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THE SOLUTIONS OF SOME OPERATOR EQUATIONS

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Abstract. In this paper we consider the solvability and describe the set of the solutions of the operator equations

$$AX + X^*C = B$$

and

$$AXB + B^*X^*A^* = C.$$

This generalizes the results of D. S. Djordjević [*Explicit solution of the operator equation* $A^*X+X^*A=B$, J. Comput. Appl. Math. **200** (2007), 701–704].

1. Introduction

Throughout the paper $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces over the same field. We denote the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ by $\mathcal{L}(\mathcal{H},\mathcal{K})$ and by $\mathcal{L}(\mathcal{H})$ when $\mathcal{H} = \mathcal{K}$. For $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$, let $\mathcal{R}(A)$ and $\mathcal{N}(A)$ be the range of A and the null space of A, respectively.

$A \in \mathcal{L}(\mathcal{H},\mathcal{K})$ is g-invertible, if there exists an operator $A^- \in \mathcal{L}(\mathcal{K},\mathcal{H})$, such that $AA^-A = A$. In this case A^- is called a g-inverse, or an inner generalized inverse of A. Recall that $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$ is g-invertible if and only if $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively, are closed and complemented subspaces of $\mathcal{K}$ and $\mathcal{H}$. If $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$, then A is g-invertible if and only if $\mathcal{R}(A)$ is closed. In this case the Moore-Penrose generalized inverse of A, denoted by $A^\dagger$, is the unique operator $A^\dagger \in L(\mathcal{K},\mathcal{H})$ which satisfies

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger \quad \text{and} \quad (A^\dagger A)^* = A^\dagger A.$$

It is well-known that if $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$ has a closed range, then using the following decompositions

$$\mathcal{H} = \mathcal{R}(A^*) \oplus \mathcal{N}(A) \quad \text{and} \quad \mathcal{K} = \mathcal{R}(A) \oplus \mathcal{N}(A^*),$$

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since A is g-invertible, we have that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $A_1 : \mathcal{R}(A^*) \to \mathcal{R}(A)$ is invertible.

In this case, the Moore-Penrose generalized inverse of A has the following matrix decomposition

$$A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.$$

For other important properties of generalized inverses see [1], [3], [2], [5].

There are many papers in which the basic aim is to find necessary and sufficient conditions for the existence of a solution to some matrix or operator equations. The reason for this is that there is a large number of applications in physics, mechanics, control theory and many other fields (see [6], [10], [9], [8])

This paper was motivated by the paper [4] in which the necessary and sufficient conditions for the existence of a solution of the equation

$$AX + X^*A^* = B, \tag{1}$$

were given and the set of the solutions was completely describe, for the operators $A \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H})$. In this paper we consider two generalizations of the equation (1) and we get the results from [4] as a corollary for both cases.

In the first section the object of our interest is the equation

$$AX + X^*C = B,$$

first in the case when $A \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ and $C \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ are invertible and then in the case when they are g-invertible. The results from [4] are obtained for $C = A^*$.

In the second section we give the necessary and sufficient conditions of the existence of a solution of the equation

$$AXB + B^*X^*A^* = C,$$

under some additional assumptions for the operators $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$,$B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ and $C \in \mathcal{L}(\mathcal{K})$. This equation is reduced to the equation (1), for $B = I$ or if at least one of the operators A and B is invertible, this is actually a special case of the equation $AX + X^*C = B$.

2. Operator equation $AX + X^*C = B$

In this section we investigate the solvability of the equation

$$AX + X^*C = B, \tag{2}$$

where $A \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, $C \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and $B \in \mathcal{L}(\mathcal{H})$ are given operators. Under some conditions we describe the set of the solutions.

The next theorem offers general conditions of the existence of a solution of the equation (2) in the case when A and C are invertible operators.

Theorem 2.1. *Let $A \in \mathcal{L}(\mathcal{K},\mathcal{H})$ and $C \in \mathcal{L}(\mathcal{H},\mathcal{K})$ be invertible operators and $B \in \mathcal{L}(\mathcal{H})$. If*

$$A(C^*)^{-1}B^* + B^*(A^*)^{-1}C = 2B, \tag{3}$$

then the operator equation (2) has a solution $X \in \mathcal{L}(\mathcal{H},\mathcal{K})$. In this case any solution of the equation (2) is represented by

$$X = \frac{1}{4}A^{-1}B + \frac{1}{4}(C^*)^{-1}B^* + ZC - Z^*A^*, \tag{4}$$

where $Z \in \mathcal{L}(\mathcal{K})$ satisfies $C^*Z^*C = AZ^*A^*$.

Proof. Suppose (3) is satisfied. Taking $Z = 0$ in (4), we have that the operator X defined by (4) is the solution of the equation (2).

Now, suppose that the equation (2) has a solution $X \in \mathcal{L}(\mathcal{H},\mathcal{K})$ and prove that it must be of the form (4). Let

$$\hat{A} = \begin{bmatrix} A & 0 \\ 0 & C^* \end{bmatrix} : \mathcal{K}\oplus\mathcal{K} \to \mathcal{H}\oplus\mathcal{H},$$

$$\hat{X} = \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} : \mathcal{H}\oplus\mathcal{H} \to \mathcal{K}\oplus\mathcal{K},$$

and

$$\hat{B} = \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix} : \mathcal{H}\oplus\mathcal{H} \to \mathcal{H}\oplus\mathcal{H}.$$

By $AX + X^*C = B$, we have

$$\begin{aligned} \hat{A}\hat{X} + \hat{X}^*\hat{A}^* &= \begin{bmatrix} A & 0 \\ 0 & C^* \end{bmatrix}\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} + \begin{bmatrix} 0 & X^* \\ X^* & 0 \end{bmatrix}\begin{bmatrix} A^* & 0 \\ 0 & C \end{bmatrix} \\ &= \begin{bmatrix} 0 & AX + X^*C \\ C^*X + X^*A & 0 \end{bmatrix} = \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix} = \hat{B}. \end{aligned}$$

By Djordjevic's result [4], it follows that $\hat{X}$ has the following representation

$$\hat{X} = \frac{1}{2}\hat{A}^{-1}\hat{B} + \hat{W}\hat{A}^*,$$

where $\hat{W} \in \mathcal{L}(\mathcal{K}\oplus\mathcal{K})$ satisfies $\hat{W} = -\hat{W}^*$. Let

$$\hat{W} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}.$$

Then by the above solution, we obtain

$$X = \frac{1}{2}A^{-1}B + W_{12}C = \frac{1}{2}(C^*)^{-1}B^* + W_{21}A^*,$$

$$W_{11}A^* = W_{22}C = 0.$$

Hence, $W_{11} = 0$ and $W_{22} = 0$. Now, by $\hat{W} = -\hat{W}^*$, we get $W_{12}^* = -W_{21}$. For $Z = W_{12}$, we have

$$X = \frac{1}{2}A^{-1}B + ZC,$$

where Z satisfying $ZC + Z^*A^* = \frac{1}{2}\{(C^*)^{-1}B^* - A^{-1}B\}$. Remark that the solution $X = \frac{1}{2}A^{-1}B + ZC$ and its condition $ZC + Z^*A^* = \frac{1}{2}\{(C^*)^{-1}B^* - A^{-1}B\}$ ensure the following equation

$$C^*Z^*C - AZ^*A^* = (X^*C + AX) - \frac{1}{2}\{B^*(A^*)^{1}C + A(C^*)^{1}B^*\}.$$

Using the condition (3) we get that X is represented by (4), where $Z \in \mathcal{L}(\mathcal{K})$ satisfies $C^*Z^*C = AZ^*A^*$. □

Note that for arbitrary $Z \in \mathcal{L}(\mathcal{K})$ which satisfies $C^*Z^*C = AZ^*A^*$, the operator X defined by (4) is the solution of (2). So, the set of the solutions of (2) is completely characterized by

$$\{X = \frac{1}{4}A^{-1}B + \frac{1}{4}(C^*)^{-1}B^* + ZC - Z^*A^* : Z \in \mathcal{L}(\mathcal{K}),\ C^*Z^*C = AZ^*A^*\}.$$

Equation $C^*ZC = AZA^*$ is equivalent to

$$Z - D^*ZD = 0,$$

where $D = A^*C^{-1}$. Equation $Z - D^*ZD = E$ is well known discrete Lyapunov equation, and it is known that if 1 does not belong to the set product $\sigma(D)\sigma(D^*)$ of the spectra, then such an equation is uniquely solvable for each operator E by Kleinecke's theorem. Specialized software is available for solving Lyapunov equations. For the discrete case, the Schur method of Kitagawa [7] is often used.

It is interesting to remark that (3) is sufficient but not necessary condition for the existence of a solution of the equation (2). That the converse of Theorem 2.1 can not be proved is illustrated by the following example:

Example. Let us consider the set of 2×2 matrices and let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

It is easy to check that $X = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ is a solution of (2), so the equation (2) is solvable but the condition (3) is not satisfied.

In the case when $C = A^*$ the equation (2) becomes

$$AX + X^*A^* = B. \tag{5}$$

which is exactly the equation considered in [4]. Note that in this case, the condition (3) becomes $B = B^*$ which is now equivalent to the existence of a solution of (5). Hence, the equation (5) is solvable if and only if $B = B^*$ and in this case the set of the solutions can be represented by

$$\begin{aligned} & \{X = \frac{1}{2}A^{-1}B + (Z - Z^*)A^* : Z \in \mathcal{L}(\mathcal{K})\} \\ = \quad & \{X = \frac{1}{2}A^{-1}B + WA^* : W \in \mathcal{L}(\mathcal{K}), W^* = -W\}. \end{aligned}$$

Now, we will consider the more general case when A and C are g-invertible.

Theorem 2.2. *Let $A \in \mathcal{L}(\mathcal{K},\mathcal{H})$ and $C \in \mathcal{L}(\mathcal{H},\mathcal{K})$ be g-invertible, and $B \in \mathcal{L}(\mathcal{H})$ such that*

$$CC^\dagger = A^\dagger A,\ C^\dagger C = AA^\dagger, \tag{6}$$

$$A^\dagger B(I - AA^\dagger) = (BC^\dagger)^*(I - AA^\dagger), \tag{7}$$

$$AA^\dagger(A(C^*)^\dagger B^* + B^*(A^*)^\dagger C)AA^\dagger = 2AA^\dagger BAA^\dagger. \tag{8}$$

Operator equation (2) *has a solution $X \in \mathcal{L}(\mathcal{H},\mathcal{K})$ if and only if*

$$(I - AA^\dagger)B(I - C^\dagger C) = 0. \tag{9}$$

Any solution of the equation (2), *if it exists, is represented by*

$$X = \frac{1}{4}A^\dagger B + \frac{1}{4}(C^*)^\dagger B^* + \frac{1}{2}A^\dagger B(I - AA^\dagger) + ZC - Z^*A^* + (I - A^\dagger A)W,$$

where $Z \in \mathcal{L}(\mathcal{K})$ satisfies $C^*Z^*C = AZ^*A^*$ and $W \in \mathcal{L}(\mathcal{H},\mathcal{K})$ is arbitrary.

Proof. First suppose that there exists a solution X of the equation (2). Then

$$(I - AA^\dagger)B(I - C^\dagger C) = (I - AA^\dagger)(AX + X^*C)(I - C^\dagger C) = 0.$$

Now, suppose that (9) is satisfied. Using the following decompositions

$$\mathcal{K} = \mathcal{R}(A^*) \oplus \mathcal{N}(A) \quad \text{and} \quad \mathcal{H} = \mathcal{R}(A) \oplus \mathcal{N}(A^*),$$

since A is g-invertible, we have that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $A_1 : \mathcal{R}(A^*) \to \mathcal{R}(A)$ is invertible.

In this case, the Moore-Penrose generalized inverse of A has the following matrix decomposition

$$A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.$$

Also, there exist suitable matrix decompositions of B, C and Y:

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

$$C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \quad \text{and}$$

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.$$

By (6) it follows that $C = A^\dagger AC = CAA^\dagger$ which implies that $C_2 = 0$, $C_3 = 0$ and $C_4 = 0$. Notice, that (6) also implies that $C_1 : \mathcal{R}(A) \to \mathcal{R}(A^*)$ is invertible. Hence,

$$C^\dagger = \begin{bmatrix} C_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}.$$

By computation we obtain

$$(I - AA^\dagger)B(I - C^\dagger C) = \begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix},$$

i.e., $B_4 = 0$. Now,

$$AX + X^*C = \begin{bmatrix} A_1X_1 + X_1^*C_1 & A_1X_2 \\ X_2^*C_1 & 0 \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \\ B_3 & 0 \end{bmatrix},$$

is equivalent to

$$A_1X_1 + X_1^*C_1 = B_1, \tag{10}$$

$$A_1X_2 = B_2 \text{ and } X_2^*C_1 = B_3, \tag{11}$$

for

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.$$

From (8) it follows that $A_1(C_1^*)^{-1}B_1^* + B_1^*(A_1^*)^{-1}C_1 = 2B_1$, so, using Theorem 2.1, we have that the equation (10) is solvable with a solution

$$X_1 = \frac{1}{4}A_1^{-1}B_1 + \frac{1}{4}(C_1^*)^{-1}B_1^* + Z_1C_1 - Z_1^*A_1^*,$$

where $Z_1 \in \mathcal{L}(\mathcal{R}(A^*))$ satisfying $C_1^*Z_1^*C_1 = A_1Z_1^*A_1^*$. By (7), it follows $A_1^{-1}B_2 = (B_3C_1^{-1})^*$, so $X_2 = A_1^{-1}B_2 = (B_3C_1^{-1})^*$. Note that if $Z = \left[\begin{smallmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{smallmatrix}\right]$ then the condition $C^*Z^*C = AZ^*A^*$ is equivalent to the condition $C_1^*Z_1^*C_1 = A_1Z_1^*A_1^*$. Finally,

$$X = \frac{1}{4}A^\dagger B + \frac{1}{4}(C^*)^\dagger B^* + \frac{1}{2}A^\dagger B(I - AA^\dagger) + ZC - Z^*A^* + (I - A^\dagger A)W$$

is a solution of the equation (2) where $Z \in \mathcal{L}(\mathcal{K})$ satisfies $C^*Z^*C = AZ^*A^*$ and $W \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is arbitrary. □

As a corollary we obtain Theorem 2.2 from [4].

Corollary 2.1. *Let $A \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ be a g-invertible and $B \in \mathcal{L}(\mathcal{H})$. The operator equation $AX + X^*A^* = B$ has a solution $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ if and only if*

$$B = B^* \text{ and } (I - AA^\dagger)B(I - AA^\dagger) = 0. \tag{12}$$

Any solution of the equation, if it exists,is represented by

$$X = \frac{1}{2}A^\dagger B + \frac{1}{2}A^\dagger B(I - AA^\dagger) + (Z - Z^*)A^* + (I - A^\dagger A)W, \tag{13}$$

where $Z \in \mathcal{L}(\mathcal{K})$ and $W \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ are arbitrary.

3. The operator equation $AXB + B^*X^*A^* = C$

Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ and $C \in \mathcal{L}(\mathcal{K})$ are given operators. The equation

$$AXB + B^*X^*A^* = C, \tag{14}$$

when B is invertible reduces to

$$(A^*B^{-1})^*X + X^*A^*B^{-1} = (B^*)^{-1}CB^{-1},$$

and, when A is invertible, to

$$X(A^{-1}B^*)^* + A^{-1}B^*X^* = A^{-1}C(A^*)^{-1}.$$

Both of equations are special cases of equation (2), so by Corollary 2.1 we get the following result:

Corollary 3.1. *Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, $C \in \mathcal{L}(\mathcal{K})$ be such that B is invertible and $D = A^*B^{-1}$ is g-invertible. The solution $X \in \mathcal{L}(\mathcal{H})$ of the equation* (14) *exists if and only if*

$$C = C^* \quad \text{and} \quad (I - D^\dagger D)E(I - D^\dagger D) = 0, \tag{15}$$

where $E = (B^*)^{-1}CB^{-1}$. In this case, any solution of equation (14) *is represented by*

$$X = \frac{1}{2}(D^*)^\dagger E + \frac{1}{2}(D^*)^\dagger E(I - D^\dagger D) + (Z - Z^*)D + (I - DD^\dagger)W,$$

where $Z \in \mathcal{L}(\mathcal{H})$ and $W \in \mathcal{L}(\mathcal{H})$ are arbitrary.

It is interesting to consider the more general case when both A and B are g-invertible operators.

Theorem 3.1. *Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, $C \in \mathcal{L}(\mathcal{K})$ be such that A, B and $D = A^*B^\dagger$ are g-invertible, and*

$$B^\dagger BA = ABB^\dagger, \tag{16}$$

$$(I - BB^\dagger)(AA^\dagger C - C)BB^\dagger = 0, \tag{17}$$

$$(I - D^\dagger D)(B^\dagger)^*CB^\dagger(I - D^\dagger D) = 0. \tag{18}$$

Operator equation (14) *has a solution $X \in \mathcal{L}(\mathcal{H})$ if and only if*

$$C = C^* \quad \text{and} \quad (I - B^\dagger B)C(I - B^\dagger B) = 0. \tag{19}$$

Any solution of equation (14), *if it exists, is represented by*

$$\begin{aligned} X = {} & \frac{1}{2}(D^*)^\dagger E + \frac{1}{2}(D^*)^\dagger E(I - D^\dagger D) + BB^\dagger(Z - Z^*)D \\ & + BB^\dagger(I - DD^\dagger)W + (I - BB^\dagger)(A^\dagger CB^\dagger + (I - A^\dagger A)Y) \\ & + L(I - BB^\dagger), \end{aligned} \tag{20}$$

where $E = (B^*)^\dagger CB^\dagger$ and $Z \in \mathcal{L}(\mathcal{H}), Y \in \mathcal{L}(\mathcal{H})$, $W \in \mathcal{L}(\mathcal{H})$ and $L \in \mathcal{L}(\mathcal{H})$ are arbitrary.

Proof. If equation (14) has a solution it is evident that (19) is satisfied. On the contrary, suppose that (19) is satisfied. Since B is g-invertible, we have that

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix},$$

where $B_1 : \mathcal{R}(B^*) \rightarrow \mathcal{R}(B)$ is invertible.

Also, there exist matrix decompositions of A and C,

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \quad \text{and}$$

$$C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix}.$$

As $B^\dagger BA = ABB^\dagger$ we have $A_2 = 0$ and $A_3 = 0$. From (19), it follows that $C_1 = C_1^*$, $C_3 = C_2^*$ and $C_4 = 0$.

If we suppose that X has the following corresponding decomposition

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix},$$

we obtain that

$$AXB + (AXB)^* = \begin{bmatrix} A_1X_1B_1 + (A_1X_1B_1)^* & (A_4X_3B_1)^* \\ A_4X_3B_1 & 0 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ C_2^* & 0 \end{bmatrix},$$

i.e.,

$$A_1X_1B_1 + (A_1X_1B_1)^* = C_1, \tag{21}$$

$$A_4X_3B_1 = C_2^*. \tag{22}$$

From the fact that

$$D = A^*B^\dagger = \begin{bmatrix} A_1^*B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

is g-invertible, we have that $D_1 = A_1^*B_1^{-1}$ is g-invertible, i.e., A_1 is g-invertible. Also, from (18), we have that

$$(I - D_1^\dagger D_1)E_1(I - D_1^\dagger D_1) = 0, \tag{23}$$

where $E_1 = (B_1^*)^{-1}C_1B_1^{-1}$. Now, using Corollary 3.1, we obtain that equation (21) has a solution

$$X_1 = \frac{1}{2}(D_1^*)^\dagger E_1 + \frac{1}{2}(D_1^*)^\dagger E_1(I - D_1^\dagger D_1) + (Z_1 - Z_1^*)D_1 + (I - D_1D_1^\dagger)W_1,$$

where $Z_1 \in \mathcal{L}(\mathcal{R}(B))$ and $W_1 \in \mathcal{L}(\mathcal{R}(B))$ are arbitrary. Notice that A_4 is g-invertible since A and A_1 are g-invertible. Equation (22) has a solution if and only if $A_4A_4^\dagger C_3 = C_3$ which is satisfied by (17) and the solution is represented by

$$X_3 = A_4^\dagger C_3B_1^{-1} + (I - A_4^\dagger A_4)Y,$$

where $Y \in \mathcal{L}(\mathcal{R}(B^*), \mathcal{N}(B))$ is arbitrary.

Now, by computation we get that X is represented by (20). □

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