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Nonlinear Regression for an Asymptotic Option Price

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Abstract

This paper approaches the problem of option pricing in an incomplete market, where the underlying asset price process follows a compound Poisson model. We assume that the price process follows a compound Poisson model under an equivalent martingale measure and it converges weakly to the Black-Scholes model. First, we express the option price as the expectation of the discounted payoff and expand it at the Black-Scholes price to obtain a pricing formula with three unknown parameters. Then we estimate those parameters using the market option data. This method can use the option data on the same stock with different expiration dates and different strike prices.

Keywords: Option pricing, compound Poisson, asymptotic expansion, nonlinear regression.

1. Introduction

The most famous option pricing model is the Black-Scholes model without any doubt. It has been used in academia and industry for a long time due to its simplicity, even though it has been well-known that the model fails to fit the real market data. But after the Black Monday in 1987, needs for alternative models has been raised in both in academia and industry. There has been extensive investigation on alternative models in finance literature and consequently, there have been many relevant and sophisticated models proposed. However, with many of those alternative models, the market becomes incomplete and unlike with the Black-Scholes model, no unique price of derivative securities exists in any incomplete market. Thus, the problem of option pricing is not a simple calculation problem any more. Since no unique price exists, some authors found a range of prices of contingent claims and since there exist many different equivalent martingale measures, several criteria on choosing a specific equivalent martingale measure have also been proposed.

Among many alternative models proposed, we would like to consider a relatively simple model with jumps, a compound Poisson model to study the option pricing problem. Compound Poisson

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processes are used for option pricing more often with diffusion processes in the form of jump-diffusion models, but they are often used to model the underlying price process on their own, as they could have such real market features that prices change at discrete random points in time (See, for example: Frey, 2000; León, *et al.*, 2002; Kirch and Runggaldier, 2004; Song and Mykland, 2006). It is also a special case of Lévy models, which appears very often in recent finance literature, being considered as a natural extension of the Black-Scholes model since the Brownian motion term is replaced by a Lévy process. Under Lévy models, the option pricing is typically studied in line of inversion, approximation and calibration. Since the characteristic function is given in an explicit form with Lévy process, we could use the inversion formula to obtain the density function, write the expected value of the option payoff with the density, set it to be equal to the observed market price, calibrate the appropriate parameters and then find the price of option of interest using those calibrated parameter values. The option pricing under a compound Poisson model can, of course, be done in the same way. But what we study in this paper has several advantages over this method. It has less parameters to calibrate or estimate and does not have to use Fast Fourier transform or any numerical methods for inversion. It is also better than general smoothing methods in the sense that we can use option data with both of different strike prices and expiration dates. The performance of smoothing spline decreases with the dimension of variables and it needs many observed data points in case of higher dimensional problems. In our case, we have two explanatory variables, strike price and expiration date, with small number of data points and thus, the performance of our method can be better than smoothing spline methods.

Here, we assume the underlying asset price process follows a compound Poisson process not only under the empricial measure but also under an equivalent martingale measure. When the underlying process follows an exponential Lévy model, it is shown that there are several equivalent martingale measures under which the underlying process still follows an exponential Lévy model with different parameters (See, Schoutens, 2003). Thus, it would be reasonable to assume that the underlying process follows a compound Poisson process under the empirical measure and a proper equivalent martingale measure. More specifically, we consider a sequence of compound Poisson processes whose limit is the Black-Scholes model. The Black-Scholes price can be, therefore, considered as the leading term of the price computed under the assumed compound Poisson model. It would be reasonable since the Black-Scholes model is still used as a reasonable approximation of the underlying process in practice and is often considered robust in theory.

The purpose of this paper is to provide a simple way to get an approximate answer for the option price for compound Poisson models using market data with different strike prices and expirations dates. We find an approximate option price by fitting the option pricing formula to the real market data.

The remainder of the paper is organized as follows. Section 2 describes the detailed model and some weak convergence results. Section 3 shows the asymptotic expansion of the option price as the expected value of the discounted payoff of an European-style option. The estimation of parameters found in Section 3 is done with nonlinear regression in Section 4. Section 5 contains concluding comments.

2. The Model

Consider a sequence of discontinuous processes that converges to a geometric Brownian motion. Each element of the sequence is a pure jump process, indexed by n. n does not have a practical meaning, but it is used as a measure of the degree of discontinuity. A larger n means that the degree of discontinuity is smaller, *i.e.*, the process is closer to a geometric Brownian motion.

We suppose that for each n, the log stock price process is defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}^{(n)})_{0\le t\le T}, \mathbf{P}^{*(n)})$ and follows a compound Poisson process such as

$$\log S_t^{(n)} = \log S_0^{(n)} + \sum_{i=1}^{N_t^{(n)}} Z_i^{(n)}, \tag{2.1}$$

where $N^{(n)}$ is a Poisson process with rate λ_n and $Z_i^{(n)}$'s are *i.i.d.* random variables that are independent of $N^{(n)}$. $\mathbf{P}^{*(n)}$ is an equivalent martingale measure that we would like to use for pricing purpose. The filtration, $\{\mathcal{F}_t^{(n)}\}$, is generated by the stock price process $S^{(n)}$ defined above. We also assume the initial stock price $S_0^{(n)}$ is the same as S_0 for all n. As n goes to ∞, we assume that λ_n[3]goes to ∞ and $Z^{(n)}$ converges to 0 in distribution. $N_t^{(n)}$ is the number of jumps in the log stock price process up to time t and $Z_i^{(n)}$ represents the size of the i^{th} jump of $\log S^{(n)}$.

We assume that the jump size distribution $Z^{(n)}$ satisfies $E^*(e^{Z^{(n)}}-1) = 1/\lambda_n r$, $E^*((e^{Z^{(n)}}-1)^2) = 1/\lambda_n\sigma^2 + O(\lambda_n^{-3/2})$, $E^*((e^{Z^{(n)}}-1)^3) = 1/(\lambda_n\sqrt{\lambda_n})k_3 + o(\lambda_n^{-3/2})$, $E^*((e^{Z^{(n)}}-1)^4) = 1/\lambda_n^2 k_4 + o(\lambda_n^{-2})$ and $E^*|(e^{Z^{(n)}}-1)^p| = o(\lambda_n^{-2})$ for $p > 4$, for the constant interest rate r and for some constants σ, k_3 and k_4. E^* represents the expectation under the martingale measure $\mathbf{P}^{*(n)}$. For example, we could use $Z^{(n)}$ such as

$$Z^{(n)} \overset{\mathcal{D}}{=} \log\left(1 + \frac{1}{\sqrt{\lambda_n}}Q + \frac{r}{\lambda_n}\right), \tag{2.2}$$

where Q is a random variable with $\mathrm{E}Q = 0$, $\mathrm{E}Q^2 = \sigma^2$, $\mathrm{E}Q^3 = k_3$ and $\mathrm{E}Q^4 = k_4$, under $\mathbf{P}^{*(n)}$. Q has a distribution that does not depend on n and it has finite moments of all orders. $\overset{\mathcal{D}}{=}$ means that both sides of the equation have the same distribution.

This definition of $Z^{(n)}$ might not look natural, but we can see that if we assume the model (2.1) under $\mathbf{P}^{*(n)}$ and want to achieve (2.3), $Z^{(n)}$ must have a distribution of this kind. For example, for the discounted stock price process to be a martingale under $\mathbf{P}^{*(n)}$, $E^*(e^{Z^{(n)}}-1) = 1/\lambda_n r$ must be satisfied because for any $s < t$,

$$\begin{aligned}
E^*\left(e^{-rt}S_t^{(n)} \,\middle|\, \mathcal{F}_s\right) &= E^*\left(e^{-rt}S_0 e^{\sum_{i=1}^{N_t^{(n)}} Z_i^{(n)}} \,\middle|\, \mathcal{F}_s\right) \\
&= e^{-rt}S_0 e^{\sum_{i=1}^{N_s^{(n)}} Z_i^{(n)}} E^*\left(e^{\sum_{i=N_s^{(n)}+1}^{N_t^{(n)}} Z_i^{(n)}} \,\middle|\, \mathcal{F}_s\right) \\
&= e^{-rt}S_s E^*\left[E^*\left(e^{\sum_{i=N_s^{(n)}+1}^{N_t^{(n)}} Z_i^{(n)}} \,\middle|\, \mathcal{F}_s, N_t^{(n)}\right) \,\middle|\, \mathcal{F}_s\right] \\
&= e^{-rt}S_s E^*\left[\left\{E^*\left(e^{Z^{(n)}}\right)\right\}^{N_t^{(n)}-N_s^{(n)}} \,\middle|\, \mathcal{F}_s\right]
\end{aligned}$$

[3]The jump intensity, λ_n, is related to the level of the trading activity of an individual stock. A heavily traded stock is modeled with a large λ_n and a less heavily traded stock is modeled with a smaller λ_n. According to the level of trading activity of a stock, we determine the value of λ_n so that the model fits with the data. Each jump occurs when there is a trading that changes the underlying stock price.

$$
\begin{aligned}
&= e^{-rt} S_s \exp\left[\lambda_n (t-s)\left\{E^*\left(e^{Z^{(n)}}-1\right)\right\}\right] \\
&= e^{-rs} S_s \exp\left[-r(t-s)+\lambda_n (t-s)\left\{E^*\left(e^{Z^{(n)}}-1\right)\right\}\right] \\
&= e^{-rs} S_s .
\end{aligned}
$$

Now, consider the asymptotics as n goes to ∞. The conditions above assure that $\log S^{(n)}$ converges in distribution to $\log S$ that is

$$
\log S_t = \log S_0 + \left(r - \frac{1}{2}\sigma^2\right) t + \sigma B_t, \tag{2.3}
$$

where B is a Brownian Motion under the limiting martingale measure (See, Song and Mykland, 2006).

We consider a market with two securities; a stock as a risky asset and a cash bond as a riskless asset. We assume that the stock has no dividend. Then consider a European style option whose payoff is $H(S_T^{(n)})$ with the expiration time T. The underlying asset price follows the compound Poisson process as in (2.1) and throughout the paper, we denote $C(x,\, t)$ the solution of the Black-Scholes PDE in (3.7) at time $t < T$, with the terminal condition, $C(x,\, T) = H(x)$. C_S, C_{SS} and C_{SSS} denote the first, second and third derivatives of $C(x,\, t)$ with respect to x, respectively.

For each n, we compute $C(S_t^{(n)},\, t)$ by plugging in the corresponding stock price process $S^{(n)}$. In other words, $C(S_t^{(n)},\, t)$ is computed by the Black-Scholes PDE, but it may be different from what we observe from the market. On the other hand, $C(S_t,\, t)$ is also computed by the Black-Scholes PDE, but it is the true market price in the limit because the limiting stock price S follows the geometric Brownian motion.

3. Asymptotic Expansion

Let us denote $C^*(S_t,\, t)$ to be the discounted Black-Scholes price, that is, $e^{-rt}C(S_t,\, t)$. Then, by Itô's formula and Taylor expansion, since $S^{(n)}$ is a pure jump process,

$$
\begin{aligned}
C^*\left(S_t^{(n)},\, t\right) = C(S_0,\, 0) &+ \int_0^t C_t^*\left(S_{u-}^{(n)},\, u\right) du + \sum_{u\le t}\left\{C_S^*\left(S_{u-}^{(n)},\, u\right)\Delta S_u^{(n)}\right. \\
&+ \frac{1}{2} C_{SS}^*\left(S_{u-}^{(n)},\, u\right)\left(\Delta S_u^{(n)}\right)^2 + \frac{1}{6} C_{SSS}^*\left(S_{u-}^{(n)},\, u\right)\left(\Delta S_u^{(n)}\right)^3 \\
&\left.+ \frac{1}{24} C_S^{*(4)}\left(\tilde{Z}_u^{(n)},\, u\right)\left(\Delta S_u^{(n)}\right)^4\right\},
\end{aligned}
$$

where $\tilde{Z}^{(n)}$ is a process satisfying $\min(S_{t-}^{(n)},\, S_t^{(n)}) \le \tilde{Z}_t^{(n)} \le \max(S_{t-}^{(n)},\, S_t^{(n)})$. $C_t^*(x,\, t)$ is the first derivative of $C^*(x,\, t)$ with respect to t. $C^*(S_t^{(n)},\, t)$ is also written as

$$
\begin{aligned}
C^*\left(S_t^{(n)},\, t\right) = C\left(S_0,\, 0\right) &+ \int_0^t C_t^*\left(S_{u-}^{(n)},\, u\right) du + \int_0^t C_S^*\left(S_{u-}^{(n)},\, u\right) dS_u^{(n)} \\
&+ \int_0^t \frac{1}{2} C_{SS}^*\left(S_{u-}^{(n)},\, u\right) d\left[S^{(n)},\, S^{(n)}\right]_u + \int_0^t \frac{1}{6} C_{SSS}^*\left(S_{u-}^{(n)},\, u\right) d\left[S^{(n)},\, S^{(n)},\, S^{(n)}\right]_u \\
&+ \int_0^t \frac{1}{24} C_S^{*(4)}\left(\tilde{Z}_u^{(n)},\, u\right) d\left[S^{(n)},\, S^{(n)},\, S^{(n)},\, S^{(n)}\right]_u ,
\end{aligned} \tag{3.1}
$$

where $[S^{(n)},\ldots,S^{(n)}]^v$ is the v^{th} order optional variation of $S^{(n)}$. It is defined as $[S^{(n)},\ldots,S^{(n)}]^v = \lim_{\max|t_{i+1}-t_i|\to 0}\sum(S^{(n)}_{t_{i+1}} - S^{(n)}_{t_i})^v$. Since $S^{(n)}$ is a pure jump process,

$$\left[S^{(n)},\ldots,S^{(n)}\right]^v_t = \sum_{i=1}^{N^{(n)}_t}\left(\Delta S^{(n)}_{\tau^{(n)}_i}\right)^v = \sum_{i=1}^{N^{(n)}_t}\left(S^{(n)}_{\tau^{(n)}_{i-}}\right)^v\left\{\exp\left(Z^{(n)}_i\right)-1\right\}^v,$$

where $\tau^{(n)}_i$ is the time of the i^{th} jump of $S^{(n)}$. By the uniqueness of Doob-Meyer decomposition,

$$\left\langle S^{(n)},\ldots,S^{(n)}\right\rangle^{*v}_t = E^*\left(\exp\left(Z^{(n)}\right)-1\right)^v\int_0^t\left(S^{(n)}_{u-}\right)^v\lambda_n du.$$

$\langle S^{(n)},\ldots,S^{(n)}\rangle^{*v}_t$ denotes the compensator of $[S^{(n)},\ldots,S^{(n)}]^v$ under $\mathbf{P}^{*(n)}$. Using (3.1) and the fact that $[S^{(n)},\ldots,S^{(n)}]^v - \langle S^{(n)},\ldots,S^{(n)}\rangle^{*v}_t$ is a martingale under $\mathbf{P}^{*(n)}$, we obtain the following.

Theorem 3.1. *Under the model given in (2.1) and assumptions in (3.3), (3.4), (3.5) and (3.6), the price of the European-style option with the payoff $H(S^{(n)}_T)$ is*

$$\begin{aligned}E^*e^{-rT}H\left(S^{(n)}_T\right) &= C(S_0,0) + \left(\alpha^{(n)}_2 - \frac{1}{2}\sigma^2\right)C^*_{SS}(S_0,0)S_0^2T\\ &\quad + \lambda_n^{-\frac{1}{2}}\alpha^{(n)}_3 C^*_{SSS}(S_0,0)S_0^3T + o\left(\lambda_n^{-\frac{1}{2}}\right),\end{aligned} \tag{3.2}$$

where $\alpha^{(n)}_2 = \lambda_n/2E^*(e^{Z^{(n)}}-1)^2$ and $\alpha^{(n)}_3 = (\lambda_n\sqrt{\lambda_n})/6E^*(e^{Z^{(n)}}-1)^3$, under the following assumptions.

Assumption 3.1. *For a process, $\tilde{Z}^{(n)}$, satisfying $\min(S^{(n)}_{t-}, S^{(n)}_t) \le \tilde{Z}^{(n)}_t \le \max(S^{(n)}_{t-}, S^{(n)}_t)$,*

$$\int_0^T E^*\left(C^{*(4)}_S\left(\tilde{Z}^{(n)}_u, u\right)\left(S^{(n)}_u\right)^4\right)du = O(1). \tag{3.3}$$

$$\int_0^T\int_0^t E^*\left(C^{*(v)}_S\left(S^{(n)}_u, u\right)\left(S^{(n)}_u\right)^v\right)dudt = O(1), \tag{3.4}$$

for $v = 2$, 3 and 4,

$$\int_0^T\int_0^t E^*\left(C^{*(v)}_S\left(\tilde{Z}^{(n)}_u, u\right)\left(\tilde{Z}^{(n)}_u\right)^{v-3}\left(S^{(n)}_u\right)^3\right)dudt = O(1), \tag{3.5}$$

for $v = 3$, 4 and 5 and

$$\int_0^T\int_0^t E^*\left[\left\{C^{*(v)}_S\left(\tilde{Z}^{(n)}_u, u\right)\left(\tilde{Z}^{(n)}_u\right)^{v-2} - C^{*(v)}_S\left(S^{(n)}_u, u\right)\left(S^{(n)}_u\right)^{v-2}\right\}\left(S^{(n)}_u\right)^2\right]dudt = o(1), \tag{3.6}$$

for $v = 3$, 4 and 5.

Proof.

$$\begin{aligned}E^*e^{-rT}H\left(S^{(n)}_T\right) &= E^*C^*\left(S^{(n)}_T, T\right)\\ &= C(S_0,0) + \int_0^T E^*C^*_t\left(S^{(n)}_{u-}, u\right)du\\ &\quad + E^*\int_0^T E^*\left(e^{Z^{(n)}}-1\right)\lambda_n S^{(n)}_{u-}C^*_S\left(S^{(n)}_{u-}, u\right)du\end{aligned}$$

$$+ E^* \int_0^T \frac{1}{2} E^* \left(e^{Z^{(n)}} - 1\right)^2 \lambda_n \left(S_{u-}^{(n)}\right)^2 C_{SS}^* \left(S_{u-}^{(n)}, u\right) du$$
$$+ E^* \int_0^T \frac{1}{6} E^* \left(e^{Z^{(n)}} - 1\right)^3 \lambda_n \left(S_{u-}^{(n)}\right)^3 C_{SSS}^* \left(S_{u-}^{(n)}, u\right) du$$
$$+ E^* \int_0^T \frac{1}{24} E^* \left(e^{Z^{(n)}} - 1\right)^4 \lambda_n \left(S_{u-}^{(n)}\right)^4 C_S^{*(4)} \left(\tilde{Z}_u^{(n)}, u\right) du.$$

By Black-Scholes PDE, we have

$$-rC(x, t) + C_t(x, t) + rxC_x(x, t) + \frac{1}{2}\sigma^2 x^2 C_{xx}(x, t) = 0, \tag{3.7}$$

for the Black-Scholes price $C(x, t)$. From that, we can easily see that

$$C_t^*(x, t) + rxC_x^*(x, t) + \frac{1}{2}\sigma^2 x^2 C_{xx}^*(x, t) = 0$$

and consequently, we get

$$E^* C^* \left(S_T^{(n)}, T\right) = C\left(S_0, 0\right) + \left\{-r + E^*\left(e^{Z^{(n)}} - 1\right)\right\} \lambda_n E^* \int_0^T S_u^{(n)} C_S^* \left(S_u^{(n)}, u\right) du$$
$$+ \frac{1}{2}\left\{E^*\left(e^{Z^{(n)}} - 1\right)^2 \lambda_n - \sigma^2\right\} E^* \int_0^T \left(S_u^{(n)}\right)^2 C_{SS}^* \left(S_u^{(n)}, u\right) du$$
$$+ \frac{1}{6} E^* \left(e^{Z^{(n)}} - 1\right)^3 \lambda_n E^* \int_0^T \left(S_u^{(n)}\right)^3 C_{SSS}^* \left(S_u^{(n)}, u\right) du$$
$$+ \frac{1}{24} E^* \left(e^{Z^{(n)}} - 1\right)^4 \lambda_n E^* \int_0^T \left(S_u^{(n)}\right)^4 C_S^{*(4)} \left(\tilde{Z}_u^{(n)}, u\right) du.$$

Since $E^*(e^{Z^{(n)}} - 1) = r/\lambda_n$, with $\alpha_2^{(n)} = \lambda_n/2\, E^*(e^{Z^{(n)}} - 1)^2$ and $\alpha_3^{(n)} = (\lambda_n\sqrt{\lambda_n})/6\, E^*(e^{Z^{(n)}} - 1)^3$, we obtain

$$E^* C^* \left(S_T^{(n)}, T\right) = C\left(S_0, 0\right) + \left(\alpha_2^{(n)} - \frac{1}{2}\sigma^2\right) E^* \int_0^T \left(S_u^{(n)}\right)^2 C_{SS}^* \left(S_u^{(n)}, u\right) du$$
$$+ \lambda_n^{-\frac{1}{2}} \alpha_3^{(n)} E^* \int_0^T \left(S_u^{(n)}\right)^3 C_{SSS}^* \left(S_u^{(n)}, u\right) du$$
$$+ \frac{1}{24} o\left(\lambda_n^{-\frac{1}{2}}\right) E^* \int_0^T \left(S_u^{(n)}\right)^4 C_S^{*(4)} \left(\tilde{Z}_u^{(n)}, u\right) du.$$

And assuming (3.3),

$$E^* C^* \left(S_T^{(n)}, T\right) = C\left(S_0, 0\right) + \left(\alpha_2^{(n)} - \frac{1}{2}\sigma^2\right) E^* \int_0^T \left(S_u^{(n)}\right)^2 C_{SS}^* \left(S_u^{(n)}, u\right) du$$
$$+ \lambda_n^{-\frac{1}{2}} \alpha_3^{(n)} E^* \int_0^T \left(S_u^{(n)}\right)^3 C_{SSS}^* \left(S_u^{(n)}, u\right) du + o\left(\lambda_n^{-\frac{1}{2}}\right).$$

Similarly to the above calculation, we can also show that

$$E^*\left(\left(S_u^{(n)}\right)^2 C_{SS}^* \left(S_u^{(n)}, n\right)\right) = C_{SS}\left(S_0, 0\right) S_0^2 + O\left(\lambda_n^{-\frac{1}{2}}\right)$$

and

$$E^*\left(\left(S_u^{(n)}\right)^3 C_{SSS}^* \left(S_u^{(n)}, n\right)\right) = C_{SSS}\left(S_0, 0\right) S_0^3 + O\left(\lambda_n^{-\frac{1}{2}}\right),$$

under Assumption 3.1 (See, Song, 2007).

Therefore,

$$E^{*}e^{-rT}H\left(S_T^{(n)}\right) = C\left(S_0, 0\right) + \left(\alpha_2^{(n)} - \frac{1}{2}\sigma^2\right) C_{SS}\left(S_0, 0\right) S_0^2 T + \lambda_n^{-\frac{1}{2}} \alpha_3^{(n)} C_{SSS}\left(S_0, 0\right) S_0^3 T + o\left(\lambda_n^{-\frac{1}{2}}\right).$$

□

We would like to drop the higher order term $o(\lambda_n^{-1/2})$ in (3.2) and use the first three terms to price the European-style option. We will denote β_1 and β_2 to be $(\alpha_2^{(n)} - 1/2\sigma^2)$ and $\lambda_n^{-1/2}\alpha_3^{(n)}$, respectively, in Section 4. Note that both of β_1 and β_2 are $O(\lambda_n^{-1/2})$. Since there is σ inside the formula for the Black-Scholes price $C(S_0, 0)$, the option price formula we obtained has three unknown parameters, σ, β_1 and β_2. In Section 4, we will estimate those parameters from the market data.

4. Estimation

In this section, we illustrate the procedure of estimating necessary parameters to obtain an approximate option price with real market data. We use the nonlinear least squares function *nls* in the statistical programming language, R. As is well known, the least squares method is the most commonly used method for regression problems and it minimizes the prediction error when we use a squared loss function (See, Bates and Watts, 1988).

We use the European call option prices on S&P 500 Index for October 29, 2007. The dataset has 52 observations with three variables; option price, strike price and time to expiration. The current index price, S_0, is \$1540.98. The response variable is the option price and the explanatory variables are strike price and time to expiration. Before we try to estimate parameters, we need to fix the value, r, for risk-free interest rate. It can be done by observing the future price on S&P 500. Using the futures data on October 25, 2007, we fix $r = 0.0269$. This would be, in fact, an estimate of the difference between the interest rate and the dividend yield rate of S&P 500 index, but we can regard this as the estimate of the short rate because we are assuming no stock dividend.

We use the model

$$E^{*}e^{-rT}H\left(S_T^{(n)}\right) \approx C(S_0, 0) + \beta_1 C_{SS}(S_0, 0) S_0^2 T + \beta_2 C_{SSS}(S_0, 0) S_0^3 T$$

and estimate σ, β_1 and β_2 as we discuss at the end of the previous section. With the European call option, $H(S_T^{(n)}) = (S_T^{(n)} - K)^{+}$, where K is the strike price at which the option is exercised and the above model can be written as

$$E^{*}e^{-rT}\left(S_T^{(n)} - K\right)^{+} \approx S_0\Phi(z_+) - e^{-rT}K\Phi(z_-) + \beta_1 S_0^2 T \frac{\phi(z_+)}{S_0\sigma\sqrt{T}} + \beta_2 S_0^3 T \frac{\phi(z_+)}{S_0^2\sigma\sqrt{T}}\left(-1 - \frac{z_+}{\sigma\sqrt{T}}\right),$$

where $z_\pm = \{\log(S_0/K) + (r \pm \sigma^2/2)T\}/(\sigma\sqrt{T})$, ϕ and Φ are the density and the cumulative distribution function of the standard normal distribution, respectively, K is the strike price and T is the time to expiration of the option. Since the left hand side of the above equation is the price of

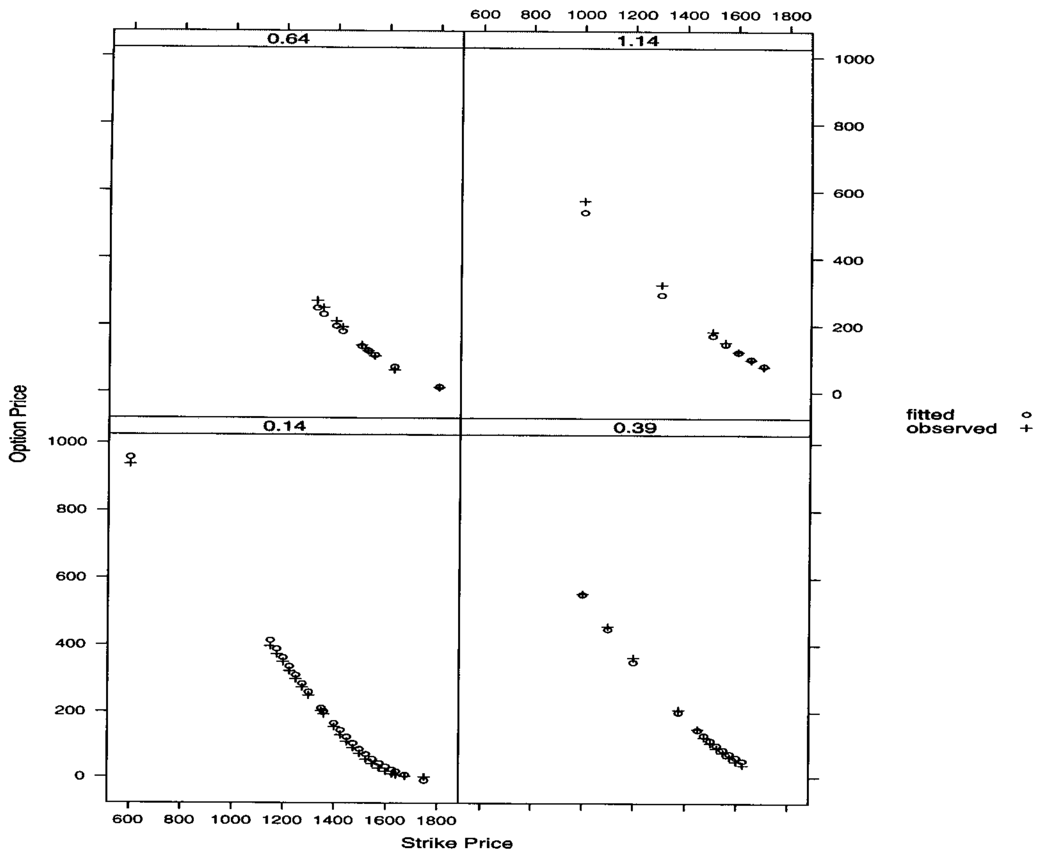

Figure 4.1. Observed and fitted option prices

the option, we set it to equal to the observed option price in the market. Observed values of S_0, K, T and r are plugged in the right hand side and then, we have σ, β_1 and β_2 to be estimated.

The results show that β_2 is insignificant, so we dropped it from the model. Without it, the estimated parameter values are $\hat{\sigma} = 0.448$ and $\hat{\beta}_1 = -0.114$. Both parameters are very significant with p-values smaller than 0.001. Using these estimated parameters, we can compute option prices for any possible combination of strike price and option maturity. Figure 4.1 shows observed and fitted option prices for different values of strike prices and option maturities. In the figure, we see four plots for each maturity of $T = 0.14$, 0.39, 0.64 and 1.14 years. The observed strike prices are between \$1000 and \$1800 with one case of \$600. As we can see, the fitted values are very close to the observed values except some points with the maturity longer than 1 year and low strike prices.

Even though we used a very simple model and simple estimation procedure, overall fitting performance is quite good. Considering that we commonly use inversion with the Fast Fourier Transform to calibrate parameters with general Lévy models, our procedure of computing option prices is fast and simple.

5. Concluding Remarks

In this paper, we found an asymptotic option price formula for a compound Poisson model that converges to the Black-Scholes model. The formula consists of the Black-Scholes price as a leading term and some correction terms. Then we use the formula to find fitted option prices by nonlinear least squares method. The procedure we used is very fast, simple and practical to use with real market data, compared to other pricing methods. We checked that the fitted values are very close to the observed option prices in Figure 4.1.

One may attempt to fit a linear interpolation or a smoothing spline method for each maturity date in Figure 4.1. It may work better than our proposed method for each fixed maturity date since it is a problem of fitting the option price with one dimensional explanatory variable. However, using this type of methods, we cannot compute an option price for maturity dates that are not observed in the market. The method we propose has advantages in the sense that we can compute an option price for any combination of strike prices and maturity dates, while we do not require to observe lots of data points.

It is also clear that we can use the same method for more general Lévy processes if we assume a sequence of exponential Lévy processes converging to the Black-Scholes model. We would, however, need several assumptions on the Lévy measure to obtain the right orders of convergence.

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