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PROJECTIONS OF BOUQUET GRAPH WITH TWO CYCLES

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ABSTRACT. In this paper we investigate the projections of bouquet graph B with two cycles. A projection of B is said to be trivial if only trivial embeddings are obtained from the projection. It is shown that, to cover all nontrivial projections of B, at least three embeddings of B are needed. We also show that a nontrivial projection of B is covered by one of some two embeddings if the image of each cycle has at most one self-crossing.

1. Introduction

Throughout this paper we work in the piecewise linear category and graphs are considered as topological spaces. Let G be a graph with finitely many vertices and edges. Let $f, f' : G \to \mathbb{R}^3$ be embeddings of G into $\mathbb{R}^3$. f is said to be *equivalent* to f', denoted by $f \sim f'$, if there exists an orientation preserving homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h \circ f = f'$. Especially we say that f is *trivial* if $f'(G) \subset \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$.

A continuous map $\varphi : G \to \mathbb{R}^2$ is called a *projection* of G if the multiple points of φ are finitely many transversal double points away from the image of vertices. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the map defined by $\pi(x, y, z) = (x, y)$. We say that φ is a *projection* of an embedding f, if there exists an equivalent embedding f' such that $\varphi = \pi \circ f'$. In fact, if we determine which strand passes over the other strand on each double point of φ, then a diagram representing an embedding of G is obtained as depicted in Figure 2. A projection is said to be *trivial*, if only trivial embeddings are obtained from the projection.

A set $\mathcal{E}_G$ of nontrivial embeddings of G is said to be *elementary with respect to projection* if it is minimal among sets satisfying a property that *every nontrivial projection of G is a projection of at least one element of $\mathcal{E}_G$.*

Note that although G may have two different elementary sets, the cardinalities of such sets are same. The cardinality of elementary set is dependent on the topological type of graph. Let K be a cycle and L be a disjoint union of more than one cycles. Here, a cycle means a graph homeomorphic to a circle.

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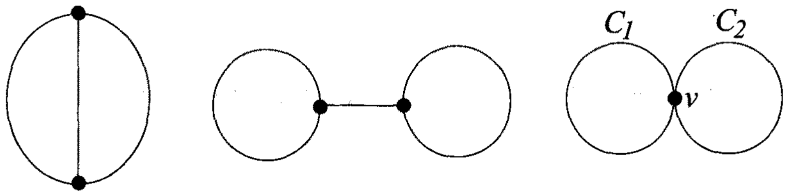

FIGURE 1. Θ_3 graph, handcuff graph and bouquet with two cycles

In [5] K. Taniyama showed that the trefoil knot found in Figure 2-(a) is the only element of $\mathcal{E}_K$. He also showed that $|\mathcal{E}_L| = 2$ [6]. In [2] it was proved that $|\mathcal{E}_{\Theta_m}| = m$ for $m \geq 3$, where Θ_m is the graph consisting of two vertices and m edges between them. On the other hand, $|\mathcal{E}_G| = \infty$, if a graph G contains a subgraph homeomorphic to handcuff graph [8]. Here, the handcuff graph is a graph consisting of two disjoint cycles and one edge joining them. These previous works on elementary sets of small graphs were utilized to study the relation between the combinatorial types of graphs and their projections [3, 4].

In this paper we investigate the elementary set of another small graph B which is the *bouquet with two cycles.* Precisely, B is a graph which consists of two cycles C_1 and C_2 sharing one vertex v as shown in Figure 1. The authors of [8] conjectured $|\mathcal{E}_B| = 2$. But the first result of this paper is

Theorem 1. *No proper subset of* $\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3\}$ *is elementary with respect to projection of* B.

$\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ will denote the embeddings of B which are represented by the diagrams in Figure 2-(b). Theorem 1 implies $|\mathcal{E}_B| \geq 3$.

A nontrivial projection of B is said to be *almost trivial* if the restriction on each cycle is a trivial projection of circle. For a projection φ of B, a double point d will be called a *self-double point of* $\varphi|_{C_i}$ if $\varphi^{-1}(d) \subset C_i$.

As an effort to determine the cardinality of $\mathcal{E}_B$, we give two more theorems in the below.

Theorem 2. *If a nontrivial projection of* B *is not almost trivial, then it is a projection of* $\mathcal{B}_1$.

Theorem 3. *An almost trivial projection* φ *of* B *is a projection of* $\mathcal{B}_2$ *or* $\mathcal{B}_3$, *if each* $\varphi|_{C_i}$ *has at most one self-double point.*

Therefore, for the complete determination of the cardinality of $\mathcal{E}_B$, we have only to investigate the case that φ is almost trivial and some $\varphi|_{C_i}$ has more than one double points. This remaining case will be dealt with in a forthcoming paper [1].

The rest of this paper will be devoted to the proofs of Theorem 1, 2 and 3.

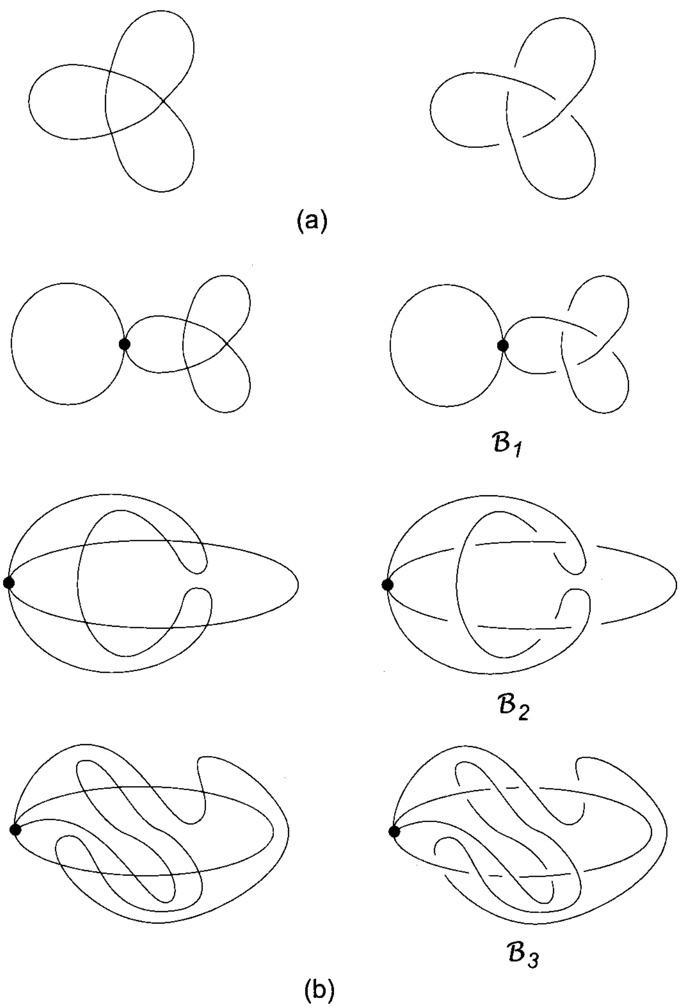

FIGURE 2. (a) A projection of circle and a diagram representing the trefoil knot. (b) Projections of bouquet graph with two cycles and diagrams representing the embeddings $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$

2. Proofs of theorems

Proof of Theorem 1. Let α, β and γ denote the first, second and third projection of B in Figure 2-(b), respectively. And let $R(\alpha)$ be the set of all nontrivial embeddings obtained from α.

The projection α has exactly three double points. Therefore, from α, we obtain eight embeddings of B among which only $\mathcal{B}_1$ and its mirror image are

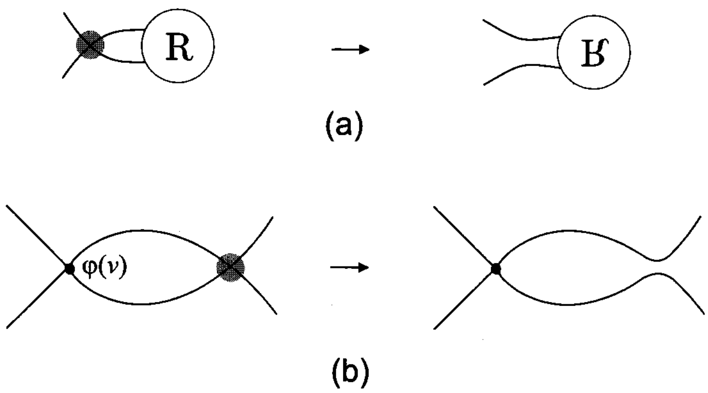

FIGURE 3. Removing a separating nugatory point and a vertex nugatory point

nontrivial. Note that the two embedded graphs contain the trefoil knot or its mirror image. In β, the image of each circle is a trivial projection of circle, for it has no self-double point. Therefore $R(\alpha) \cap R(\beta) = \emptyset$ and similarly $R(\alpha) \cap R(\gamma) = \emptyset$.

The number of nontrivial embeddings from β is four, including $\mathcal{B}_2$. And γ produces thirty two nontrivial embeddings of B, including $\mathcal{B}_3$. To complete the proof, we will show that $R(\beta) \cap R(\gamma) = \emptyset$. For this purpose, we utilize a polynomial invariant of embedded graphs in $\mathbb{R}^3$. In [9] Y. Yokota defined a family of polynomial invariants based on the linear skein theory. The invariants in the family are determined by choosing weight system. Among them, let

$$Z : \{\text{embeddings of } B\} \to \mathbb{Z}[t, t^{-1}]$$

be the polynomial invariant determined by the constant weight 1. Note that if two embeddings f and f' are equivalent, then $Z(f) = Z(f')$. And $Z(f) = 1$ if f is trivial. The width of a polynomial is defined to be the difference between the maximal and minimal degrees of the polynomial.

We traced the widths of polynomials of all nontrivial embeddings from β and γ. The width of $Z(\mathcal{B}_2)$ is 12. Also the polynomials of the other three nontrivial embeddings from β have width 12. The nontrivial embeddings from γ have polynomials with width at least 14, which implies $R(\beta) \cap R(\gamma) = \emptyset$. □

For the proof of the other two theorems we introduce some necessary definitions. For a projection φ of B, let $\varphi_i : [0,1] \to \mathbb{R}^2$ be a parametrization of the curve $\varphi(C_i)$. Then $\varphi_i(0) = \varphi_i(1) = \varphi(v)$. If d is a self-double point of $\varphi|_{C_i}$ then we can choose t_1 and t_2 so that $d = \varphi_i(t_1) = \varphi_i(t_2)$ and $t_1 < t_2$. We call $\varphi_i([t_1, t_2])$ a *loop on* $\varphi(C_i)$ *based at* d.

A *separating nugatory point* is a double point separating $\varphi(B)$ into two disjoint parts. A *vertex nugatory point* is a double point near which a local

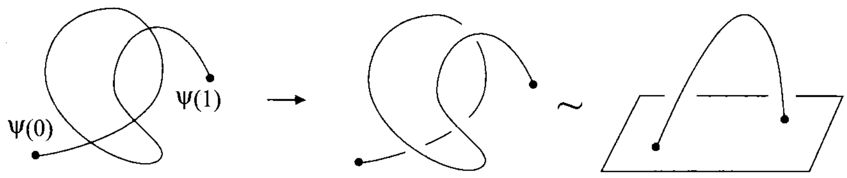

FIGURE 4. Trivialization of a projected arc

picture looks like the left side of Figure 3-(b). If d is a vertex nugatory point then we can find two subarcs c_1 and c_2 of $\varphi(B)$ such that $\partial c_i = \{\varphi(v), d\}$ and there is no double point on the interior of each c_i. We call $c_1 \cup c_2$ a *bigon near* d. A projection will be said to be *reduced* if it has no nugatory point. Suppose that φ has a nugatory point and ψ is a projection obtained by removing the nugatory point as illustrated in Figure 3. If φ is a projection of an embedding f, then we can isotope f so that it is projected onto ψ, that is, ψ is also a projection of f. Therefore, in the proofs of the theorems, we assume that φ is reduced.

Let $\psi : [0,1] \to \mathbb{R}^2$ be a projection of the unit interval. Take a function $h : [0,1] \to [0,\infty)$ so that h is strictly increasing on $[0, 1-\epsilon]$, decreasing on $[1-\epsilon, 1]$ and $h(0) = h(1) = 0$. Then the map $f : [0,1] \to \mathbb{R}^3$ defined by $f(t) = (\psi(t), h(t))$ is an embedding of the unit interval for a small enough ϵ. And f can be isotoped to some f' with boundary fixed so that $\pi \circ f'$ has no double point. We call f a *trivialization of* ψ *into* $\mathbb{R}^3_+$. The trivialization into $\mathbb{R}^3_-$ is defined similarly by taking h into $(-\infty, 0]$.

Proof of Theorem 2. Let φ be a projection of B which is not almost trivial. We may assume that the restriction of φ on the cycle C_1 is a nontrivial projection of circle. Then $\varphi|_{C_1}$ is a projection of the trefoil knot [5]. Select the over/under-passing on each self-double point of $\varphi(C_1)$ so that the resulting embedding of C_1 be the trefoil knot in $\mathbb{R}^2 \times [0,\infty)$. Trivialize the closure of $\varphi(C_2) - N(\varphi(v))$ into $\mathbb{R}^3_-$, where $N(\varphi(v))$ is a small neighborhood of $\varphi(v)$ in $\varphi(C_2)$. And put the rest of $\varphi(C_2)$ into $\mathbb{R}^2 \times \{0\}$. Then the resulting embedding of B is equivalent to $\mathcal{B}_1$. □

Remark. Let ψ be a parametrized projection of circle. In [5] it was shown that ψ is a projection of the trefoil knot if and only if there exists a double point d which is not separating-nugatory, that is, for some $t_1 < t_2$

$$\psi(t_1) = \psi(t_2) = d, \quad \psi([t_1, t_2]) \cap \psi([0, t_1) \cup (t_2, 1]) \neq \emptyset.$$

If every double point of ψ is separating-nugatory then ψ can be reduced to a trivial projection with no double point.

Proof of Theorem 3. Let φ be an almost trivial projection of B. Then Theorem 3 easily follows from the lemmas in the below. □

Lemma 4. *If each $\varphi|_{C_i}$ has no self-double point, then φ is a projection of $\mathcal{B}_2$ or $\mathcal{B}_3$.*

Lemma 5. *If $\varphi|_{C_1}$ has no self-double point and $\varphi|_{C_2}$ has only one self-double point, then φ is a projection of $\mathcal{B}_2$ or $\mathcal{B}_3$.*

Lemma 6. *If each $\varphi|_{C_i}$ has only one self-double point, then φ is a projection of $\mathcal{B}_2$ or $\mathcal{B}_3$.*

In the rest of this paper we will prove the lemmas.

3. Proofs of lemmas

To prove each lemma, considering the conditions given in the lemma, all possible shapes of $\varphi(B)$ will be listed. And we show that the desired embeddings $\mathcal{B}_2$ or $\mathcal{B}_3$ can be obtained from each shape.

Proof of Lemma 4. Let w_{i1} and w_{i2} be the first and last double point along the orientation of φ_i for $i = 1, 2$. Since φ is assumed to be reduced, it has no vertex nugatory point and therefore all these four points are distinct each other. Considering the order in which the four points appear on each φ_i, we have four cases in the below.

	φ_1	φ_2
Case 1	$w_{11}w_{21}w_{22}w_{12}$	$w_{21}w_{11}w_{12}w_{22}$
Case 2	$w_{11}w_{21}w_{22}w_{12}$	$w_{21}w_{12}w_{11}w_{22}$
Case 3	$w_{11}w_{22}w_{21}w_{12}$	$w_{21}w_{11}w_{12}w_{22}$
Case 4	$w_{11}w_{22}w_{21}w_{12}$	$w_{21}w_{12}w_{11}w_{22}$

If we reparametrize φ_2 so that the orientation is reversed, then case 3 and 4 are identical to case 2 and 1, respectively. Now we consider how φ_2 intersects φ_1 in each of case 1 and 2. Without loss of generality, φ_2 is assumed to penetrate φ_1 through w_{21} from the unbounded component D_2 of $\mathbb{R}^2 - \varphi(C_1)$ into the bounded component D_1.

Case 1: Suppose φ_2 penetrates φ_1 through w_{12} from D_1 into D_2. Considering that there is no self-double point, we know that φ_2 penetrates φ_1 through w_{11} from D_2 into D_1, and through w_{22} from D_1 to D_2. The situation is illustrated in the left side of Figure 5-(a). In the figure a subarc is drawn by a dotted curve, when we don't know the relative position of the subarc precisely. Choose t_1, t_2, t_3 and t_4 so that $\varphi_2(t_1) = w_{21}$, $\varphi_2(t_2) = w_{11}$, $\varphi_2(t_3) = w_{12}$ and $\varphi_2(t_4) = w_{22}$. Then, it holds that $t_1 < t_2 < t_3 < t_4$. We assume that φ is put into the plane $\mathbb{R}^3_0 = \mathbb{R}^2 \times \{0\}$. Now trivialize each of $\varphi_2[0, t_1+\epsilon]$, $\varphi_2[t_2-\epsilon, t_3+\epsilon]$ and $\varphi_2[t_4 - \epsilon, 1]$ into $\mathbb{R}^3_+$. Let the other two subarcs of φ_2 be trivialized into $\mathbb{R}^3_-$. Keep φ_1 into $\mathbb{R}^3_0$ as it is. Then we obtain $\mathcal{B}_2$ as illustrated in the right side of Figure 5-(a). In the figure, a subarc that should be trivialized into $\mathbb{R}^3_+$ (*resp.* $\mathbb{R}^3_-$) is represented by a black thick (*reps.* gray) curve. A black thin curve represents a subarc that should remain in $\mathbb{R}^3_0$.

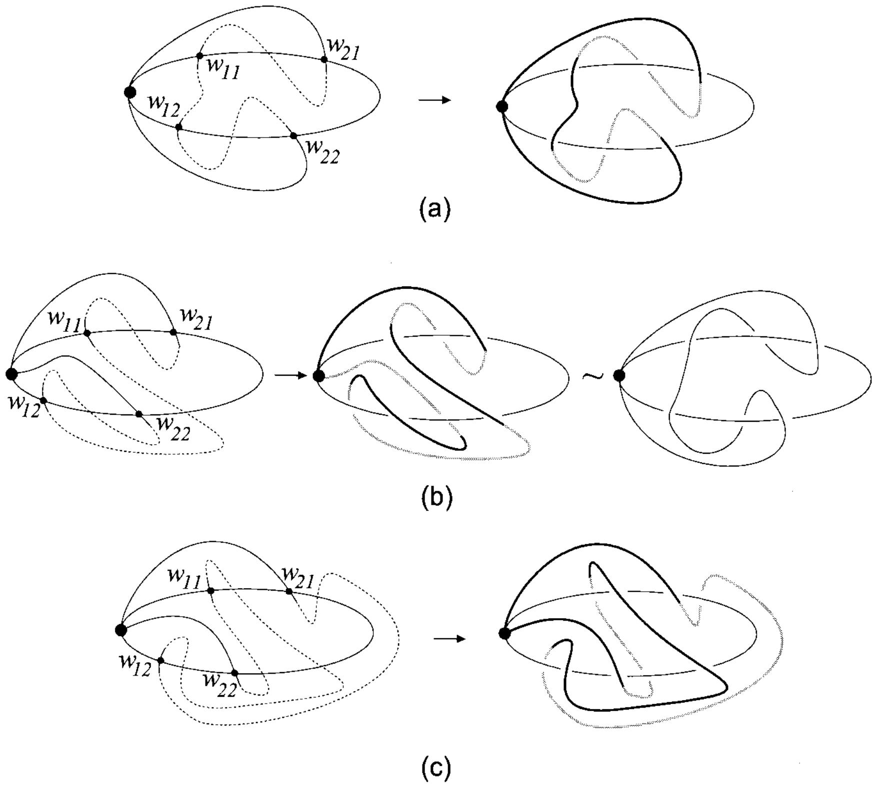

FIGURE 5. Realization of $\mathcal{B}_2$ and $\mathcal{B}_3$ from a projection with no self-double point

If φ_2 penetrates φ_1 through w_{12} from D_2 into D_1, then φ_2 should pass through w_{22} from D_2 to D_1. We can also obtain $\mathcal{B}_2$ as illustrated in Figure 5-(b).

Case 2: In this case, it is easy to know that φ_2 should penetrate φ_1 through w_{12}, w_{11} and w_{22} from D_2 into D_1. Therefore $\mathcal{B}_3$ is obtained as illustrated in Figure 5-(c). $\square$

Proof of Lemma 5. Let w be the self-double point of $\varphi(C_2)$ and l be the loop based at w. Let φ' be a projection of B such that

$$\varphi'(C_1) = \varphi(C_1), \quad \varphi'(C_2) = \text{closure of } \varphi(C_2) - l.$$

Note that each of $\varphi'(C_i)$ has no self-double point. Let D_1 and D_2 be the bounded and unbounded components of $\mathbb{R}^2 - \varphi'(C_1)$, respectively.

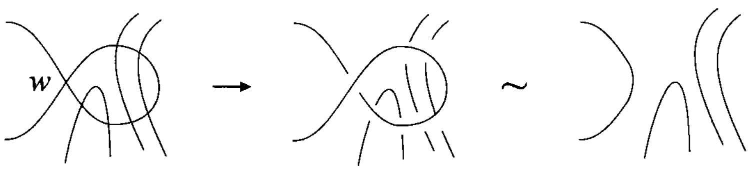

FIGURE 6

Case 1: Suppose that φ' is a nontrivial projection of B. Then, by Lemma 4, an embedding $\mathcal{B}$ equivalent to $\mathcal{B}_2$ or $\mathcal{B}_3$ can be obtained from the projection. Now we will determine over/under-passings at the double points of φ so that the resulting embedding is equivalent to $\mathcal{B}$. Consider $\varphi'(B)$ as a subspace of $\varphi(B)$. At each double point between $\varphi(C_1)$ and $\varphi(C_2) - l$, set the over/under-passing in the same way with φ'. At each double point between $\varphi(C_1)$ and l, let l pass over $\varphi(C_1)$. If we isotope the resulting embedding to reduce the lift of l as illustrated in Figure 6, then $\mathcal{B}$ is obtained.

Case 2: Suppose that φ' is a trivial projection. Then, by Lemma 4, φ' should not be reduced, that is, have nugatory points, all of which should be vertex-nugatory.

If there exist two vertex nugatory points in φ', we may assume that the first (also the last) double points of the parametrizations φ'_1 and φ'_2 coincide with each other, by reversing the orientation of φ'_1 if necessary. Select t_1 and s_1 (*resp.* t_2 *and* s_2) so that $\varphi'_1(t_1) = \varphi'_2(s_1)$ (*resp.* $\varphi'_1(t_2) = \varphi'_2(s_2)$) is the first (*resp.* *last*) double point. Now we consider how the loop l should be added to $\varphi'(B)$ so that the resulting image be identical with $\varphi(B)$. Because φ is reduced, $\varphi'_2(s_1)$ and $\varphi'_2(s_2)$ should not be vertex nugatory any longer after adding l, which implies that l should have nonempty intersection with each bigon near $\varphi'_2(s_1)$ and $\varphi'_2(s_2)$. The base point w of l may be located on one of $\varphi'_2(0, s_1)$, $\varphi'_2(s_1, s_2)$ or $\varphi'_2(s_2, 1)$. According to the positions of w and the subarc $\varphi'_2(s_2, 1)$, we can list possible shapes of $\varphi(B)$. Considering $\mathbb{R}^2$ as a subspace of the sphere $\mathbb{S}^2$ and reversing the orientation of φ'_2, all possible shapes can be reduced to three shapes, from each of which an embedding equivalent to $\mathcal{B}_2$ is obtained as illustrated in Figure 7. Note that $\varphi'_2(0, s_1)$ can be assumed to be contained in the unbounded component D_2 without loss of generality. In Figure 7-(b) it looks as if the base point w is located in D_1. Even in the case that w is located in D_2, we obtain the desired embedding by the same trivializations.

When there is only one vertex nugatory point in φ', we may also assume that the first double points of φ'_1 and φ'_2 are identical with each other. Note that if there exists only one double point in φ', then four bigons exist in the projection and there is no way to add l so that φ is reduced. Let $\varphi'_1(t_1) = \varphi'_2(s_1)$ be the first double point. And let $\varphi'_1(t_2) = \varphi'_2(s_2)$ be the last double point of φ'_2.

Then we can select $\{u_{ij}\}$ such that

$$t_2 < u_{11} < u_{12}, \quad u_{21} < u_{22} < s_2,$$
$$\varphi_1'(u_{11}) = \varphi_2'(u_{22}) \quad \text{and} \quad \varphi_1'(u_{12}) = \varphi_2'(u_{21})$$

because there should exist only one vertex nugatory point. Note that $\varphi_2'(0, s_1)$ can be assumed to be contained in D_2 without loss of generality. Then we have two possible shapes of $\varphi'(B)$ as illustrated in Figure 8. If we add l onto $\varphi'(B)$ to recover $\varphi(B)$, the base point w may be located on one of $\varphi_2'(0, s_1)$, $\varphi_2'(s_1, u_{21})$, $\varphi_2'(u_{21}, u_{22})$, $\varphi_2'(u_{22}, s_2)$ and $\varphi_2'(s_2, 1)$. Figure 9 shows the possible shapes of $\varphi(B)$, when $\varphi_2'(s_2, 1)$ is contained in D_2. From each of the shapes we can obtain $\mathcal{B}_2$ by the trivializations illustrated in the figure. In the cases of (c), (d) and (e), because we don't know the exact position of w relative to $\varphi'(C_1)$, the subarcs connecting w are lifted into the same half space so that they have relative heights as seen in the figure. If $w \in \varphi_2'(u_{22}, s_2)$ and $l \cap \varphi_1'(u_{12}, 1) \neq \emptyset$, then the shape corresponds to (e). If $w \in \varphi_2'(u_{22}, s_2)$ and $l \cap \varphi_1'(u_{12}, 1) = \emptyset$, then we can find another $\{u_{ij}\}$ so that the shape corresponds to (c).

When $\varphi_2'(s_2, 1)$ is contained in the bounded region D_1, we can apply similar arguments to get $\mathcal{B}_2$. □

Proof of Lemma 6. Let w_i be the self-double point of $\varphi(C_i)$ for $i = 1, 2$. The loop based at w_i is denoted by l_i. For each i, select t_{i1} and t_{i2} so that $\varphi_i(t_{i1}) = \varphi_i(t_{i2}) = w_i$ with $t_{i1} < t_{i2}$. Let m_i be the closure of $\varphi(C_i) - l_i$, that is, $m_i = \varphi_i([0, t_{i1}] \cup [t_{i2}, 1])$. And let D_1 and D_2 be the bounded and unbounded connected component of $\mathbb{R}^2 - m_1$, respectively. Without loss of generality we may assume that $\varphi_1(t_{11}, t_{12})$ is contained in D_2.

Note that each of $m_1 \cup \varphi(C_2)$ and $\varphi(C_1) \cup m_2$ can be considered as the image of another projection of B. If we can obtain a nontrivial embedding from $m_1 \cup \varphi(C_2)$, then an equivalent embedding is obtained also from $\varphi(B)$ as discussed in the proof of Lemma 5. And the obtained embedding should be equivalent to $\mathcal{B}_2$ or $\mathcal{B}_3$. Therefore we may assume that each of $m_1 \cup \varphi(C_2)$ and $\varphi(C_1) \cup m_2$ is the image of a trivial projection.

Case 1. Suppose $l_1 \cap l_2 = \emptyset$. Then $l_1 \cap m_2$ and $m_1 \cap l_2$ should be nonempty because φ is reduced. We may assume that $\varphi_2(t') \in \varphi_1(t_{11}, t_{12})$ for some $t' \in (0, t_{21})$, by reversing the orientation of φ_2 if necessary. Now we consider how $\varphi(C_2)$ should be added onto $\varphi(C_1)$ in $\mathbb{R}^2$ so that φ is reduced. The base point w_2 of the loop l_2 may be located in D_1 or in D_2, but should not be inside of l_1. According to the position and orientation of w_2, we have four possible shapes of $\varphi(B)$ from each of which $\mathcal{B}_2$ is obtained by the trivializations illustrated in Figure 10. Let e_1 and e_2 be two simple subarcs of $\varphi(C_1)$ which are contained in a small neighborhood of w_1 and intersect each other at w_1. The two subarcs were lifted into the small 3-ball with center $(w_1, 0) \in \mathbb{R}^2 \times \{0\}$, so that they have relative heights as seen in the figure. For simplicity, their lifts were represented by thin curves.

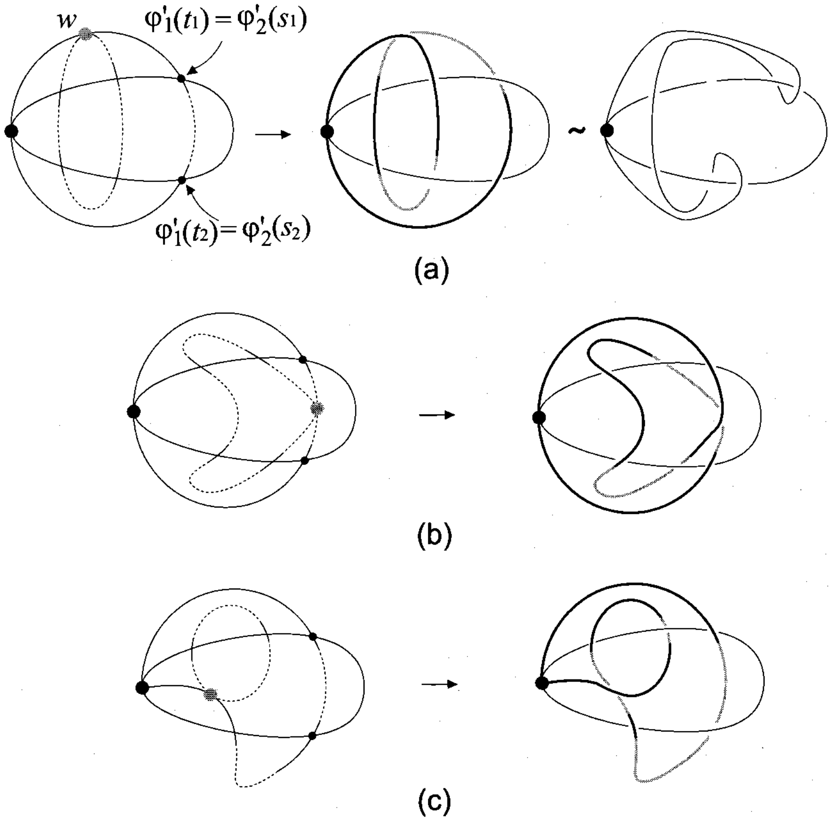

FIGURE 7

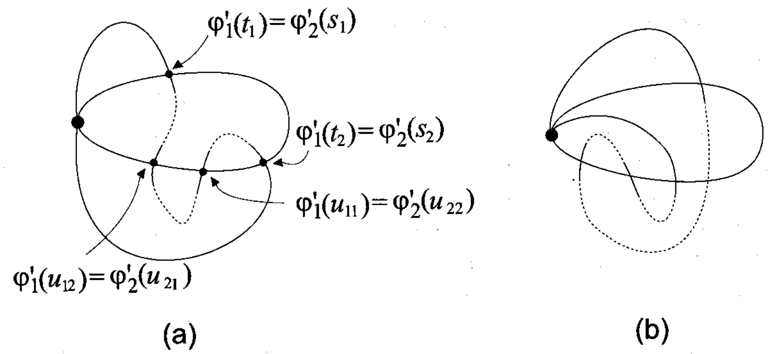

FIGURE 8

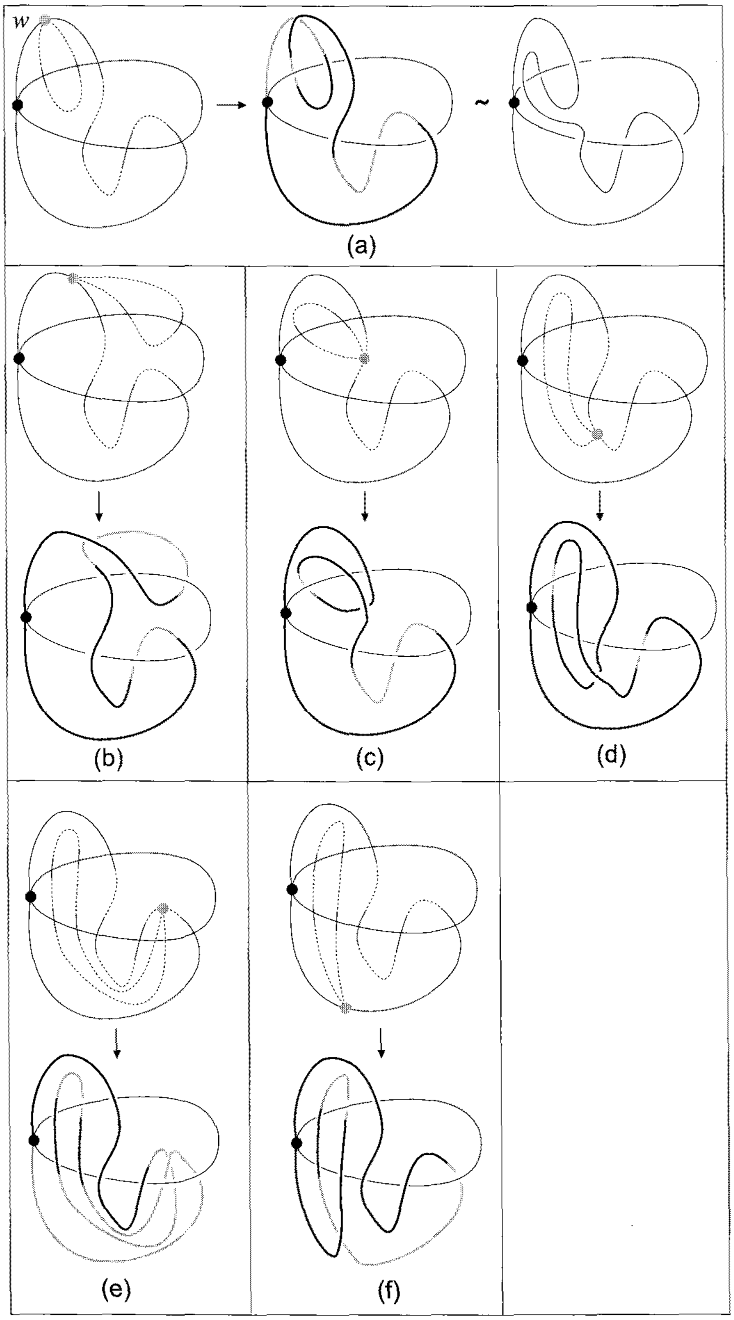

FIGURE 9

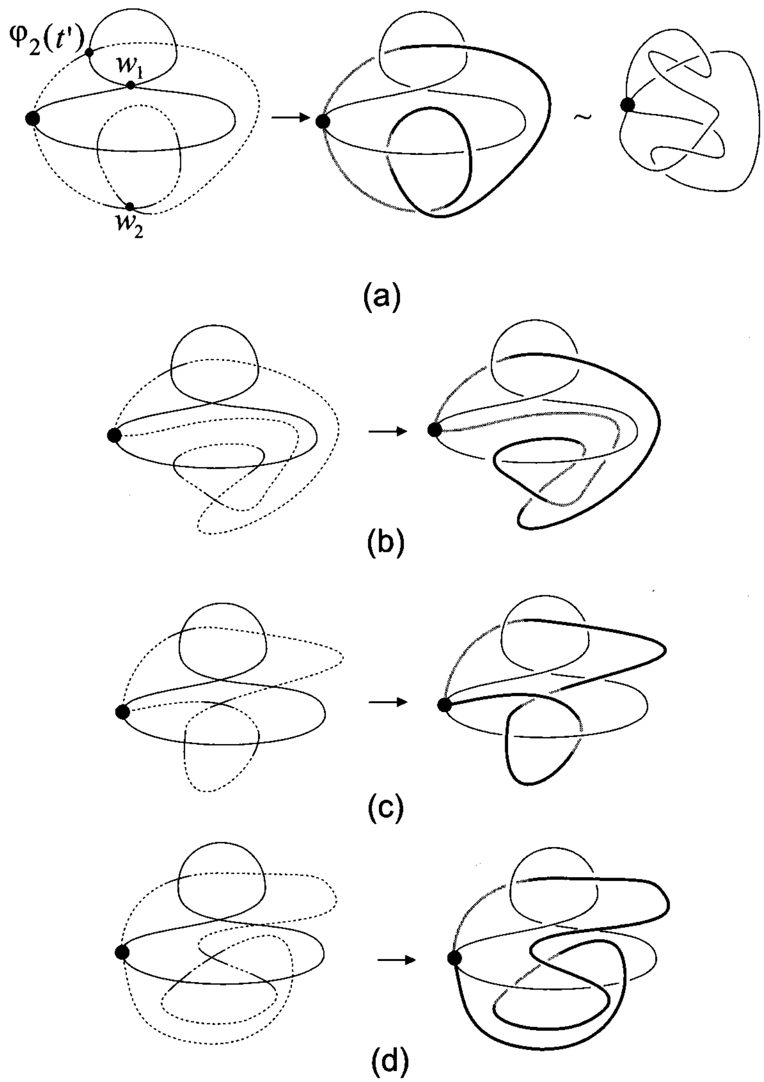

FIGURE 10

Case 2. Suppose that $l_1 \cap l_2 \neq \emptyset$, and that each of $\varphi(C_1) \cup m_2$ and $m_1 \cup \varphi(C_2)$ has a vertex nugatory point. Then we have three cases to consider.

Case 2-1. Suppose that there exist two vertex nugatory points in $\varphi(C_1) \cup m_2$. Let v_1 and v_2 be the vertex nugatory points. Then we can find s_{i1} and s_{i2} such that $v_1 = \varphi_1(s_{11}) = \varphi_2(s_{21})$, $v_2 = \varphi_1(s_{12}) = \varphi_2(s_{22})$ and $s_{i1} < s_{i2}$. If we consider $\varphi(C_1) \cup m_2$ as a subspace of $\varphi(B)$, then l_2 should be added onto

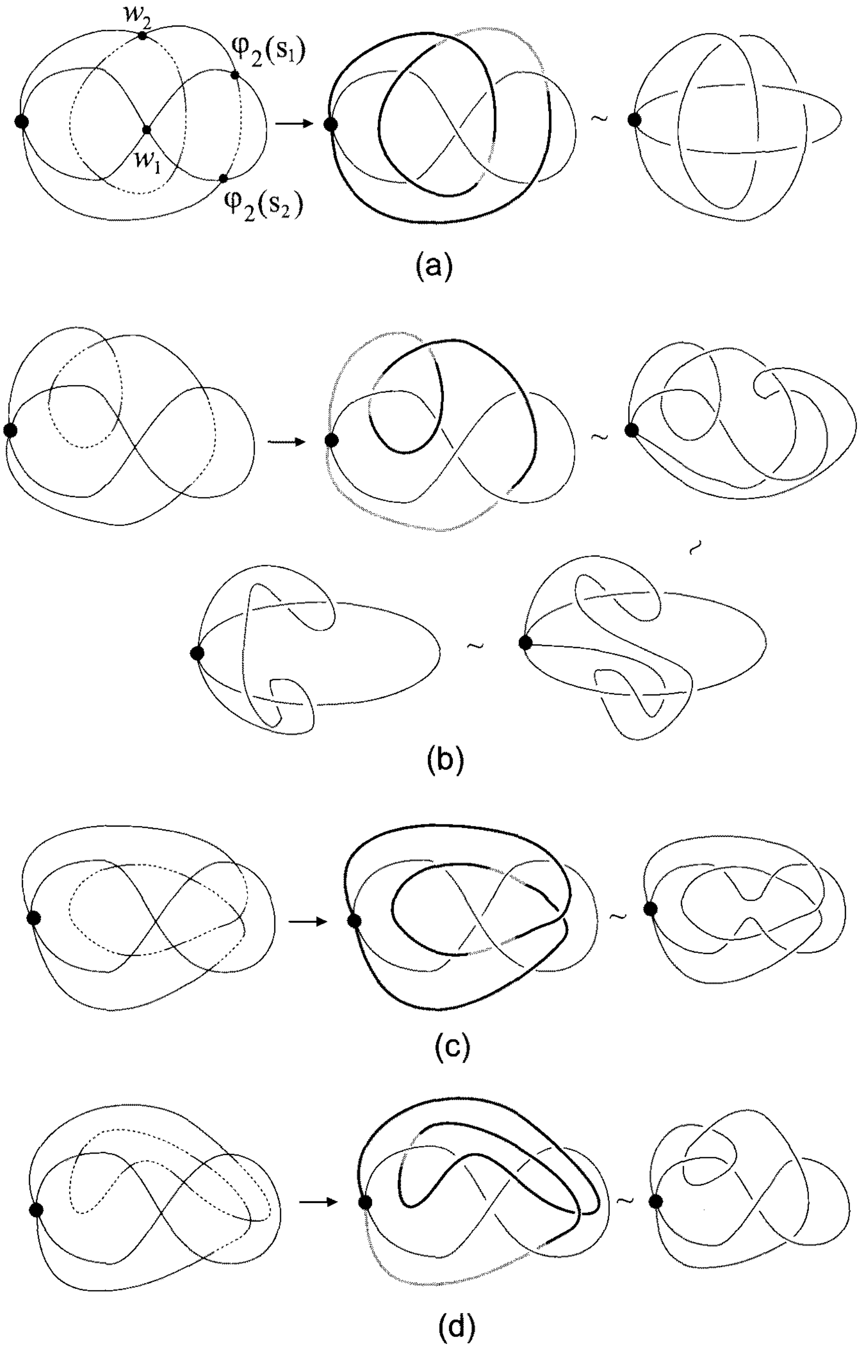

FIGURE 11

$\varphi(C_1) \cup m_2$ so that no vertex nugatory point is allowed. This consideration implies that one of three subcases happens as in the below:

(i) $l_2 \cap \varphi_1(0, s_{11}) \neq \emptyset$ and $l_2 \cap \varphi_1(s_{12}, 1) \neq \emptyset$,
(ii) $w_2 \in \varphi_2(0, s_{21})$ and $l_2 \cap \varphi_1(s_{12}, 1) \neq \emptyset$,
(iii) $l_2 \cap \varphi_1(0, s_{11}) \neq \emptyset$ and $w_2 \in \varphi_2(s_{21}, 1)$.

In any of the subscases, $m_1 \cup \varphi(C_2)$ is reduced. Therefore, by Lemma 5, we know that our case can not happen because $m_1 \cup \varphi(C_2)$ was assumed to be trivial.

Case 2-2. Suppose that w_1 is the only vertex nugatory point in $\varphi(C_1) \cup m_2$. Then $m_1 \cap m_2 = \emptyset$. Because φ is reduced, none of $m_1 \cap l_2$ and $l_1 \cap m_2$ is empty. Now we imagine how l_2 should be added onto $\varphi(C_1) \cup m_2$ so that the resulting image is identical with $\varphi(B)$. Choose s_1 and s_2 from the set $S = (0, t_{21}) \cup (t_{22}, 1)$ so that

- $\varphi_2(s_1), \varphi_2(s_2) \in \varphi(C_1)$ and
- if $\varphi_2(s) \in \varphi(C_1)$ for some $s \in S$, then $s_1 \leq s \leq s_2$.

Suppose $t_{21}, t_{22} < s_2$. Then, according to whether $t_{21}, t_{22} < s_1$ and whether l_2 is trivial as an element of $\pi_1(\mathbb{R}^2 - w_1)$, we have four types of shapes as depicted in Figure 11. From each type of the projected image, $\mathcal{B}_2$ is obtained by the trivialization as illustrated in the figure. In (c) and (d) of the figure, it looks as if w_2 is located inside of l_1. The four subarcs of $\varphi(C_2)$ connecting w_2 are trivialized into $\mathbb{R}^3_+$ so that, near w_2, they have relative heights as seen in the figure. Then $\mathcal{B}_2$ can be obtained irrespective of the position of w_2 relative to l_1.

Similarly we can obtain $\mathcal{B}_2$ in the case that $t_{21}, t_{22} > s_2$.

Case 2-3. By the discussion in Case 2-1, it suffices to deal with the case that there is only one vertex nugatory point in each of $\varphi(C_1) \cup m_2$ and $m_1 \cup \varphi(C_2)$. And by Case 2-2 we may assume that the vertex nugatory points are not any of w_1 and w_2. Let v_1 (*resp.* v_2) be the vertex nugatory point of $\varphi(C_1) \cup m_2$ (*resp.* $m_1 \cup \varphi(C_2)$). Then, after reversing orientation if necessary, we can select s_1 and s_2 so that

- $\varphi_1(s_1) = \varphi_2(s_2) = v_1$,
- $\varphi_1(0, s_1) \cap m_2 = \emptyset$,
- $s_1 < t_{11}, t_{12}$ and
- $\varphi_2((0, s_2) - (t_{21}, t_{22})) \cap \varphi(C_1) = \emptyset$.

Suppose that $s_2 > t_{21}, t_{22}$, equivalently, the base point w_2 of l_2 is located on a bigon of $\varphi(C_1) \cup m_2$ near v_1. Now we imagine how φ_2 will go forward along its orientation after passing v_1.

Firstly observe the case that $\varphi_2((0, s_2) - (t_{21}, t_{22}))$ is contained in the unbounded region D_2. Let $\varphi_2(s) = \varphi_1(s')$ be the double point such that $s > s_2$ and $\varphi_2(s_2, s) \cap \varphi(C_1) = \emptyset$. If $s = 1$, then a bigon should exist near v_1 in $\varphi(B)$. Therefore $s < 1$ and there are three possible subcases as given in the below:

(i) $\varphi_2(s) \in \varphi_1(s_1, t_{11})$,
(ii) $\varphi_2(s) \in \varphi_1(t_{12}, 1)$ and $\varphi_2(s, 1) \cap m_1 = \emptyset$,
(iii) $\varphi_2(s) \in \varphi_1(t_{12}, 1)$ and $\varphi_2(s, 1) \cap m_1 \neq \emptyset$.

In subcase (ii), $\varphi(s, 1)$ should have intersection with l_1, for there exists only one vertex nugatory point in $\varphi(C_1) \cup m_2$. Figure 12-(a) and (b) represent the shapes corresponding to subcase (i) and (ii), respectively. In subcase (iii), if we can find u_{i1} and u_{i2} such that

$$t_{12} < u_{11} < u_{12},\ s \leq u_{21} < u_{22},$$
$$\varphi_1(u_{11}) = \varphi_2(u_{22}) \text{ and } \varphi_1(u_{12}) = \varphi_2(u_{21}),$$

then the shape of $\varphi(B)$ looks like Figure 12-(c). If there is no such $\{u_{ij}\}$, then $\varphi(s_2, 1) \cap l_1 \neq \emptyset$ and the shape of $\varphi(B)$ corresponds to Figure 12-(b). In each subcase we can obtain $\mathcal{B}_2$ by the trivialization illustrated in the figure. Also when $\varphi_2((0, s_2) - (t_{21}, t_{22}))$ is contained in D_1, we can apply similar arguments to get $\mathcal{B}_2$.

Suppose that the base point of l_2 (*resp.* l_1) is not located on any bigon of $\varphi(C_1) \cup m_2$ (*resp.* $m_1 \cup \varphi(C_2)$) near v_1 (*resp.* v_2). This assumption gives us two restrictions on the shape of $m_1 \cup m_2$. Firstly, $m_1 \cup m_2$ should have at least two double points. Otherwise we can not find any point at which l_2 is based because every connected component of $m_2 - \{\varphi(v), v_1\}$ is a subarc of some bigon near v_1 in $\varphi(C_1) \cup m_2$. Secondly, there exist at least two bigons in $m_1 \cup m_2$. A bigon in $\varphi(C_1) \cup m_2$ (*resp.* $m_2 \cup \varphi(C_2)$) is also a bigon in $m_1 \cup m_2$. If there is only one bigon in $m_1 \cup m_2$ then it should be a bigon in both of $\varphi(C_1) \cup m_2$ and $m_1 \cup \varphi(C_2)$, and therefore even in $\varphi(B)$, which is contradictory. The two constraints imply that $m_1 \cup m_2$ has two vertex-nugatory points. Figure 13-(a) shows two possible shapes of $m_1 \cup m_2$ according to the relative positions of the two bigons. Now we consider how to add l_1 and l_2 onto $m_1 \cup m_2$ to recover φ. Each bigon should intersects at least one of the loops. And each loop should intersects at least one of the bigons without its base point on any of the bigons. Therefore we have two possible shapes of $\varphi(B)$, in each of which $\mathcal{B}_2$ is obtained by the trivializations as illustrated in Figure 13-(b) and (c). In Figure 13-(b), all of the subarcs connecting w_2 are trivialized into $\mathbb{R}^3_+$ so that, near w_2, they have relative heights as illustrated. In Figure 13-(c), they are trivialized into $\mathbb{R}^3_-$. In the two figures it looks as if w_2 is located in D_2. Also when the point is located inside of m_1, we obtain $\mathcal{B}_2$ by the same trivializations.

Before going into *Case 3*, we recall a notation. For a projection ψ, $R(\psi)$ denotes the set of all nontrivial embeddings obtained from ψ.

Case 3. Suppose that $l_1 \cap l_2 \neq \emptyset$ and $\varphi(C_1) \cup m_2$ has no vertex nugatory point. Because $\varphi(C_1) \cup m_2$ is assumed to the image of a trivial projection, there exists a nugatory point in $\varphi(C_1) \cup m_2$. By the assumption of this case, the nugatory point should be the separating nugatory point w_1, which implies

$$l_1 \cap m_2 = \emptyset \quad \text{and} \quad R(m_1 \cup m_2) = R(\varphi(C_1) \cup m_2) = \emptyset.$$

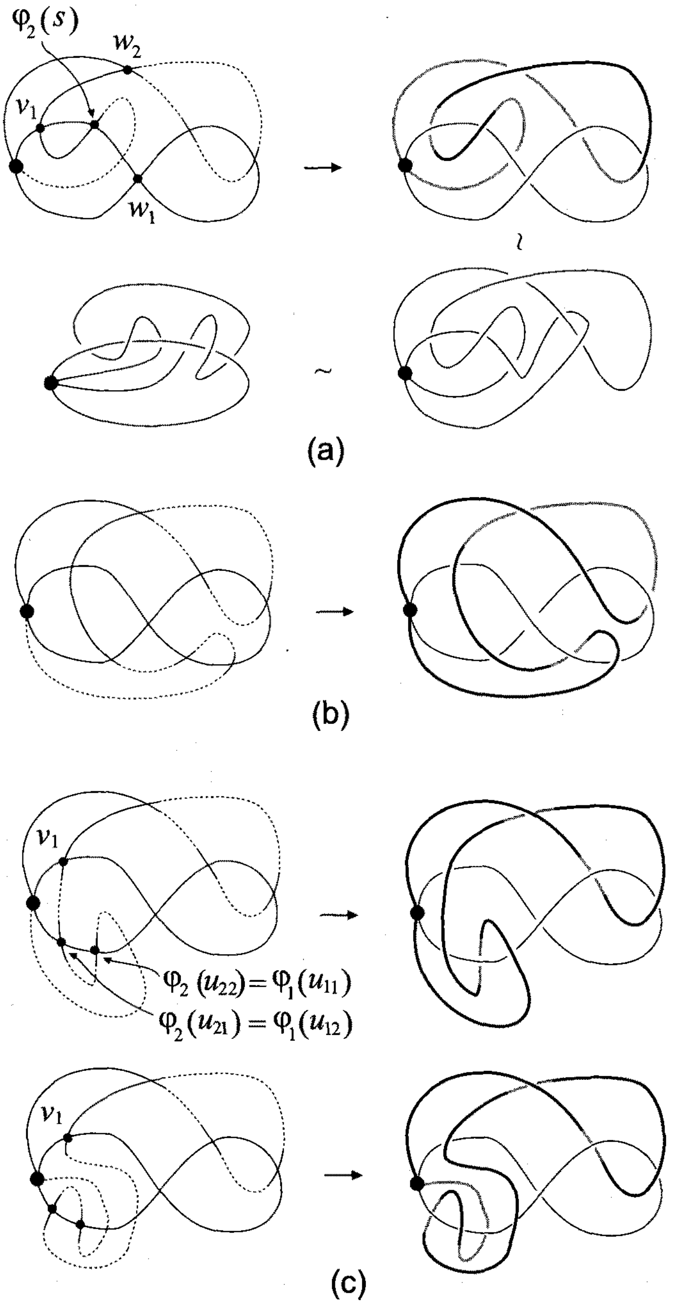

FIGURE 12. (a) subcase-(i). (b) subcase-(ii). (c) subcase-(iii)

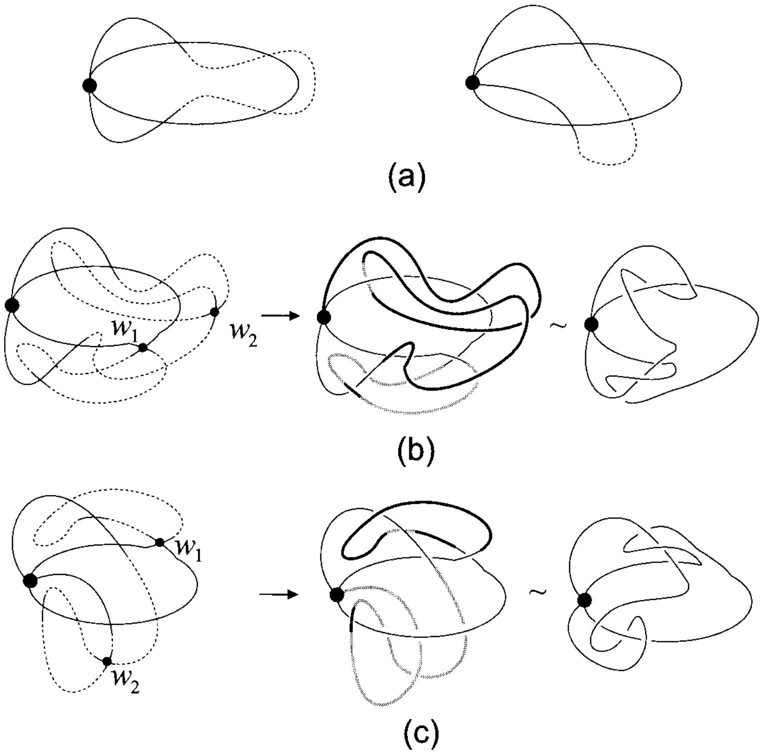

FIGURE 13. (a) two possible shapes of $m_1 \cup m_2$. (b),(c) Realization of $\mathcal{B}_2$ from each shape.

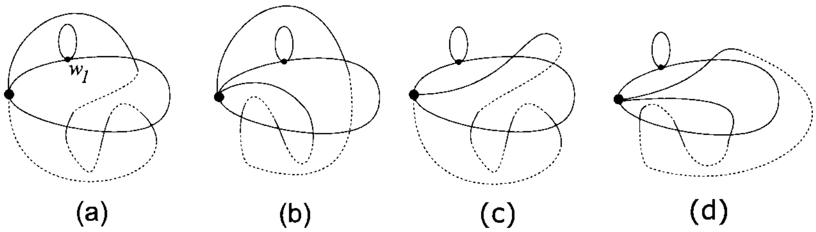

FIGURE 14

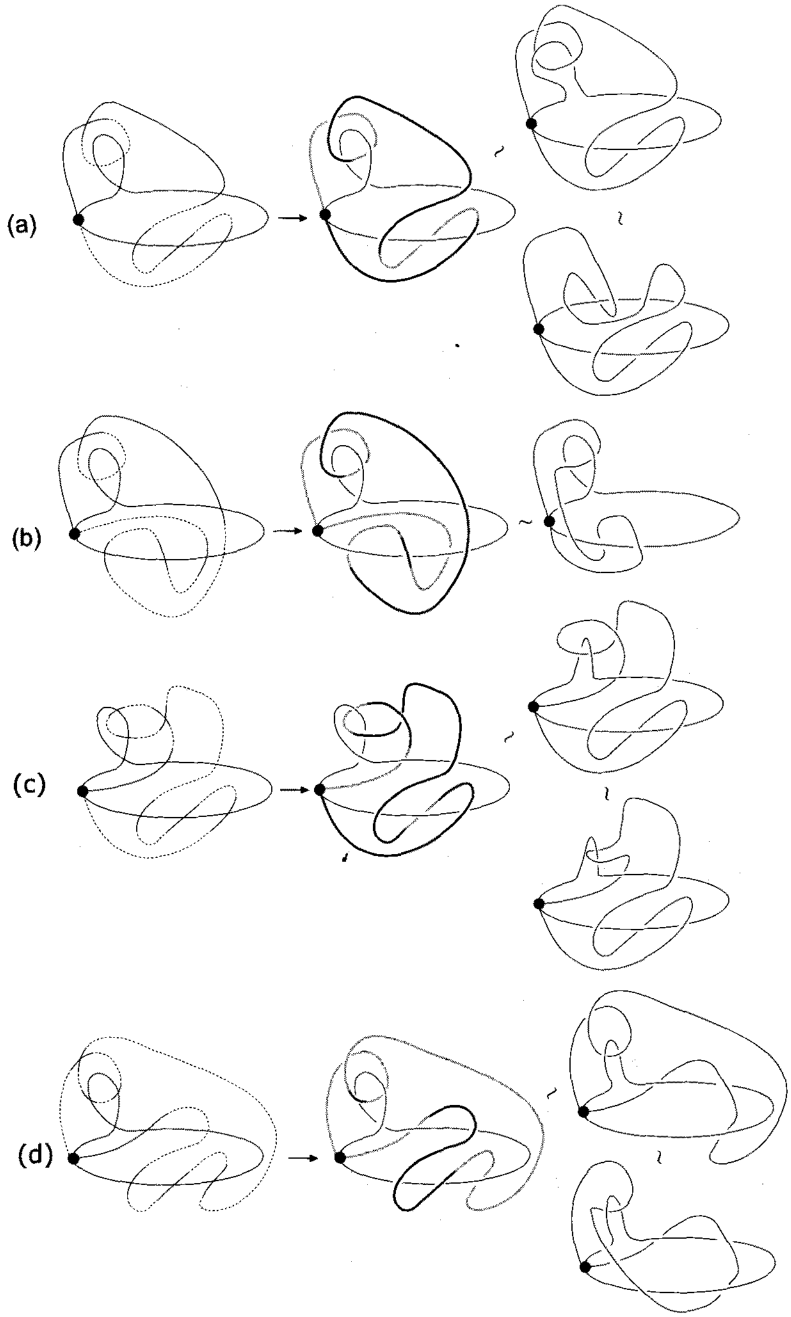

FIGURE 15

If $m_1 \cap m_2 = \emptyset$, then m_2 is a bigon in $\varphi(B)$, which is contradictory to that φ is reduced. If $m_1 \cap m_2 \neq \emptyset$, by Lemma 4, there exists a bigon in $m_1 \cup m_2$. Again by the assumption of this case, none of such bigons is a bigon in $\varphi(C_1) \cup m_2$. Because $l_1 \cap m_2 = \emptyset$, the base point w_1 of l_1 should be located on a bigon b which is the only bigon in $m_1 \cup m_2$. Let v_1 be the vertex nugatory point in $m_1 \cup m_2$ corresponding to the bigon b. And let $\varphi_1(s_1) = \varphi_2(s_2) = v_1$. Because there is only one bigon in $m_1 \cup m_2$, we can select $\{u_{ij}\}$ such that

$$t_{12} < s_1 < u_{11} < u_{12} < u_{13}, \quad s_2 < u_{21} < u_{22} < u_{23},$$

$$\varphi_1(u_{11}) = \varphi_2(u_{23}), \quad \varphi_1(u_{12}) = \varphi_2(u_{22}) \quad \text{and} \quad \varphi_1(u_{13}) = \varphi_2(u_{21}).$$

Then according to the orientations of $\varphi_2(s_2)$ and $\varphi_2(u_{23})$, we have four possible shapes of $\varphi(C_1) \cup m_2$ as given in Figure 14. Note that the interior of l_2 should not intersect b, so that $m_1 \cup \varphi(C_2)$ is not reduced. If $\varphi(C_1) \cup m_2$ looks like Figure 14-(a) or (b), then $t_{22} < s_2$ and therefore the shape of $\varphi(B)$ corresponds to Figure 15-(a) or (b) from which $\mathcal{B}_2$ is obtained as illustrated.

In case of (c), t_{22} may be contained in one of the open intervals (s_2, u_{21}), (u_{21}, u_{22}), (u_{22}, u_{23}) and $(u_{23}, 1)$. If $t_{22} \in (s_2, u_{21})$ or $(u_{23}, 1)$, then $\varphi(B)$ can be lifted to $\mathcal{B}_2$ by the trivializations illustrated in Figure 15-(c) or (d). The four subarcs of $\varphi(C_2)$ connecting w_2 were lifted into the same half space because we don't know the position of w_2 relative to m_1. When t_{22} is contained in (u_{21}, u_{22}), we set the over/under passsings so that l_2 passes over $\varphi_1(s_1, 1)$ and the subarcs connected to w_2 are lifted into $\mathbb{R}^3_+$. And the under/over passings at the other double points are chosen in the same way with the case that $t_{22} \in (s_2, u_{21})$. When $t_{22} \in (u_{22}, u_{23})$, we set the over/under passsings so that l_2 passes under $\varphi_1(s_1, 1)$ and the subarcs connected to w_2 are lifted into $\mathbb{R}^3_-$. The under/over passings at the other double points are chosen in the same way with the case that $t_{22} \in (u_{23}, 1)$. Then, in both cases, we obtain an embedding which has same conformation with the embedding in Figure 15-(c) or (d), after moving the lifts of w_2 toward $\varphi_2(u_{21} - \epsilon)$ or $\varphi_2(u_{23} + \epsilon)$ isotopically. In case of (d), we can apply similar arguments to obtain $\mathcal{B}_2$. □

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