J. Korean Math. Soc. **45** (2008), No. 5, pp. 1379–1391

# ON THE INFINITE PRODUCTS DERIVED FROM THETA SERIES II

Daeyeoul Kim and Ja Kyung Koo

Abstract. Let $k$ be an imaginary quadratic field, $\mathfrak{h}$ the complex upper half plane, and let $\tau \in \mathfrak{h} \cap k$, $q = e^{\pi i \tau}$. For $n, t \in \mathbb{Z}^+$ with $1 \le t \le n-1$, set $n = \mathfrak{z} \cdot 2^l$ ($\mathfrak{z} = 2, 3, 5, 7, 9, 13, 15$) with $l \ge 0$ integer. Then we show that $q^{\frac{n}{12} - \frac{t}{2} + \frac{t^2}{2n}} \prod_{m=1}^{\infty}(1 - q^{nm-t})(1 - q^{nm-(n-t)})$ are algebraic numbers.

## § 1. Introduction

Ramanujan discovered important $q$-series and theta series, and he further developed several profound theorems in the study of theta series. In this paper we shall examine certain family of algebraic numbers as values of infinite products constructed from his theta series by using Berndt's idea. Precisely speaking, let us consider the following theta series

$$f(a,b) = 1 + \sum_{m=1}^{\infty} (ab)^{m(m-1)/2}(a^m + b^m) = \sum_{m=-\infty}^{\infty} a^{m(m+1)/2} b^{m(m-1)/2},$$

where $|ab| < 1$. If we set $a = qe^{2iz}$ and $b = qe^{-2iz}$ with $z$ a complex number and $\mathrm{Im}(\tau) > 0$, then $f(a,b)$ is none other than a classical theta series $\theta_3(z,\tau)$ in its standard notation ([10, p.464]). Berndt then found many interesting formulas for Ramanujan's theta series in [1], [2], [3]. Of these formulas we list the following two identities for later use:

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \quad ([2, \text{p.35}])$$
$$f(a,b) + f(-a,-b) = 2f(a^3b, ab^3), \quad ([2, \text{p.46}])$$

where $(a;b)_\infty = \prod_{m=0}^{\infty}(1 - ab^m)$.

We first obtain from these properties two lemmas (Lemma 2.1, Lemma 2.2) about algebraic numbers which can be derived from certain infinite products.

Next, Gelfond and Schneider ([7], [9]) independently solved in 1949 the famous Hilbert 7-th problem concerning the transcendence of $2^{\sqrt{2}}$. They actually

Received January 17, 2007.

2000 *Mathematics Subject Classification.* 11Jxx, 11R04, 11F11.

*Key words and phrases.* algebraic number, theta series, Rogers-Ramanujan identities.

This work was partially supported by the SRC Program of KOSEF Research Grant R11-2007-035-01001-0.

proved the following strong transcendence criterion. For $\alpha$, $\beta \in \overline{\mathbb{Q}}$ with $\alpha \neq 0, 1$ and $\beta \notin \mathbb{Q}$, $\alpha^\beta$ is transcendental. Therefore, for $\tau \in k \cap \mathfrak{h}$ the Gelfond-Schneider theorem yields that $e^{\pi\alpha} = (-1)^{-i\alpha}$ is transcendental whenever $i\alpha$ is algebraic of degree at least 2 over $\mathbb{Q}$. This leads us to the fact that

$$q = e^{\pi i \tau} \text{ is a transcendental number.} \tag{1.1}$$

Let $\hat{K}(A, B) := (A; B)_\infty(\frac{B}{A}; B)_\infty$. We obtained in [5] that $q^a\hat{K}(q^t, q^n)$ are algebraic numbers, where the 3-tuple $(a, n, t)$ runs over the following cases:

$$\begin{aligned}
&(-\frac{1}{12}, 2, 1), (-\frac{1}{12}, 3, 1), (-\frac{1}{24}, 4, 1), (\frac{1}{60}, 5, 1), (-\frac{11}{60}, 5, 2),\\
&(\frac{1}{12}, 6, 1), (\frac{13}{84}, 7, 1), (-\frac{11}{84}, 7, 2), (-\frac{23}{84}, 7, 3),\\
&(\frac{11}{48}, 8, 1), (-\frac{13}{48}, 8, 3), (\frac{11}{36}, 9, 1), (-\frac{1}{36}, 9, 2), (-\frac{13}{36}, 9, 4),\\
&(\frac{23}{60}, 10, 1), (-\frac{13}{60}, 10, 3), (\frac{13}{24}, 12, 1), (-\frac{11}{24}, 12, 5), (\frac{59}{84}, 14, 1),\\
&(-\frac{1}{84}, 14, 3), (-\frac{37}{84}, 14, 5), (\frac{37}{36}, 18, 1), (-\frac{11}{36}, 18, 5), (-\frac{23}{36}, 18, 7).
\end{aligned} \tag{1.2}$$

In Theorem 3.1 and Theorem 4.2 we prove that $q^{\frac{n}{12} - \frac{t}{2} + \frac{t^2}{2n}}\hat{K}(q^t, q^n)$ is an algebraic number for $n = \mathfrak{z} \cdot 2^l$ ($\mathfrak{z} = 2, 3, 5, 7, 9, 13, 15$) and $l \geq 0$ using the properties of theta series and (1.1), which would be a generalization of our previous works ((1.2) and Lemma 2.0). On the other hand Sill recently investigated in [9] several properties for double sums and infinite products. By using his result we get 21 algebraic numbers as values of some double sums (Example 4.3). We also find in §5 certain algebraic numbers derived from the infinite products twisted by some root of unity (Theorem 5.1, 5.2, 5.3, 5.4).

Throughout the article we adopt the following notations:

- $k$ an imaginary quadratic field
- $\mathfrak{h}$ the complex upper half plane
- $\tau \in \mathfrak{h} \cap k$
- $q = e^{\pi i \tau}$
- $f(a, b) = 1 + \sum_{m=1}^{\infty}(ab)^{m(m-1)/2}(a^m + b^m) = \sum_{m=-\infty}^{\infty} a^{m(m+1)/2}b^{m(m-1)/2}$
- $\overline{\mathbb{Q}}$ the field of algebraic numbers
- $(a; b)_\infty = \prod_{m=0}^{\infty}(1 - ab^m)$
- $\Delta(\tau) = (2\pi)^{12}q^2(q^2; q^2)_\infty^{24}$
- $(a, b, \ldots, c; d)_\infty = (a; d)_\infty(b; d)_\infty \cdots (c; d)_\infty$
- $\hat{K}(A, B) := (A; B)_\infty(\frac{B}{A}; B)_\infty$
- $\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$

## § 2. Algebraic numbers from theta series

Let $\alpha = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $b$ mod $d$ and $|\alpha|$ be the determinant of $\alpha$, and let

$$\phi_\alpha(\tau) := |\alpha|^{12} \frac{\Delta\left(\alpha\binom{\tau}{1}\right)}{\Delta\left(\binom{\tau}{1}\right)} = |\alpha|^{12} d^{-12} \frac{\Delta(\alpha\tau)}{\Delta(\tau)}.$$

Then, for later use, we recall the following fact that

(2.0) for any $\tau \in k \cap \mathfrak{h}$ the value $\phi_\alpha(\tau)$ is an algebraic integer,

which divides $|\alpha|^{12}$([6]).

**Lemma 2.0** ([4, Theorem 2.2]). *Let $\tau \in k \cap \mathfrak{h}$. Then $q^{1/24}(-q;q)_\infty \in \overline{\mathbb{Q}}$ and $q^{-1/12}\hat{K}(\pm q, q^2) \in \overline{\mathbb{Q}}$.*

**Lemma 2.1.** *Let $t_a \in \overline{\mathbb{Q}}$.*

(a) *Let $n \in \mathbb{Z}^+$ and $a\,(\leq n-1)$ be a positive integer. If $q^{t_a}\hat{K}(q^a, q^n) \in \overline{\mathbb{Q}}$, then $q^{t_a}\hat{K}(-q^a, q^n) \in \overline{\mathbb{Q}}$.*

(b) *Let $n$ be an even integer and $a$ be an odd integer with $1 \leq a \leq n-1$. If $q^{t_a}\hat{K}(-q^a, q^n) \in \overline{\mathbb{Q}}$, then $q^{t_a}\hat{K}(q^a, q^n) \in \overline{\mathbb{Q}}$.*

*Proof.* (a) Assume $q^{t_a}\hat{K}(q^a, q^n) \in \overline{\mathbb{Q}}$. Since $q^{2t_a}\hat{K}(q^{2a}, q^{2n}) \in \overline{\mathbb{Q}}$ and the set of algebraic numbers is a field,

$$q^{t_a}\hat{K}(-q^a, q^n) = \frac{q^{2t_a}\hat{K}(q^{2a}, q^{2n})}{q^{t_a}\hat{K}(q^a, q^n)} \in \overline{\mathbb{Q}}.$$

(b) Assume $\hat{K}(-q^a, q^n) \in \overline{\mathbb{Q}}$. If $\tau \in \mathfrak{h} \cap k$, then $\tau + 1 \in \mathfrak{h} \cap k$; hence $q^{t_a}\hat{K}(q^a, q^n) = (-1)^{t_a} q^{t_a}\hat{K}(-q^a, q^n) \in \overline{\mathbb{Q}}$. □

**Lemma 2.2.** *Let $t_a \in \overline{\mathbb{Q}}$.*

(a) *Let $a, n \in \mathbb{Z}^+$ with $n \equiv 0 \pmod 4$ and $1 \leq a \leq n-1$. If $q^{t_a}\hat{K}(q^a, q^n) \in \overline{\mathbb{Q}}$, then $q^{\frac{t_a}{2} - \frac{n}{16}}\hat{K}(q^{\frac{n+2a}{2}}, q^{2n}) \in \overline{\mathbb{Q}}$.*

(b) *Let $a, n \in \mathbb{Z}^+$ with $a, n \equiv 2 \pmod 4$ and $1 \leq a \leq n-1$. If $q^{t_a}\hat{K}(q^a, q^n) \in \overline{\mathbb{Q}}$, then $q^{\frac{t_a}{2} - \frac{n}{16}}\hat{K}(q^{\frac{n+2a}{2}}, q^{2n}) \in \overline{\mathbb{Q}}$.*

*Proof.* (a) From the Berndt's formulae ([2, pp.35,46]) we deduce that

$$\begin{aligned} f(-q^a, -q^{n-a}) + f(q^a, q^{n-a}) &= 2f(-q^{3a+n-a}, -q^{a+3n-3a}) \\ &= 2f(-q^{n+2a}, -q^{3n-2a}), \end{aligned}$$

that is,

$$\begin{aligned} &\hat{K}(q^a, q^n)(q^n; q^n)_\infty + \hat{K}(-q^a, q^n)(q^n; q^n)_\infty \\ &= 2\hat{K}(q^{n+2a}, q^{4n})(q^{4n}; q^{4n})_\infty. \end{aligned} \tag{2.1}$$

Multiplying both sides by $q^{t_a-\frac{n}{8}}(q^{4n},q^{4n})_\infty^{-1}$ in (2.1), we see that

$$\tag{2.2} \begin{aligned} &q^{t_a-\frac{n}{8}}\hat{K}(q^a,q^n)\frac{(q^n,q^n)_\infty}{(q^{4n},q^{4n})_\infty}+q^{t_a-\frac{n}{8}}\hat{K}(-q^a,q^n)\frac{(q^n,q^n)_\infty}{(q^{4n},q^{4n})_\infty}\\ &=2q^{t_a-\frac{n}{8}}\hat{K}(q^{n+2a},q^{4n}). \end{aligned}$$

It then follows from (2.0) that

$$\frac{q^{\frac{n}{24}}(q^n;q^n)_\infty}{q^{\frac{4n}{24}}(q^{4n};q^{4n})_\infty}\in\overline{\mathbb{Q}}.$$

And, by the assumption, (2.0), (2.2) and Lemma 2.1, we have

$$q^{t_a-\frac{n}{8}}\hat{K}(q^{n+2a},q^{4n})\in\overline{\mathbb{Q}}.$$

However, one can readily check that $n+2a\equiv 3n-2a\equiv 2 \pmod 4$. Thus, replacing $q^2$ by $q$ we get the conclusion.

(b) It is similar to (a). □

**Corollary 2.3.** *Let $l,n,t\in\mathbb{Z}^+$ with $1\le t<n$. If $q^{t_a}\hat{K}(q^a,q^n)\in\overline{\mathbb{Q}}$, then*

$$\tag{2.3} \sum_{m=1}^{\infty}\frac{q^{tm}+q^{(n-t)m}}{1+q^{nm}} \quad \textit{and} \quad \sum_{m=1}^{\infty}\frac{(-q^t)^m+(-q^{(n-t)})^m}{1+(-q^n)^m}$$

*are transcendental numbers except* 0.

*Proof.* By Lemma 2.1 we get that $q^{t_a}\hat{K}(-q^a,q^n)\in\overline{\mathbb{Q}}$. And we see from [2, p.54] that

$$\phi^2(-ab)\frac{f(a,b)}{f(-a,-b)}=1+2\sum_{m=1}^{\infty}\frac{a^m+b^m}{1+a^mb^m}.$$

Now, replace $a$ by $\pm q^t$ and $b$ by $\pm q^{n-t}$. Then we know from [5] that $\phi(-q^n)$ is a transcendental number. Since $\dfrac{f(\pm q^t,\pm q^{n-t})}{f(\mp q^t,\mp q^{n-t})}\in\overline{\mathbb{Q}}$ by the assumption, we have the corollary. □

Put $n=2$ and 3, respectively, in the first of (2.3). Then we obtain that

$$\sum_{m=1}^{\infty}\frac{q^m}{1+q^{2m}}=\sum_{m=1}^{\infty}(q^m+q^{-m})^{-1} \text{ and } \sum_{m=1}^{\infty}\frac{q^m+q^{2m}}{1+q^{3m}}=\sum_{m=1}^{\infty}(q^m+q^{-m}-1)^{-1}$$

are transcendental numbers.

## § 3. The case of $2^n$

**Theorem 3.1.** *Let $l,t\in\mathbb{Z}^+$ with $1\le t\le 2^l-1$. Then*

$$q^{\frac{2^l}{12}-\frac{t}{2}+\frac{t^2}{2^{l+1}}}\hat{K}(q^t,q^{2^l})\in\overline{\mathbb{Q}}.$$

*Proof.* If $t$ is an even integer, then there exists $u$ such that $\frac{t}{2^u}$ is an odd integer. Hence, we can check that

$$2^u\left(\frac{\frac{2^l}{2^u}}{12}-\frac{\frac{t}{2^u}}{2}+\frac{(\frac{t}{2^u})^2}{2\frac{2^l}{2^u}}\right)=\frac{2^l}{12}-\frac{t}{2}+\frac{t^2}{2^{l+1}}.$$

And, if $q^{\frac{2^{l-u}}{12}-\frac{\frac{t}{2^u}}{2}+\frac{(\frac{t}{2^u})^2}{2^{l-u+1}}}\hat{K}(q^{\frac{t}{2^u}},q^{2^{l-u}})\in\overline{\mathbb{Q}}$, then $q^{\frac{2^l}{12}-\frac{t}{2}+\frac{t^2}{2^{l+1}}}\hat{K}(q^t,q^{2^l})\in\overline{\mathbb{Q}}$. Thus we may assume that $t$ is an odd integer with $1\le t\le 2^l-1$.

We will proceed by induction on $l$. By Lemma 2.0 we know that

$$q^{\frac{-1}{12}}\hat{K}(q,q^2)\in\overline{\mathbb{Q}} \tag{3.1}$$

with $-\frac{1}{12}=\frac{1}{6}-\frac{1}{2}+\frac{1}{4}$. Here we must consider the case $\frac{2^l}{12}-\frac{t}{2}+\frac{t^2}{2^{l+1}}$ by using Lemma 2.2. Then by (3.1) we may write $q^{-\frac{1}{24}}\hat{K}(q,q^4)\in\overline{\mathbb{Q}}$ with $-\frac{1}{24}=\frac{4}{12}-\frac{1}{2}+\frac{1}{8}$.

Now, we assume that $q^{\frac{4\cdot2^l}{12}-\frac{t}{2}+\frac{t^2}{2\cdot4\cdot2^l}}\hat{K}(q^t,q^{4\cdot2^l})\in\overline{\mathbb{Q}}$ with $1\le t\le 2^{l+1}-1$. From Lemma 2.2 we can deduce that

$$q^{\frac{1}{2}(\frac{4\cdot2^l}{12}-\frac{t}{2}+\frac{t^2}{4\cdot2^{l+1}})-\frac{4\cdot2^l}{16}}\hat{K}(q^{\frac{4\cdot2^l+2t}{2}},q^{4\cdot2^{l+1}})\in\overline{\mathbb{Q}} \tag{3.2}$$

with

$$\frac{1}{2}(\frac{4\cdot2^l}{12}-\frac{t}{2}+\frac{t^2}{4\cdot2^{l+1}})-\frac{4\cdot2^l}{16}=\frac{4\cdot2^{l+1}}{12}-\frac{(\frac{4\cdot2^l+2t}{2})}{2}+\frac{(\frac{4\cdot2^l+2t}{2})^2}{2\cdot4\cdot2^{l+1}}.$$

Also, it follows from (3.2) that

$$q^{\frac{4\cdot2^l}{12}-\frac{2\cdot2^l-t}{2}+\frac{(2\cdot2^l-t)^2}{2\cdot4\cdot2^l}}\hat{K}(q^{2\cdot2^l-t},q^{4\cdot2^l})\in\overline{\mathbb{Q}}$$

and

$$\frac{q^{\frac{4\cdot2^l}{12}-\frac{2\cdot2^l-t}{2}+\frac{(2\cdot2^l-t)^2}{2\cdot4\cdot2^l}}\hat{K}(q^{2\cdot2^l-t},q^{4\cdot2^l})}{q^{\frac{4\cdot2^{l+1}}{12}-\frac{(\frac{4\cdot2^l+2t}{2})}{2}+\frac{(\frac{4\cdot2^l+2t}{2})^2}{2\cdot4\cdot2^{l+1}}}\hat{K}(q^{\frac{4\cdot2^l+2t}{2}},q^{4\cdot2^{l+1}})} \tag{3.3}$$
$$=q^{\frac{4\cdot2^{l+1}}{12}-\frac{2\cdot2^l-t}{2}+\frac{(2\cdot2^l-t)^2}{2\cdot4\cdot2^{l+1}}}\hat{K}(q^{(2\cdot2^l-t)},q^{4\cdot2^{l+1}})\in\overline{\mathbb{Q}}.$$

We then see from (3.2) and (3.3) that each case of $t$ $(1\le t<2\cdot2^l)$ in $\hat{K}(q^t,q^{4\cdot2^l})$ gives rise to $s_1=\frac{4\cdot2^l+2t}{2}$ and $s_2=2\cdot2^l-t$ in $\hat{K}(q^{s_i},q^{4\cdot2^{l+1}})(i=1,2)$. Thus all these are algebraic numbers for $1\le t<4\cdot2^l$ in $\hat{K}(q^t,q^{4\cdot2^{l+1}})$. This completes the proof. □

**Corollary 3.2.** *Let $l,n,t$ be positive integers with $1\le t\le 2^l-1$.*

(a) *$\hat{K}(q^t,q^{2^l})$ is a transcendental number.*

(b) $\dfrac{\hat{K}(q^t,q^{2^l})}{\hat{K}(-q^t,q^{2^l})}\in\overline{\mathbb{Q}}$ *and* $\dfrac{\hat{K}(-q^t,q^{2^l})}{\hat{K}(q^t,q^{2^l})}\in\overline{\mathbb{Q}}.$

(c)

$$\sum_{m=1}^{\infty} \frac{q^{tm} + q^{(2^l - t)k}}{1 + q^{2^l m}} \quad and \quad \sum_{m=1}^{\infty} \frac{(-q^t)^m + (-q^{(2^l - t)})^k}{1 + (-q)^{2^l m}}$$

*are transcendental numbers except* 0.

*Proof.* (a) If $\hat{K}(q^t, q^{2^l}) \in \overline{\mathbb{Q}}$, then there should be a positive integer $t$ satisfying $\frac{2^l}{12} - \frac{t}{2} + \frac{t^2}{2^{l+1}} = 0$ by Theorem 3.1 and (1.1). Equating $\frac{2^l}{12} - \frac{t}{2} + \frac{t^2}{2^{l+1}} = 0$, we have

$$t = \frac{(3+\sqrt{3})2^l}{6} \quad \text{or} \quad t = \frac{(3-\sqrt{3})2^l}{6},$$

which is impossible.

(b), (c) It is immediate from Lemma 2.1, Corollary 2.3, and Theorem 3.1. □

**Examples.** $q^{-\frac{1}{12}}\hat{K}(q,q^2)$, $q^{-\frac{1}{24}}\hat{K}(q,q^4)$, $q^{-\frac{2}{12}}\hat{K}(q^2,q^4)$, $q^{\frac{11}{48}}\hat{K}(q, q^8)$, $q^{\frac{-1}{12}}\hat{K}(q^2, q^8)$, $q^{\frac{-13}{48}}\hat{K}(q^3,q^8)$ and $q^{-\frac{4}{12}}\hat{K}(q^4,q^8)$ are algebraic numbers. And, we note that Theorem 3.1 and Lemma 2.1 would be a generalization of Lemma 2.0.

## § 4. The case of $\mathfrak{z} \cdot 2^l$ ($\mathfrak{z} = 3, 5, 7, 9, 13, 15$)

**Lemma 4.0.** *Let $n > 1$ be an odd integer. If*

$$q^{\frac{n}{12} - \frac{t}{2} + \frac{t^2}{2n}}\hat{K}(q^t, q^n) \in \overline{\mathbb{Q}} \ (1 \le t \le n-1),$$

*then* $q^{\frac{2n}{12} - \frac{n-t}{2} + \frac{(n-t)^2}{4n}}\hat{K}(q^{(n-t)}, q^{2n}) \in \overline{\mathbb{Q}}$.

*Proof.* We know that $t$ or $n-t$ is an odd integer. Thus we may assume that $n-t$ is an odd integer. Since $t$ and $2n-t$ are even integers, $q^{\frac{2n}{12} - \frac{t}{2} + \frac{t^2}{4n}}\hat{K}(q^t, q^n) \in \overline{\mathbb{Q}}$. So is

$$\frac{q^{\frac{n}{12} - \frac{t}{2} + \frac{t^2}{2n}}\hat{K}(q^t, q^n)}{q^{\frac{2n}{12} - \frac{t}{2} + \frac{t^2}{4n}}\hat{K}(q^t, q^{2n})} = q^{\frac{2n}{12} - \frac{n-t}{2} + \frac{(n-t)^2}{4n}}\hat{K}(q^{(n-t)}, q^{2n}).$$

□

**Theorem 4.1.** *Let $l > 2$ be an odd integer. If $q^{\frac{l}{12} - \frac{t}{2} + \frac{t^2}{2l}}\hat{K}(q^t, q^l) \in \overline{\mathbb{Q}}$ with $1 \le t \le l-1$, then $q^{\frac{2^a \cdot l}{12} - \frac{t}{2} + \frac{t^2}{2 \cdot 2^a l}}\hat{K}(q^t, q^{2^a l}) \in \overline{\mathbb{Q}}$ with $1 \le t \le 2^a \cdot (l-1)$ and $a \in \mathbb{Z}^+$.*

*Proof.* By Lemma 4.0, Lemma 2.2 and the similar arguments as in Theorem 3.1 we get the theorem. □

Observe that by using the properties of theta series in [1], [2], [3] we obtain from (1.2) that

$$(4.0)\qquad \begin{aligned} &q^{\frac{-1}{12}}\hat{K}(q,q^3) = \frac{q^{\frac{1}{24}}(q;q)_\infty}{q^{\frac{3}{24}}(q^3;q^3)_\infty},\\ &q^{\frac{1}{60}}\hat{K}(q,q^5),\quad q^{\frac{-11}{60}}\hat{K}(q^2,q^5),\\ &q^{\frac{13}{84}}\hat{K}(q,q^7),\quad q^{\frac{-11}{84}}\hat{K}(q^2,q^7), q^{\frac{-23}{84}}\hat{K}(q^3,q^7),\\ &q^{\frac{11}{36}}\hat{K}(q,q^9), q^{\frac{-1}{36}}\hat{K}(q^2,q^9),\quad q^{\frac{-13}{36}}\hat{K}(q^4,q^9) \end{aligned}$$

are algebraic numbers.

Next, by the Entry 36 ([3, p. 188]) we have

$$(4.1)\qquad f^3(-q^7,-q^8)+q^3f^3(-q^2,-q^{13}) = f^3(-q^5)\frac{f(-q^6,-q^9)}{f(-q^3,-q^{12})}$$

and

$$(4.2)\qquad f^3(-q^4,-q^{11})-q^3f^3(-q,-q^{14}) = f^3(-q^5)\frac{f(-q^3,-q^{12})}{f(-q^6,-q^9)},$$

where $f(-q)=(q;q)_\infty$. It then follows from (4.1) and (4.2) that

$$(4.3)\qquad \begin{aligned} &\left(q^{-\frac{37}{60}}\hat{K}(q^7,q^{15})\right)^3+\left(q^{\frac{23}{60}}\hat{K}(q^2,q^{15})\right)^3\\ &= \left(\frac{q^{\frac{5}{24}}(q^5;q^5)_\infty}{q^{\frac{15}{24}}(q^{15};q^{15})_\infty}\right)^3\cdot\frac{q^{-\frac{33}{60}}\hat{K}(q^6,q^{15})}{q^{\frac{3}{60}}\hat{K}(q^3,q^{15})}\in\overline{\mathbb{Q}} \end{aligned}$$

and

$$(4.4)\qquad \begin{aligned} &\left(q^{-\frac{13}{60}}\hat{K}(q^4,q^{15})\right)^3-\left(q^{\frac{47}{60}}\hat{K}(q,q^{15})\right)^3\\ &= \left(\frac{q^{\frac{5}{24}}(q^5;q^5)_\infty}{q^{\frac{15}{24}}(q^{15};q^{15})_\infty}\right)^3\cdot\frac{q^{\frac{3}{60}}\hat{K}(q^3,q^{15})}{q^{-\frac{33}{60}}\hat{K}(q^6,q^{15})}\in\overline{\mathbb{Q}}. \end{aligned}$$

And, one can readily check that

$$(4.5)\qquad \begin{aligned} &\left(q^{\frac{47}{60}}\hat{K}(q,q^{15})\right)\left(q^{\frac{23}{60}}\hat{K}(q^2,q^{15})\right)\\ &\cdot\left(q^{-\frac{13}{60}}\hat{K}(q^4,q^{15})\right)\left(q^{-\frac{37}{60}}\hat{K}(q^7,q^{15})\right)\\ &= \left(\frac{q^{\frac{1}{24}}(q;q)_\infty}{q^{\frac{15}{24}}(q^{15};q^{15})_\infty}\right)\cdot\left(\frac{q^{\frac{15}{24}}(q^{15};q^{15})_\infty}{q^{\frac{3}{24}}(q^3;q^3)_\infty}\right)\cdot\left(\frac{q^{\frac{15}{24}}(q^{15};q^{15})_\infty}{q^{\frac{5}{24}}(q^5;q^5)_\infty}\right)\in\overline{\mathbb{Q}} \end{aligned}$$

and

$$(4.6)\qquad \begin{aligned} &\left(q^{\frac{47}{60}}\hat{K}(q,q^{15})\right)\left(q^{-\frac{13}{60}}\hat{K}(q^4,q^{15})\right)\\ &= \prod_{m=1}^{\infty}\frac{q^{\frac{1}{60}}\hat{K}(q,q^{15})}{q^{-\frac{33}{60}}\hat{K}(q^6,q^{15})}\in\overline{\mathbb{Q}}. \end{aligned}$$

Thus, we derive from (4.3)∼(4.6) that

$$q^{\frac{47}{60}}\hat{K}(q,q^{15}),\qquad q^{\frac{23}{60}}\hat{K}(q^2,q^{15}),$$
$$q^{-\frac{13}{60}}\hat{K}(q^4,q^{15})\quad\text{and}\quad q^{-\frac{37}{60}}\hat{K}(q^7,q^{15}) \tag{4.7}$$

are algebraic numbers.

Similarly, by using the Entry 8(i)([2, p.373]) we obtain that

$$q^{\frac{97}{156}}\hat{K}(q,q^{13}),\ q^{\frac{37}{156}}\hat{K}(q^2,q^{13}),\ q^{-\frac{11}{156}}\hat{K}(q^3,q^{13}),$$
$$q^{-\frac{47}{156}}\hat{K}(q^4,q^{13}),\ q^{-\frac{71}{156}}\hat{K}(q^5,q^{13}),\ q^{-\frac{83}{156}}\hat{K}(q^6,q^{13}) \tag{4.8}$$

are algebraic numbers. Therefore we are ready to justify the following theorem.

**Theorem 4.2.** *Let $n,t,l\in\mathbb{Z}^+$ with $1\le t\le n-1$ and let $n=\mathfrak{z}\cdot 2^l$ ($\mathfrak{z}=3,5,7,9,13,15$).*

(a) $q^{\frac{n}{12}-\frac{t}{2}+\frac{t^2}{2n}}\hat{K}(q^t,q^n)\in\overline{\mathbb{Q}}$.

(b) *$\hat{K}(q^t,q^n)$ is a transcendental number.*

(c) $\dfrac{\hat{K}(q^t,q^n)}{\hat{K}(-q^t,q^n)}\in\overline{\mathbb{Q}}$.

*Proof.* (a) We will proceed by induction on $l$. It follows from (1.2), (4.7) and (4.8) that $q^a\hat{K}(q^t,q^n)\in\overline{\mathbb{Q}}$ for $n=3,5,7,9,13,15$. Then by using the same arguments as in Theorem 3.1 we can derive the conclusion.

(b), (c) As for (b) and (c), it follows from (a) by using the same ideas in Corollary 3.2, Corollary 2.3, and Lemma 2.1. □

Sills found in [8] a list of 26 new double sum-product Rogers-Ramanujan type identities. From these identities we establish the following 21 examples by using Theorem 3.1 and Theorem 4.2:

**Example 4.3.** Let $(a;b)_r=\prod_{m=0}^{r-1}(1-ab^m)$.

$$q^{\frac{1}{24}}\sum_{n\ge0}\sum_{r\ge0}\frac{(-1)^r q^{3n(n-1)/2+r^2-2nr}}{(q;q^2)_n(q^2;q^2)_r(q;q)_{n-2r}}=q^{\frac{1}{24}}(-q;q)_\infty\in\overline{\mathbb{Q}}. \tag{4.9}$$

$$\begin{aligned}&1+\sum_{n\ge1}\sum_{r\ge0}\frac{q^{n(n+1)/2+3r(r-1)/2}(-1;q)_n(q^3;q^3)_{n-r-1}}{(q;q)_{2n-1}(q^3;q^3)_r(q;q)_{n-3r}}\\&=\frac{(q^6,q^6,q^{12};q^{12})_\infty(-q;q)_\infty}{(q;q)_\infty}\in\overline{\mathbb{Q}}.\end{aligned} \tag{4.10}$$

(4.11)
$$\sum_{n\ge0}\sum_{r\ge0}\frac{q^{n(n+1)/2+2r^2}(-1;q)_n}{(q;q^2)_n(q^2;q^2)_r(q;q)_{n-2r}}=\frac{(q^7,q^7,q^{14};q^{14})_\infty(-q;q)_\infty}{(q;q)_\infty}\in\overline{\mathbb{Q}}.$$

(4.12)
$$q^{\frac{95}{72}}\sum_{n\geq 0}\sum_{r\geq 0}\frac{q^{n^2+2n+2r^2+2r}}{(q;q^2)_{n+1}(q^2;q^2)_r(q;q)_{n-2r}}=q^{\frac{95}{72}}\frac{(q^2,q^{16},q^{18};q^{18})_\infty}{(q;q)_\infty}\in\overline{\mathbb{Q}}.$$

(4.13)
$$q^{\frac{47}{72}}\sum_{n\geq 0}\sum_{r\geq 0}\frac{q^{n^2+2n+2r^2+2r}(1+q^{2r+2})}{(q;q^2)_{n+1}(q^2;q^2)_r(q;q)_{n-2r}}=q^{\frac{47}{72}}\frac{(q^4,q^{14},q^{18};q^{18})_\infty}{(q;q)_\infty}\in\overline{\mathbb{Q}}.$$

$$q^{\frac{5}{24}}\sum_{n\geq 0}\sum_{r\geq 0}\frac{q^{n^2+2r^2+2r}}{(q;q^2)_{n}(q^2;q^2)_r(q;q)_{n-2r}}=q^{\frac{5}{24}}\frac{(q^6,q^{12},q^{18};q^{18})_\infty}{(q;q)_\infty}\in\overline{\mathbb{Q}}. \tag{4.14}$$

(4.15)
$$q^{-\frac{1}{72}}\sum_{n\geq 0}\sum_{r\geq 0}\frac{q^{n^2+2r^2}}{(q;q^2)_{n}(q^2;q^2)_r(q;q)_{n-2r}}=q^{-\frac{1}{72}}\frac{(q^8,q^{10},q^{18};q^{18})_\infty}{(q;q)_\infty}\in\overline{\mathbb{Q}}.$$

(4.16)
$$q^{\frac{109}{84}}\sum_{n\geq 0}\sum_{r\geq 0}\frac{q^{n^2+3n+3r(r-1)/2}(q^3;q^3)_{n-r}}{(q;q)_{2n+2}(q^3;q^3)_r(q;q)_{n-3r}}=q^{\frac{109}{84}}\frac{(q^3,q^{18},q^{21};q^{21})_\infty}{(q;q)_\infty}\in\overline{\mathbb{Q}}.$$

$$\begin{aligned}&q^{\frac{37}{84}}\left(1+\sum_{n\geq 1}\sum_{r\geq 0}\frac{q^{n^2+3r(r-3)/2}(q^3;q^3)_{n-r-1}}{(q;q)_{2n-1}(q^3;q^3)_r(q;q)_{n-3r}}\right)\\&=q^{\frac{37}{84}}\frac{(q^6,q^{15},q^{21};q^{21})_\infty}{(q;q)_\infty}\in\overline{\mathbb{Q}}.\end{aligned} \tag{4.17}$$

$$\begin{aligned}&q^{\frac{1}{84}}\left(1+\sum_{n\geq 1}\sum_{r\geq 0}\frac{q^{n^2+3r(r-1)/2}(q^3;q^3)_{n-r-1}}{(q;q)_{2n-1}(q^3;q^3)_r(q;q)_{n-3r}}\right)\\&=q^{\frac{1}{84}}\frac{(q^9,q^{12},q^{21};q^{21})_\infty}{(q;q)_\infty}\in\overline{\mathbb{Q}}.\end{aligned} \tag{4.18}$$

$$\begin{aligned}&q^{\frac{25}{16}}\left(1+\sum_{n\geq 1}\sum_{r\geq 0}\frac{q^{n^2+3r(r-3)}(-q;q^2)_n(q^6;q^6)_{n-r-1}}{(q^2;q^2)_{2n-1}(q^6;q^6)_r(q^2;q^2)_{n-3r}}\right)\\&=q^{\frac{25}{16}}\frac{(q^3,q^{21},q^{24};q^{24})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}\in\overline{\mathbb{Q}}.\end{aligned} \tag{4.19}$$

$$\begin{aligned}&q^{\frac{1}{16}}\left(1+\sum_{n\geq 1}\sum_{r\geq 0}\frac{q^{n^2+3r(r-1)}(-q;q^2)_n(q^6;q^6)_{n-r-1}}{(q^2;q^2)_{2n-1}(q^6;q^6)_r(q^2;q^2)_{n-3r}}\right)\\&=q^{\frac{1}{16}}\frac{(q^9,q^{15},q^{24};q^{24})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}\in\overline{\mathbb{Q}}.\end{aligned} \tag{4.20}$$

$$\begin{aligned}(4.21)\qquad & 1+\sum_{n\geq 1}\sum_{r\geq 0}\frac{q^{n(n+1)/2+3r^2}(-1;q)_n(q^3;q^3)_{n-r-1}}{(q;q)_{2n-1}(q^3;q^3)_r(q;q)_{n-3r}}\\ &=\frac{(q^{12},q^{12},q^{24};q^{24})_\infty(-q;q)_\infty}{(q;q)_\infty}\in\overline{\mathbb{Q}}.\end{aligned}$$

$$\begin{aligned}(4.22)\qquad & \sum_{n\geq 0}\sum_{r\geq 0}\frac{q^{n^2+4r^2+4r+\frac{29}{56}}}{(q;q^2)_n(q^4;q^4)_r(q^2;q^2)_{n-2r}}\\ &=q^{\frac{29}{56}}\frac{(q^8,q^{20},q^{28};q^{28})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}\in\overline{\mathbb{Q}}.\end{aligned}$$

$$\begin{aligned}(4.23)\qquad & \sum_{n\geq 0}\sum_{r\geq 0}\frac{q^{n^2+2r^2-\frac{3}{56}}}{(q;q^2)_n(q^4;q^4)_r(q^2;q^2)_{n-2r}}\\ &=q^{-\frac{3}{56}}\frac{(q^{12},q^{16},q^{28};q^{28})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}\in\overline{\mathbb{Q}}.\end{aligned}$$

$$\begin{aligned}(4.24)\qquad & 1+\sum_{n\geq 1}\sum_{r\geq 0}\frac{(-1)^{n+r}q^{n^2+n+r^2+r-2nr}(q^4;q^4)_{n+r-1}(-1;q)_{2n}}{(q;q)_{4n-1}(q;q)_{2r}(q^2;q^2)_{n-r}}\\ &+\sum_{n\geq 1}\sum_{r\geq 0}\frac{(-1)^{n+r}q^{n^2+n+r^2+r-2nr+1}(q^4;q^4)_{n+r}(-1;q)_{2n+1}}{(q;q)_{4n+1}(q;q)_{2r+1}(q^2;q^2)_{n-r}}\\ &=\frac{(q^{18},q^{18},q^{36};q^{36})_\infty(-q;q)_\infty}{(q;q)_\infty}\in\overline{\mathbb{Q}}.\end{aligned}$$

(4.25)

$$\begin{aligned} & q^{\frac{71}{32}}\left(1+\sum_{n\geq 1}\sum_{r\geq 0}\frac{q^{n^2+6r^2-6}(-q;q^2)_n(q^6;q^6)_{n-r-1}(q^{6r}+q^{12r+6}-1)}{(q^2;q^2)_{2n-1}(q^6;q^6)_r(q^2;q^2)_{n-3r}}\right)\\ &=q^{\frac{71}{32}}\frac{(q^9,q^{39},q^{48};q^{48})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}\in\overline{\mathbb{Q}}.\end{aligned}$$

$$\begin{aligned}(4.26)\qquad & q^{\frac{23}{32}}\left(1+\sum_{n\geq 1}\sum_{r\geq 0}\frac{q^{n^2+6r^2+6r}(-q;q^2)_n(q^6;q^6)_{n-r-1}}{(q^2;q^2)_{2n-1}(q^6;q^6)_r(q^2;q^2)_{n-3r}}\right)\\ &=q^{\frac{23}{32}}\frac{(q^{15},q^{33},q^{48};q^{48})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}\in\overline{\mathbb{Q}}.\end{aligned}$$

$$\begin{aligned}(4.27)\qquad & q^{-\frac{1}{32}}\left(1+\sum_{n\geq 1}\sum_{r\geq 0}\frac{q^{n^2+6r^2}(-q;q^2)_n(q^6;q^6)_{n-r-1}}{(q^2;q^2)_{2n-1}(q^6;q^6)_r(q^2;q^2)_{n-3r}}\right)\\ &=q^{-\frac{1}{32}}\frac{(q^{21},q^{27},q^{48};q^{48})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}\in\overline{\mathbb{Q}}.\end{aligned}$$

(4.28)
$$q^{-\frac{1}{312}}\left(1+\sum_{n\geq 1}\sum_{r\geq 0}\frac{(-1)^{n+r}q^{3n^2+r^2+r-2nr}(q^4;q^4)_{n+r-1}}{(q;q)_{4n-1}(q;q)_{2r}(q^2;q^2)_{n-r}}\right.$$
$$\left.+\sum_{n\geq 1}\sum_{r\geq 0}\frac{(-1)^{n+r}q^{3n^2+2n+r^2+r-2nr+1}(q^4;q^4)_{n+r}}{(q;q)_{4n+1}(q;q)_{2r+1}(q^2;q^2)_{n-r}}\right)$$
$$=q^{-\frac{1}{312}}\frac{(q^{24},q^{28},q^{52};q^{52})_\infty}{(q;q)_\infty}\in\overline{\mathbb{Q}}.$$

(4.29)
$$q^{-\frac{1}{72}}\left(1+\sum_{n\geq 1}\sum_{r\geq 0}\frac{(-1)^{n+r}q^{2n^2+2r^2+2r-4nr}(q^8;q^8)_{n+r-1}(-q;q^2)_{2n}}{(q^2;q^2)_{4n-1}(q;q)_{2r}(q^4;q^4)_{n-r}}\right.$$
$$\left.+\sum_{n\geq 1}\sum_{r\geq 0}\frac{(-1)^{n+r}q^{2n^2+2r^2+2r-4nr+1}(q^8;q^8)_{n+r}(-q;q^2)_{2n+1}}{(q^2;q^2)_{4n+1}(q^2;q^2)_{2r+1}(q^4;q^4)_{n-r}}\right)$$
$$=q^{-\frac{1}{72}}\frac{(q^{32},q^{40},q^{72};q^{72})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}\in\overline{\mathbb{Q}}.$$

## § 5. More algebraic numbers

In this section we consider a family of algebraic numbers for the infinite products twisted by some root of unity.

**Theorem 5.1.** *Let $n\geq 2$ be a positive integer, $1\leq t<n$ and $\omega$ a primitive cube root of unity. If $q^{\frac{n}{12}-\frac{t}{2}+\frac{t^2}{2n}}\hat{K}(q^t,q^n)\in\overline{\mathbb{Q}}$ and $q^{-\frac{n}{12}-\frac{t}{6}+\frac{t^2}{6n}}\hat{K}(q^{n+t},q^{3n})\in\overline{\mathbb{Q}}$, then*

$$q^{\frac{(n-2t)^2}{12n}}\prod_{m=1}^{\infty}(1\pm\omega q^{3nm-(3n-t)})(1\pm\omega^2 q^{3nm-(n-t)})(1\pm\omega q^{3nm-(2n+t)})$$
$$\cdot(1\pm\omega^2 q^{3nm-t})(1-\omega q^{3nm-n})(1-\omega^2 q^{3nm-2n})$$

*are algebraic numbers with double signs in the same order.*

*Proof.* If $q^{\frac{n}{12}-\frac{t}{2}+\frac{t^2}{2n}}\hat{K}(q^t,q^n)\in\overline{\mathbb{Q}}$ and $q^{-\frac{n}{12}-\frac{t}{6}+\frac{t^2}{6n}}\hat{K}(q^{n+t},q^{3n})\in\overline{\mathbb{Q}}$, then $q^{\frac{n}{12}-\frac{t}{2}+\frac{t^2}{2n}}\hat{K}(-q^t,q^n)\in\overline{\mathbb{Q}}$ and $q^{-\frac{n}{12}-\frac{t}{6}+\frac{t^2}{6n}}\hat{K}(-q^{n+t},q^{3n})\in\overline{\mathbb{Q}}$ by Lemma 2.1. On the other hand, we see from [3, p.144] that

(5.1)
$$f(\omega a,\omega b)=\omega f(a,b)+(1-\omega)f(a^6b^3,a^3b^6)$$

with $|ab| < 1$. Thus, let $a = q^t$ and $b = q^{n-t}$ in (5.1). Then, multiplying both sides of (5.1) by $\dfrac{q^{\frac{(n-2t)^2}{12n}}}{\hat{K}(-q^{n+t}, q^{3n})(q^{3n}; q^{3n})_\infty}$ we get that

(5.2)

$$q^{\frac{(n-2t)^2}{12n}} \prod_{m=1}^{\infty} (1 + \omega q^{3nm-(3n-t)})(1 + \omega^2 q^{3nm-(n-t)})(1 + \omega q^{3nm-(2n+t)}) \cdot (1 + \omega^2 q^{3nm-t})(1 - \omega q^{3nm-n})(1 - \omega^2 q^{3nm-2n})$$
$$= \omega q^{\frac{(n-2t)^2}{12n}} \frac{\hat{K}(-q^t, q^n)(q^n; q^n)_\infty}{\hat{K}(-q^{n+t}, q^{3n})(q^{3n}; q^{3n})_\infty} + (1-\omega) q^{\frac{(n-2t)^2}{12n}} \frac{\hat{K}(-q^{3n+3t}, q^{9n})(q^{9n}; q^{9n})_\infty}{\hat{K}(-q^{n+t}, q^{3n})(q^{3n}; q^{3n})_\infty}.$$

Since $\frac{n}{12} - \frac{t}{2} + \frac{t^2}{2n} - (-\frac{n}{12} - \frac{t}{6} + \frac{t^2}{6n}) - \frac{2n}{24} = \frac{(n-2t)^2}{12n}$, the value on the right hand side of (5.2) is an algebraic number. Hence, we obtain the result. Similarly, setting $a = -q^t$ and $b = -q^{n-t}$ in (5.1) we can complete the proof. □

**Corollary 5.2.** *Let $n = \mathfrak{z} \cdot 2^l$ ($\mathfrak{z} = 2, 3, 5$) be positive integers with $l \geq 0$ and $\omega$ a primitive cube root of unity. Then*

$$q^{\frac{(n-2t)^2}{12n}} \prod_{m=1}^{\infty} (1 \pm \omega q^{3nm-(3n-t)})(1 \pm \omega^2 q^{3nm-(n-t)})(1 \pm \omega q^{3nm-(2n+t)}) \cdot (1 \pm \omega^2 q^{3nm-t})(1 - \omega q^{3nm-n})(1 - \omega^2 q^{3nm-2n})$$

*are algebraic numbers with double signs in the same order.*

*Proof.* It is immediate from Theorem 3.1, Theorem 4.2, and Theorem 5.1. □

**Theorem 5.3.** *Let $n \geq 2$ be a positive integer. If $q^{\frac{n}{12} - \frac{t}{2} + \frac{t^2}{2n}} \hat{K}(q^t, q^n) \in \overline{\mathbb{Q}}$, then*

$$q^{\frac{n}{12} - \frac{t}{2} + \frac{t^2}{2n}} \prod_{m=1}^{\infty} (1 \pm i q^{2nm-(2n-t)})(1 \mp i q^{2nm-(n-t)})(1 \pm i q^{2nm-(n+t)})(1 \mp i q^{2nm-t})$$

*are algebraic numbers with double signs in the same order.*

*Proof.* By the Entry 9 ([3, p.146]) we claim that

$$f(ai, bi) = \frac{1}{2}(1+i) f(a,b) + (1-i) f(-a,-b) \tag{5.3}$$

with $|ab| < 1$. Set $a = \pm q^t$ and $b = \pm q^{n-t}$ in (5.3) with double signs in the same order. Then, by adopting the same arguments as in Theorem 5.1 we can derive the conclusion. □

**Corollary 5.4.** *Let $n = \mathfrak{z} \cdot 2^l$ ($\mathfrak{z} = 2, 3, 5, 7, 9, 13, 15$) be positive integers with $l \geq 0$. Then we see that*

$$q^{\frac{n}{12}-\frac{t}{2}+\frac{t^2}{2n}} \prod_{m=1}^{\infty} (1 \pm iq^{2nm-(2n-t)})(1 \mp iq^{2nm-(n-t)})(1 \pm iq^{2nm-(n+t)})(1 \mp iq^{2nm-t})$$

*are algebraic numbers with double signs in the same order.*

*Proof.* It is immediate from Theorem 3.1, Theorem 4.2, and Theorem 5.3. □

## References

[1] B. C. Berndt, *Ramanujan's Notebooks. Part II*, Springer-Verlag, New York, 1989.

[2] ———, *Ramanujan's Notebooks. Part III*, Springer-Verlag, New York, 1991.

[3] ———, *Ramanujan's Notebooks. Part IV*, Springer-Verlag, New York, 1994.

[4] D. Kim and J. K. Koo, *Algebraic integers as values of elliptic functions*, Acta Arith. **100** (2001), no. 2, 105–116.

[5] ———, *On the infinite products derived from theta series I*, J. Korean Math. Soc. **44** (2007), no. 1, 55–107.

[6] S. Lang, *Elliptic Functions*, Addison-Wesley Publishing Co., Inc., Reading, Mass.-London-Amsterdam, 1973.

[7] C. L. Siegel, *Transcendental Numbers*, Annals of Mathematics Studies, no. 16. Princeton University Press, Princeton, N. J., 1949.

[8] A. V. Sills, *On identities of the Rogers-Ramanujan type*, Ramanujan J. **11** (2006), no. 3, 403–429.

[9] M. Waldschmidt, *Nombres Transcendants*, Lecture Notes in Mathematics, Vol. 402. Springer-Verlag, Berlin-New York, 1974.

[10] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, An introduction to the general theory of infinite processes and of analytic functions: with an account of the principal transcendental functions. Fourth edition. Reprinted Cambridge University Press, New York 1962.

Daeyeoul Kim
National Institute for Mathematical Sciences 385-16
Doryong-dong, Yuseong-gu Tajeon, 305-340, Korea
*E-mail address*: daeyeoul@nims.re.kr

Ja Kyung Koo
Korea Advanced Institute of Science and Technology
Department of Mathematical Sciences
Tajeon 305-701, Korea
*E-mail address*: jkkoo@math.kaist.ac.kr