{1 - 2\gamma + \lambda^2} \\
+ \sqrt{1 - 2\rho\alpha + \rho^2\xi^2\lambda_a^2} + \rho\zeta\lambda_b + \rho\beta\big] + \mu.
\end{aligned}$$

It follows from one of (4.3), (4.4), and (4.6) that $\theta < 1$ and that F is a contraction mapping and that it has a unique fixed point $x^* \in H$ such that $x^* = F(x^*)$, that is,

$$gx^* = J_{\rho,h}^{\Delta\phi(\cdot,x^*)}[hgx^* - \rho(N(ax^*, bx^*) - M(cx^*, dx^*) - f)].$$

Thus x^* is a unique solution of the problem (2.1) in H.

Now we show that $\lim_{n\to\infty} x_n = x^*$. Notice that

$$
\begin{aligned}
x^* &= (1-a_n-b_n)x^* \\
&\quad + a_n\big[x^* - gx^* + J_{\rho,h}^{\Delta\phi_n(\cdot,x^*)}(Ex^*)\big] + b_n x^* \\
&= (1-a_n'-b_n')x^* \\
&\quad + a_n'\big[x^* - gx^* + J_{\rho,h}^{\Delta\phi_n(\cdot,x^*)}(Ex^*)\big] + b_n' x^* \\
&= (1-a_n''-b_n'')x^* \\
&\quad + a_n''\big[x^* - gx^* + J_{\rho,h}^{\Delta\phi_n(\cdot,x^*)}(Ex^*)\big] + b_n'' x^*, \quad n \ge 0.
\end{aligned} \tag{4.7}
$$

Put

$$
\begin{aligned}
d_n &= \|J_{\rho,h}^{\Delta\phi_n(\cdot,x)}(Eu) - J_{\rho,h}^{\Delta\phi(\cdot,y)}(Eu)\|, \quad \forall x,y,u \in H, \\
L &= \sup\{\|p_n - x^*\|, \|q_n - x^*\|, \|r_n - x^*\| : n \ge 0\|\}.
\end{aligned}
$$

It follows from (4.1) that

$$
\lim_{n\to\infty} d_n = 0. \tag{4.8}
$$

Using (3.2), (3.3), and (4.7), we get that

$$
\begin{aligned}
&\|u_{n+1} - x^*\| \\
&\le (1-a_n-b_n)\|u_n - x^*\| + a_n\big[\|v_n - x^* - (gv_n - gx^*)\| \\
&\quad + \big\|J_{\rho,h}^{\Delta\phi_n(\cdot,v_n)}(Ev_n) - J_{\rho,h}^{\Delta\phi_n(\cdot,v_n)}(Ex^*)\big\| \\
&\quad + \big\|J_{\rho,h}^{\Delta\phi_n(\cdot,v_n)}(Ex^*) - J_{\rho,h}^{\Delta\phi_n(\cdot,x^*)}(Ex^*)\big\| \\
&\quad + \big\|J_{\rho,h}^{\Delta\phi_n(\cdot,x^*)}(Ex^*) - J_{\rho,h}^{\Delta\phi(\cdot,x^*)}(Ex^*)\big\|\big] + b_n\|p_n - x^*\| \\
&\le (1-a_n-b_n)\|u_n - x^*\| + a_n\sqrt{1-2\sigma^2+\lambda_g}\,\|v_n - x^*\| \\
&\quad + \frac{\tau}{\delta}a_n\|Ev_n - Ex^*\| + a_n\mu\|v_n - x^*\| + a_n d_n + b_n L \\
&\le (1-a_n-b_n)\|u_n - x^*\| + a_n\sqrt{1-2\sigma^2+\lambda_g}\,\|v_n - x^*\| \\
&\quad + \frac{\tau}{\delta}a_n\|v_n - x^* - (gv_n - gx^*)\| \\
&\quad + \frac{\tau}{\delta}a_n\|gv_n - gx^* - (hgv_n - hgx^*)\| \\
&\quad + \frac{\tau}{\delta}a_n\|v_n - x^* - \rho(N(av_n, bv_n) - N(ax^*, bx^*))\| \\
&\quad + \frac{\tau}{\delta}a_n\rho\|M(cv_n, dv_n) - M(cx^*, dx^*)\| + a_n\mu\|v_n - x^*\| \\
&\quad + a_n d_n + b_n L
\end{aligned} \tag{4.9}
$$

$$\begin{aligned}&\leq (1-a_n-b_n)\|u_n-x^*\| + a_n\Big\{\Big(1+\frac{\tau}{\delta}\Big)\sqrt{1-2\sigma+\lambda_g^2}\\&\quad + \frac{\tau}{\delta}\big[\lambda_g\sqrt{1-2\gamma+\lambda^2} + \frac{\tau}{\delta}\sqrt{1-2\rho\alpha+\rho^2\xi^2\lambda_a^2}\\&\quad + \rho\lambda_b\zeta+\rho\beta\big] + \mu\Big\}\|v_n-x^*\| + a_nd_n + b_nL\\&= (1-a_n-b_n)\|u_n-x^*\| + \theta a_n\|v_n-x^*\|\\&\quad + a_nd_n + b_nL, \quad n\geq 0.\end{aligned}$$

Similarly, we deduce that

$$\begin{aligned}\|v_n-x^*\| &\leq (1-a_n'-b_n')\|u_n-x^*\| + \theta a_n'\|w_n-x^*\|\\&\quad + a_n'd_n + b_n'L,\\\|w_n-x^*\| &\leq (1-a_n''-b_n'')\|u_n-x^*\| + \theta a_n''\|u_n-x^*\|\\&\quad + a_n''d_n + b_n''L, \quad n\geq 0.\end{aligned}\tag{4.10}$$

Substituting (4.10) into (4.9), we infer that

$$\begin{aligned}&\|u_{n+1}-x^*\|\\&\leq [1-a_n-b_n+a_n\theta(1-a_n'-b_n'+a_n'\theta(1-a_n''-b_n'')\\&\quad + a_n''\theta)]\|u_n-x^*\| + a_n[\theta a_n'(a_n''\theta d_n+\theta Lb_n''+d_n)\\&\quad + \theta Lb_n'+d_n] + b_nL\\&\leq (1-(1-\theta)a_n)\|u_n-x^*\| + a_n[a_n'(2d_n+Lb_n'')\\&\quad + Lb_n'+d_n] + b_nL, \quad n\geq 0.\end{aligned}\tag{4.11}$$

It follows from Lemma 2.3, (3.4), (4.8), (4.11) and one of (3.5) and (3.6) that $\lim_{n\to\infty} x_n = x^*$.

Now we assume that (4.5) holds. Observe that each of (3.5) and (3.6) implies that

$$\lim_{n\to\infty} b_n = 0.\tag{4.12}$$

As in the proof of (4.9), by (4.5), we conclude that

$$\begin{aligned}&\|(1-a_n-b_n)x_n + a_n\big(y_n - gy_n + J_\rho^{\Delta\phi_n(\cdot,y_n)}(Ey_n)\big) + b_np_n - x^*\|\\&\leq (1-(1-\theta)a_n)\|x_n-x^*\| + a_n[a_n'(2d_n+Lb_n'') + Lb_n'+d_n] + b_nL\\&\leq (1-(1-\theta)A)\|x_n-x^*\| + a_n'(2d_n+Lb_n'')\\&\quad + Lb_n'+d_n+b_nL, \quad n\geq 0.\end{aligned}\tag{4.13}$$

Suppose that $\lim_{n\to\infty} x_n = x^*$. By virtue of (3.4), (4.2), (4.8), (4.13) and one of (3.5) and (3.6), we obtain that

$$\begin{aligned}\varepsilon &\le \|x_{n+1} - x^*\| + \big\|(1 - a_n - b_n)x_n \\ &\quad + a_n(y_n - gy_n + J_{\rho,h}^{\Delta\phi_n(\cdot,y_n)}(Ey_n)) + b_n p_n - x^*\big\| \\ &\le \|x_{n+1} - x^*\| + (1-(1-\theta)A)\|x_n - x^*\| \\ &\quad + a_n'(2d_n + Lb_n'') + Lb_n' + d_n + b_n L \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

That is, $\lim_{n\to\infty} \varepsilon_n = 0$. Conversely, suppose that $\lim_{n\to\infty} \varepsilon_n = 0$. In light of (4.2) and (4.13), we know that

$$\begin{aligned}&\|x_{n+1} - x^*\| \\ &\le \big\|(1 - a_n - b_n)x_n + a_n(y_n - gy_n + J_{\rho,h}^{\Delta\phi_n(\cdot,y_n)}(Ey_n)) \\ &\quad + b_n p_n - x^*\big\| + \varepsilon_n \\ &\le (1-(1-\theta)A)\|x_n - x^*\| + a_n'(2d_n + Lb_n'') \\ &\quad + Lb_n' + d_n + b_n L + \varepsilon_n, \quad n \ge 0.\end{aligned} \tag{4.14}$$

It follows from Lemma 2.1, (4.8), (4.12), (4.14) and one of (3.5) and (3.6) that $\lim_{n\to\infty} x_n = x^*$. This completes the proof. □

Remark 4.1. If h is the identity mapping in H and

$$N(x,y) = x - y, M(x,y) = 0 \text{ for all } x,y \in H$$

in Theorem 4.1, then Theorem 4.1 generalizes Theorems 3.6~3.8 in [6]. Furthermore, if $\eta(x,y) = x - y$ for all $x,y \in H$, then Theorem 4.1 extends Theorem 3.3 in [4].

Theorem 4.2. *Let $a,b,g,h,k,\eta,\phi,\{\phi_n\}_{n\ge0},\{\varepsilon_n\}_{n\ge0}$ and $\{x_n\}_{n\ge0}$ be as in Theorem 4.1. Suppose that c and $d : H \to H$ are Lipschizt continuous with constants λ_c and λ_d, respectively, $N : H \times H \to H$ is relaxed coercive with respect to a in the first argument with constants $\gamma > 0$ and $r > 0$, and Lipschitz continuous in the first and second arguments with constants ξ and $\zeta > 0$, respectively. Assume that $M : H \times H \to H$ is generalized pseudocontractive with respect to c in the first argument with constant ν and Lipschitz continuous in the first and second arguments with constants $\beta > 0$ and $l > 0$, respectively. Let*

$$\begin{aligned}K &= \Big(1 + \frac{\tau}{\delta}\Big)\sqrt{1 - 2\sigma + \lambda_g^2} + \frac{\tau}{\delta}\lambda_g\sqrt{1 - 2\gamma + \lambda^2} + \mu, \\ P_1 &= (\lambda_a\xi + \lambda_c\beta))^2\tau^2 - (\lambda_b\zeta + \lambda_d l)^2\delta^2, \\ Q_1 &= \tau^2[(r - \gamma\lambda_a^2)^2 - \lambda_c\nu].\end{aligned}$$

If there exists a constant $\rho > 0$ satisfying (4.1), (4.2) and one of the following conditions:

$$\begin{aligned}
&P_1 > 0,\\
&\left[(1-K)(\lambda_b\zeta + \lambda_d l)\delta^2 - Q_1\right]^2 > P_1(\tau^2 - (1-K)^2\delta^2),\\
&\left|\rho - \frac{Q_1 - (1-K)(\lambda_b\zeta + \lambda_d l)\delta^2}{P_1}\right|\\
&< \frac{\sqrt{[Q_1 - (1-K)(\lambda_b\zeta + \lambda_d l)\delta^2]^2 - P_1(\tau^2 - (1-K)^2\delta^2)}}{P_1};
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
&P_1 < 0,\\
&\left|\rho - \frac{Q_1 - (1-K)(\lambda_b\zeta + \lambda_d l)\delta^2}{P_1}\right|\\
&> \frac{\sqrt{[Q_1 - (1-K)(\lambda_b\zeta + \lambda_d l)\delta^2]^2 - P_1(\tau^2 - (1-K)^2\delta^2)}}{-P_1},
\end{aligned} \tag{4.16}$$

then the problem (2.1) has a unique solution $x^* \in H$ and the iterative sequence $\{x_n\}_{n\geq 0}$ defined by Algorithm 3.1 converges strongly to x^*. Moreover, if there exists a constant $A > 0$ satisfying (4.5), then $\lim_{n\to\infty} x_n = x^*$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0$.

Proof. As in the proof of Theorem 4.1, for any $x, y \in H$, we obtain that

$$\begin{aligned}
&\|Fx - Fy\|\\
&\leq \|x - y - (gx - gy)\|\\
&\quad + \|J_{\rho,h}^{\Delta\phi(\cdot,x)}(Ex) - J_{\rho,h}^{\Delta\phi(\cdot,x)}(Ey)\|\\
&\quad + \|J_{\rho,h}^{\Delta\phi(\cdot,x)}(Ey) - J_{\rho,h}^{\Delta\phi(\cdot,y)}(Ey)\|\\
&\leq \left(1 + \frac{\tau}{\delta}\right)\|x - y - (gx - gy)\|\\
&\quad + \frac{\tau}{\delta}\|gx - gy - (hgx - hgy)\|\\
&\quad + \frac{\tau}{\delta}[\|x - y - \rho(N(ax, bx) - N(ay, by))\\
&\quad + \rho(M(cx, dx) - M(cy, dy))\|] + \mu\|x - y\|\\
&\leq \left(1 + \frac{\tau}{\delta}\right)\sqrt{1 - 2\sigma + \lambda_g^2}\|x - y\| + \frac{\tau}{\delta}\lambda_g\sqrt{1 - 2\gamma + \lambda^2}\|x - y\|\\
&\quad + \frac{\tau}{\delta}\{\sqrt{1 - 2\rho[(r - \gamma\lambda_a^2) - \nu] + \rho^2(\xi\lambda_a + \lambda_c\beta)^2}\\
&\quad + \rho\lambda_b\zeta + \rho\lambda_d l\}\|x - y\| + \mu\|x - y\|\\
&= \theta_1\|x - y\|,
\end{aligned} \tag{4.17}$$

where

$$\begin{aligned}\theta_1 = &\Big(1+\frac{\tau}{\delta}\Big)\sqrt{1-2\lambda_g+\sigma^2}+\frac{\tau}{\delta}\lambda_g\sqrt{1-2\gamma+\lambda^2}\\ &+\frac{\tau}{\delta}\big[\sqrt{1-2\rho[(r-\gamma\lambda_a^2)-\lambda_c\nu]+\rho^2(\xi\lambda_a+\lambda_c\beta)^2}\\ &+\rho\lambda_b\zeta+\rho\lambda_d l\big]+\mu.\end{aligned}$$

It follows from one of (4.15) and (4.16) that $\theta_1 < 1$. From (4.17) we infer that F is a contraction mapping in H and it has a unique fixed point $x^* \in H$, which is a unique solution of the problem (2.1). Similarly, we can show that

$$\begin{aligned}\|u_{n+1}-x^*\| \le &(1-(1-\theta_1)a_n)\|u_n-x^*\|\\ &+a_n[a_n'(2d_n+Lb_n'')+Lb_n'+d_n]+b_nL, \quad n\ge 0.\end{aligned}$$

The rest of argument follows as in the proof of Theorem 4.1 and is therefore omitted. This completes the proof. □

Remark 4.2. Theorems 4.1 and 4.2 provide the convergence and stability of iterative sequence generated by Algorithm 3.1 under certain conditions.

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