

Exact solutions to the boundary value problems by VIM

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Abstract

In this paper, we have employed the variational iteration method to solve the boundary value problems. Numerical results reveal that it is a very effective method compared with the results obtained by using the Adomian decomposition method in Wazwaz, A. M. (2000).

Keywords: Adomain decomposition; Variational iteration.

1. Introduction

It is well known that many physical phenomena can be described by linear or nonlinear partial differential equations. Even if there are many numerical methods such as finite element or finite difference method, it is still difficult to solve those problems numerically. Specially, to solve nonlinear problems these discretized methods require a lot of computational work to obtain accurate approximate solutions. Thus, there has been much attention to develop more efficient method to solve nonlinear problems.

Adomian decomposition method (ADM) (Adomian, G. (1994), Adomian, G. (1992), Jang, B. (2007), Wazwaz, A. M. (2000)) has been widely applied to solve the nonlinear problems and it has successfully obtained accurate approximate solutions. However, in solving boundary value problems Lui, G. L. (1995) found that ADM could not be always successful to obtain an accurate approximation because ADM leads to an error at its boundary. Wazwaz, A. M. (2000) overcame such difficulty by using ADM combined with Taylor expansion.

However, it is easy to see that the ADM introduced by Wazwaz is not

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appropriate method to solve complex boundary value problems.

In this work, we have employed the variational iteration method(VIM) (He, J-H (1997), He, J-H (2007), Jang, B. (2008)) to solve the boundary value problems. Illustrative examples show that the proposed method is very effective compared with the results obtained by ADM.

2. Description of the previous methods

In this section, we describe the Adomian decomposition method(ADM). In order to describe the ADM, let us consider the following equation in an operator form as

$$Lu + Nu = f, \quad (1)$$

where L is a differential operator, N is a nonlinear operator and f is a source function.

Applying the suitable inverse operator L^{-1} to the both sides (1) yields

$$u = g + L^{-1}f - L^{-1}Nu,$$

where g is the function arising from the boundary condition.

It is well known (Adomian, G. (1994), Adomian, G. (1992)) that the ADM decomposes the solution u by an infinite series

$$u = \lim_{n \rightarrow \infty} s_n, \quad s_n = \sum_{k=0}^n u_k,$$

where each component can be determined recursively:

$$u_0 = g + L^{-1}f, \quad (2)$$

$$u_{n+1} = -L^{-1}Nu_n, \quad n \geq 0. \quad (3)$$

The nonlinear operator term Nu_n can be also decomposed by an infinite series of polynomials as follows:

$$Nu_n = \sum_{n=0}^{\infty} A_n(u_0, u_1, \dots, u_n),$$

where $A_n(u_0, u_1, \dots, u_n)$ are the Adomian polynomials. The polynomials are defined by

$$A_n(u_0, u_1, \dots, u_n) = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N \left(\sum_{i=0}^{\infty} \lambda^i u_i \right) \right]_{\lambda=0}. \quad (4)$$

Even if ADM has been successfully employed to obtain very accurate approximate solutions of linear and nonlinear equations without using discretization and linearization, it has also limitations. For instance, let us consider L is a second order partial derivative with respect to x direction; $L = \partial^2 / \partial x^2$. Taking the inverse operator L^{-1} given by $L^{-1} = \int_0^x dx' \int_0^{x'} dx''$ to the both sides (1) yields

$$u(x, y) = u(0, y) + xu_x(0, y) + L^{-1}f - L^{-1}Nu,$$

It is worth noting that it is impossible to use ADM directly unless the condition $u_x(0, y)$ is given because the zero-th component contains $u_x(0, y)$.

In order to apply ADM to Dirchlet type boundary value problems, Wazwaz introduced a modified ADM in Wazwaz, A. M. (2000). Firstly, $u_x(0, y)$ is defined to be a function $h(y)$ and compute each component by using recursive relations (2), (3). Imposing $y=0$ to n th partial sum $s_n(x, y)$ and equating the coefficients of powers of x determine all coefficients of derivative terms $h^{(n)}(0)$, $n \geq 0$. Lastly, using the Taylor expansion of $h(y)$ at $y=0$ combined with $h^{(n)}(0)$ yields the function $h(y)$. However, this approach is not appropriate in determining the unknown function $h(y)$ if either the complicated boundary condition is assigned or all conditions of $h^{(n)}(0)$ are not given.

3. Variational iteration method

To illustrate the basic idea of the variational iteration method, let us recall (1)

$$Lu + Nu = f \text{ in } (a, b) \times (c, d). \quad (5)$$

Then the correctional function to the x -direction for the system (5) can be constructed as follows:

$$u_{n+1}(x, y) = u_n(x, y) + \int_a^x \lambda \{Lu_n(s, y) + N\tilde{u}_n(s, y) - f(s, y)\} ds, n \geq 0, \quad (6)$$

where u_n is the n th approximation, λ is a general Lagrange multiplier which can be optimally obtained by the variational theory, and \tilde{u}_n denotes the restricted variation, that is, $\delta\tilde{u}_n = 0$. Calculating variation with respect to u_n with $\delta\tilde{u}_n = 0$ yields

$$\delta u_{n+1}(x, y) = \delta u_n(x, y) + \delta \int_a^x \lambda \{Lu_n(s, y)\} ds, \quad n \geq 0,$$

which leads the stationary conditions. One can obtain a general Lagrange multiplier λ by solving stationary conditions. By substituting the general Lagrange multiplier λ into (6) each component u_n can be obtained iteratively. Compared with the ADM, VIM does not need to have extra term such as the Adomian polynomial which requires a lot of computational work to obtain each component. Thus it is straightforward to compute each component.

4. Examples

Example 4. 1. Let us consider the following nonlinear PDE

$$\nabla^2 u + (u_y)^2 = 2y + x^4 \quad \text{in } (0,1)^2 \quad (7)$$

subject to boundary conditions

$$\begin{aligned} u(0,y) &= 0, & u(1,y) &= y + a, \\ u(x,0) &= ax, & u(x,1) &= x(x+a), \end{aligned}$$

where a is a constant. Let us consider (7) in operator form

$$L_{xx}u + L_{yy}u + Nu = f, \quad (8)$$

where $L_{xx} = \partial^2/\partial x^2$, $L_{yy} = \partial^2/\partial y^2$, $N = (\partial/\partial y)^2$ and $f = 2y + x^4$.

Adomian decomposition method: The inverse operator L_{xx}^{-1} is defined by

$$L_{xx}^{-1} = \int_0^x dx' \int_0^{x'} dx''. \quad (9)$$

Applying L_{xx}^{-1} to both sides of (8) yields

$$u(x,y) = u(0,y) + xu_x(0,y) + L_{xx}^{-1}f(x,y) - L_{xx}^{-1}(L_{yy}u + Nu). \quad (10)$$

Then, we have the following recursive relation

$$\begin{aligned} u_0(x,y) &= xh(y) + L_{xx}^{-1}f(x,y), \\ u_{n+1} &= -L_{xx}^{-1}(L_{yy}u_n + A_n), \quad n \geq 0, \end{aligned}$$

where $h(y) = u_x(0,y)$ and the Adomian polynomials A_n can be determined by (4).

$$A_0 = (u_{0_y})^2, A_1 = 2u_{0_y}u_{1_y}, A_2 = 2u_{0_y}u_{2_y} + (u_{1_y})^2$$

This gives

$$u(x,y) = xh(y) + x^2y - \frac{1}{6}h''(y)x^3 - \frac{1}{12}(h'(y))^2x^4 - \frac{1}{10}h'(y)x^5 + \dots$$

In order to obtain $h(y)$ let us use the boundary condition at $y=0$ and equate coefficients of all powers of x in both sides yields $h(0) = 0$, $h^{(n)}(0) = 0$, $n \geq 1$. Since the Taylor expansion of $h(y)$ at $y=0$ is given by

$$h(x) = h(0) + h'(0)y + \frac{1}{2!}h''(0)y^2 + \dots$$

we have $h(y) = a$. This implies immediately the exact solution

$$u(x,y) = x(xy + a).$$

Variational iteration method: The correctional function to the x -direction can be obtained as follows

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^x \lambda \left\{ \frac{\partial^2 u_n(s,y)}{\partial s^2} + \frac{\partial^2 \tilde{u}(s,y)}{\partial y^2} + \left(\frac{\partial \tilde{u}_n(s,y)}{\partial y} \right)^2 - (2y + s^4) \right\} ds, \quad (11)$$

where \tilde{u}_n is a restricted variation. Using the variation theory yields the following stationary condition

$$1 - \lambda'(s)|_{s=x} = 0, \lambda(s)|_{s=x} = 0, \lambda''(s) = 0.$$

Solving the stationary condition gives the general Lagrange multiplier λ

$$\lambda = s - x.$$

Substituting the general Lagrange multiplier λ into (11) produces each component $u_n(x,y)$. Starting with the zero-th component $u_0 = A + Bx$ and using

the boundary condition at $x=0,1$ gives $A=0$, $B=y+a$. It is easy to see that computation of (11) yields $u_n(x,y)=u_2(x,y)$, $n \geq 2$ which is given by

$$u_n(x,y) = \frac{1}{60}x(-6x^4 - 5x^3 + 60xy + 60y + 60a).$$

It is clear that $u_n(x,y)$ is not the exact solution. In order to find the exact solution, instead of determining the unknown constants A and B in the zeroth component u_0 by using the boundary condition, let us first obtain each component by using (11). Simple computation shows the followings

$$u_n(x,y) = A + Bx + x^2y, \quad n \geq 2.$$

Using the boundary condition at $x=0,1$ gives $A=0$ and $B=a$ which is the exact solution

$$u(x,y) = x(xy+a).$$

Example 4. 2. Let us consider the following Helmholtz PDE

$$\nabla^2 u - u = 2e^{-y} \quad \text{in } (0,1)^2 \quad (12)$$

subject to boundary conditions

$$\begin{aligned} u(0,y) &= 0, & u(1,y) &= e^{-y}, \\ u(x,0) &= x^2, & u(x,1) &= x^2/e, \end{aligned}$$

Let us consider (12) in operator form

$$L_{xx}u + L_{yy}u - Nu = f, \quad (13)$$

where $L_{xx} = \partial^2/\partial x^2$, $L_{yy} = \partial^2/\partial y^2$, $N = u$ and $f = 2e^{-y}$.

Adomian decomposition method: In a similar manner for the previous example, taking the inverse operator L_{xx}^{-1} in (9) to both sides of (13) gives the following recursive relation:

$$\begin{aligned} u_0 &= xh(y) + L_{xx}^{-1}f, \\ u_{n+1} &= -L_{xx}^{-1}(L_{yy}u_n - N_n), \quad n \geq 0, \end{aligned}$$

where $h(y) = u_x(0,y)$. Calculating each component yields

$$u(x,y) = h(y)x + e^{-y}x^2 + \frac{1}{6}(h''(y) - h(y))x^3 + \frac{1}{120}(h^{(4)}(y) - 2h''(y) + h(y))x^5 + \dots$$

Imposing $y=0$ and equating coefficients of all powers of x yields $h(0) = 0, h^{(2n)}(0) = 0$.

Let us note that the Taylor expansion of $h(y)$ at $y=0$ contains all $h^n(0)$. However, there are only even order derivatives of $h(y)$ at $y=0$ so that $h(y)$ can not be determined explicitly. Thus, it is concluded that ADM is not appropriate method to obtain the exact solution in this problem.

Variational iteration method: The correctional function to the x -direction can be obtained as follows

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^x \lambda \left\{ \frac{\partial^2 u_n(s,y)}{\partial s^2} + \frac{\partial^2 \tilde{u}_n(s,y)}{\partial y^2} - \tilde{u}(s,y) - 2e^{-y} \right\} ds,$$

where \tilde{u}_n is a restricted variation. In a similar manner in the previous example, the variation theory yields the general Lagrange multiplier λ

$$\lambda = s - x$$

Thus we have the following iteration formula to the x -direction.

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^x (s-x) \left\{ \frac{\partial^2 u_n(s,y)}{\partial s^2} + \frac{\partial^2 u_n(s,y)}{\partial y^2} - \tilde{u}(s,y) - 2e^{-y} \right\} ds.$$

Starting with the zero-th component $u_0 = A + Bx$ yields the following few

components:
$$u_1(x,y) = A + Bx + \frac{1}{6}Bx^3 + \frac{1}{2}Ax^2 + x^2e^{-y},$$

$$u_2(x,y) = A + Bx + \frac{1}{6}Bx^3 + \frac{1}{2}Ax^2 + x^2e^{-y} + \frac{1}{120}Bx^5 + \frac{1}{24}Ax^4.$$

By using the boundary condition at $x=0,1$, both constants are equal to $A=0$ and $B=0$. Thus we have

$$u(x,y) = x^2e^{-y},$$

which is the exact solution. It is worth noting that only one iteration yields the exact solution. Let us remark that if the unknown constants A, B are determined in the zeroth components, the exact solution can not be obtained.

5. Conclusions

We present the variational iteration method to obtain the exact solution for the boundary value problems. Numerical results have been compared with the results obtained by the modified Adomian decomposition method. Even if ADM has successfully obtained the exact solution for the Dirchlet boundary problem, it regures a lot of information to determine a derivative function. Thus it could be concluded that ADM is not a appropriate method to find the exact or accurate approximate solution for the complex Dirchlet boundary value problem. However, VIM is easy to implement without producing complex terms, Adomian polynomials. It is worth noting that since the zeroth component affects all other components, the suitable zeroth component should be selected to obtain the exact or rigorous approximate solution. We hope that our work can be applied to many challenging problems in science and engineer fields.

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