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A Robust Pricing/Lot-sizing Model and A Solution Method Based on Geometric Programming*

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ABSTRACT

The pricing/lot-sizing problem of determining the robust optimal order quantity and selling price is discussed. The uncertainty of parameters characterized by an ellipsoid is explicitly incorporated into the problem. An approximation scheme is proposed to transform the problem into a geometric program, which can be efficiently and reliably solved using interior-point methods.

Keywords: Robust Optimization, Pricing/Lot-sizing Model, Geometric Programming

1. Introduction

This paper deals with the pricing/lot-sizing problem of determining the robust optimal order quantity and selling price that maximize the profit for a retailer. Unlike the classical EOQ model, this problem assumes that demand can change depending on selling price rather than being constant over time, and purchase cost can also change depending on order quantity.

There have been several studies of the pricing/lot-sizing problem, among which [1, 6, 7] are representative. Especially, Lee [8] developed a solution method based on geometric programming, which guarantees to find a global optimal solution. In those studies, however, all parameters such as scaling constants, price elasticity, and quan-

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tity discount factor involved in the problem were assumed to be known for sure, which is somewhat unrealistic in practice.

We explicitly incorporate the uncertainties underlying these parameters into the problem, and seek to find out the *robust* optimal solution using *robust optimization.*

Robust optimization models can be classified into two categories depending on how the uncertainty of parameters is incorporated into models; stochastic robust optimization and worst-case robust optimization. To illustrate the two models, consider the following optimization problem:

$$\begin{aligned} \min \quad & f_0(x, u) \\ s.t. \quad & f_i(x, u) \leq 0, i = 1, \cdots, m \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^n$ is the decision variable, the function f_0: $\mathbf{R}^n \rightarrow \mathbf{R}$ is the objective function, the functions f_i: $\mathbf{R}^n \rightarrow \mathbf{R}$, $i = 1, \cdots, m$, are the constraint functions, and $u \in R^k$ is an uncertain parameter vector.

In stochastic robust optimization models, the parameter vector u is modeled as a random variable with a known distribution, and we work with the expected values of the constraints and objective functions as follows:

$$\begin{aligned} \min \quad & \mathbf{E} f_0(x, u) \\ s.t. \quad & \mathbf{E} f_i(x, u) \leq 0, i = 1, \cdots, m \end{aligned} \tag{2}$$

where the expectation is with respect to u.

In worst-case robust optimization models, we are given a set U that u is known to lie in, and we work with the worst-case values of the constraints and objective functions as follows [3]:

$$\begin{aligned} \min \quad & \sup_{u \in U} \ f_0(x, u) \\ s.t. \quad & \sup_{u \in U} \ f_i(x, u) \leq 0, i = 1, \cdots, m \end{aligned} \tag{3}$$

Worst-case robust optimization models are useful in such situations where one wants to get a solution to his optimization problem, which is *robust and less vulnerable* to uncontrollable uncertainties involved in the problem. Putting it another way, if the failure of the operating system from which the given optimization problem is derived results in, say, a catastrophic disaster, then it is quite natural for the decision maker to seek a robust solution whose performance is least vulnerable to undesirable realizations of the uncertainties. More recent and widely accepted definition of robust optimization refers to this worst-case model.

We will develop a worst-case robust optimization model for the pricing/lot-sizing problem with uncertain parameters, and propose a solution method based on geometric programming. A geometric programming problem (GP) is a type of mathematical optimization problem characterized by objective and constraint functions that have a special form. The objective must be posynomial and it must be minimized; the equality constraints can only have the form of a monomial equal to one, and the inequality constraints can only have the form of a posynomial less than or equal to one. Recently developed solution methods (interior-point methods) can solve even large-scale GPs extremely efficiently and reliably; at the same time a number of practical problems have been found to be equivalent to (or well approximated by) GPs [2]. We will show that the worst-case robust optimization model for the pricing/lot-sizing problem with uncertain parameters can be well approximated by a GP and thus can be solved efficiently using interior-point methods, which is a main contribution of the paper.

2. The Model

In the pricing/lot-sizing problem discussed in the paper, profit (π (P, Q)) for a retailer is defined as revenue less ordering cost, inventory holding cost, and purchase cost;

$$\pi\ (P, Q) = PD - aD/Q - icQ/2 - cD, \tag{4}$$

where P and Q are the decision variables representing selling price and order quantity, respectively. a is ordering cost, i is inventory carrying rate per unit, and c is unit purchase cost.

Differently from the classical EOQ model, demand (D) is a decreasing function of product price (P) with constant elasticity; that is, $D = kP^{-\alpha}$, $\alpha > 1$, where k is the scaling constant and α is the price elasticity, representing relative change in demand with respect to corresponding relative change in price. Furthermore, when quantity discount is offered by the supplier, the unit purchase cost c is a decreasing function of the order quantity; that, $c = c(Q)$, with $dc/dQ \leq 0$. We assume a continuous quantity discount, and it is represented by the function $c = rQ^{-\beta}$, $0 < \beta < 1$, which reflects an inverse relationship between unit purchase cost and order quantity. The discount factor β is similar to the price elasticity α; it measures the relative change in purchase cost with respect to corresponding relative change in order quantity, and would be a very small value, say, around .01 or .02. Any $\beta > 1$ represents too much discounting and would be unrealistic [8]. The usual assumptions of the classical EOQ model are made, except for the assumption of constant demand. These include instantaneous replenishment, no stock-outs, and a fixed order cost. By incorporating the demand function and the purchase cost function, we get the following profit formula:

$$\pi(P, Q) = kP^{-\alpha+1} - akP^{-\alpha}Q^{-1} - 0.5irQ^{-\beta+1} - krP^{-\alpha}Q^{-\beta} \tag{5}$$

There are several papers that present the above model, which include [1, 7, 8], but in this work we mitigate the strong assumption which was made in those papers that the parameters k and α are known for sure. Notice that the constants r and β can be determined with relatively high certainty since order quantity discount schedule is usually planned in advance between trading partners. So, it is not unrealistic to assume that the two constants are known for sure. But the parameters α and k may have significant variability since demand cannot be exactly predicted simply based on price. Therefore, it is more realistic to allow uncertainty in those parameters. We also assume that the uncertainty is described by an ellipsoid.

Let us have the ellipsoidal representation of the uncertainty, F, given by

$$\begin{pmatrix} \log k \\ \alpha \end{pmatrix} \in F = \left\{ \begin{pmatrix} \overline{\log k} \\ \bar{\alpha} \end{pmatrix} + M \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \mid \| w \|_2 \leq 1 \right\} \tag{6}$$

where $M \in R^{2\times 2}$ is a symmetric positive definite matrix. This sort of uncertainty set can be obtained by applying linear regression on past demand data. Then, the robust optimization version of the problem is formulated as follows:

$$\max_{P,Q} \inf_{(\log k,\alpha)'\in F} kP^{-\alpha+1} - akP^{-\alpha}Q^{-1} - 0.5irQ^{-\beta+1} - krP^{-\alpha}Q^{-\beta} \tag{7}$$

which can be rewritten as

$$\max_{P,Q} (\inf_{(\log k,\alpha)'\in F} kP^{-\alpha})(P - aQ^{-1} - rQ^{-\beta}) - 0.5irQ^{-\beta+1} \tag{8}$$

The following lemma shows that the optimal value of $\inf_{(\log k,\, \alpha)'\in F} kP^{-\alpha}$ can be found analytically.

Lemma 1: $\inf_{(\log k,\, \alpha)'\in F} kP^{-\alpha} = e^{(1\ \ -\log P)\begin{pmatrix}\overline{\log k}\\ \bar{\alpha}\end{pmatrix} - \left\| M^T \begin{pmatrix}-1\\ \log P\end{pmatrix}\right\|_2}$

Proof: First, $\inf_{(\log k,\, \alpha)'\in F} kP^{-\alpha} = 1/(\sup_{(\log k,\, \alpha)'\in F} P^{\alpha}/k)$. If logarithm is applied to the supremum, then we get the following equivalent problem:

$$\sup_{(\log k,\, \alpha)'\in F} (\alpha \log P - \log k) = \begin{array}{ll} & \max_w\ \alpha \log P - \log k \\ s.t. & \begin{pmatrix}\log k\\ \alpha\end{pmatrix} = \begin{pmatrix}\overline{\log k}\\ \bar{\alpha}\end{pmatrix} + M\begin{pmatrix}w_1\\ w_2\end{pmatrix}, \\ & \| w \|_2 \le 1. \end{array}$$

Furthermore, the above problem can be transformed into the following:

$$\begin{array}{ll} \max_w & (-1\ \ \log P)\begin{pmatrix}\overline{\log k}\\ \bar{\alpha}\end{pmatrix} + (-1\ \ \log P)\, M \begin{pmatrix}w_1\\ w_2\end{pmatrix} \\ s.t. & \| w \|_2 \le 1, \end{array}$$

of which optimal value is

$$(-1\ \ \log P)\begin{pmatrix}\overline{\log k}\\ \bar{\alpha}\end{pmatrix} + \left\| M^T \begin{pmatrix}-1\\ \log P\end{pmatrix}\right\|_2$$

hence the result. ■

Based on the lemma, the problem is reduced to

$$\text{(P)} \qquad \max_{P,Q}\ \bar{k}P^{-\bar{\alpha}}E(P)(P - aQ^{-1} - rQ^{-\beta}) - 0.5irQ^{-\beta+1}$$

where $\bar{k} = e^{\overline{\log k}}$ and $E(P) = e^{-\left\| M^T \begin{pmatrix} -1 \\ \log P \end{pmatrix} \right\|_2}$.

3. Solution Method

The problem (P) is not a GP, and it is not obvious how to solve it directly (say, whether it is convex or not). However, if we could closely approximate the exponential function $E(P)$ in the problem by a certain monomial function that has the form of a power function of P, sP^{-t} ($s > 0$ and $t > 0$ are coefficients), then (P) can be cast as an unconstrained signomial problem as follows:

$$\text{(SP)} \qquad \begin{array}{ll} \max_{P,Q,z} & z \\ \text{s.t.} & s\bar{k}P^{-\bar{\alpha}-t+1} - as\bar{k}P^{-\bar{\alpha}-t}Q^{-1} - rs\bar{k}P^{-\bar{\alpha}-t}Q^{-\beta} - 0.5irQ^{-\beta+1} \ge z. \end{array}$$

Following Duffin *et al.* [4], this problem can be transformed into a GP:

$$\text{(TP)} \qquad \begin{array}{ll} \min_{P,Q,z} & z^{-1} \\ \text{s.t.} & s^{-1}\bar{k}^{-1}zP^{\bar{\alpha}+t-1} + aP^{-1}Q^{-1} + rP^{-1}Q^{-\beta} + 0.5irs^{-1}\bar{k}^{-1}P^{\bar{\alpha}+t-1}Q^{-\beta+1} \le 1. \end{array}$$

As mentioned in the introduction, this GP can be solved very efficiently and reliably using interior-point methods since it can be transformed into a convex optimization problem [2].

Now, let us show how the approximation goes. When the defining matrix M (it defines the ellipsoid that describes the uncertainty of parameters) is given by

$$M = \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix}, \ m_1 > 0,\ m_3 > 0,\ m_1 m_3 - m_2^2 > 0 \qquad (9)$$

the exponent of $E(P)$ can be written as

$$-\left\| M^T \begin{pmatrix} -1 \\ \log P \end{pmatrix} \right\|_2 = \sqrt{f(\log P)^2 + g\log P + h} \tag{10}$$

where $f = m_2^2 + m_3^2$, $g = -2(m_1 m_2 + m_2 m_3)$, $h = m_1^2 + m_2^2$.

Then the approximation we are seeking goes as follows:

$$\begin{aligned}
& E(P) = e^{-\left\| M^T \begin{pmatrix} -1 \\ \log P \end{pmatrix} \right\|_2} \approx sP^{-t} \quad \text{(applying logarithm)} \\
\Rightarrow\ & -\sqrt{f(\log P)^2 + g\log P + h} \approx \log s - t\log P \quad \text{(squaring)} \\
\Rightarrow\ & f(\log P)^2 + g\log P + h \approx t^2(\log P)^2 - 2t\log s \log P + (\log s)^2 \\
\Rightarrow\ & t = \sqrt{f},\ s = e^{-g/2\sqrt{f}}
\end{aligned}$$

Thus, it is quite reasonable to approximate $E(P)$ by $\overline{E(P)} = e^{-g/2\sqrt{f}} P^{-\sqrt{f}}$. For approximation error analysis, let's see the ratio $E(P)/\overline{E(P)}$. First of all, the following theorem shows that $E(P)$ is strictly less than $\overline{E(P)}$ for all P, and the quality of approximation is improved as P goes up.

Theorem 1: $E(P)/\overline{E(P)} < 1$ *for all P and the ratio is increasing in P and converges to one as P goes to infinity.*

Proof: Setting $r_1 := -\sqrt{f(\log P)^2 + g\log P + h}$ and $r_2 := -g/2\sqrt{f} - \sqrt{f}\log P$, we get the following:

$$\begin{aligned}
r_1^2 - r_2^2 &= h - g^2/4f \\
&= (m_1 m_3 - m_2^2)^2/(m_2^2 + m_3^2) \\
&= \det(M)^2/(m_2^2 + m_3^2) > 0.
\end{aligned}$$

This indicates $r_1 < r_2 < -r_1$, which leads to $r_1 - r_2 < 0$ and $r_1 + r_2 < 0$. Therefore, $E(P)/\overline{E(P)} = e^{r_1 - r_2} < 1$. In addition, we get from the above

$$r_1 - r_2 = \frac{\det(M)^2}{(m_2^2 + m_3^2)(r_1 + r_2)}.$$

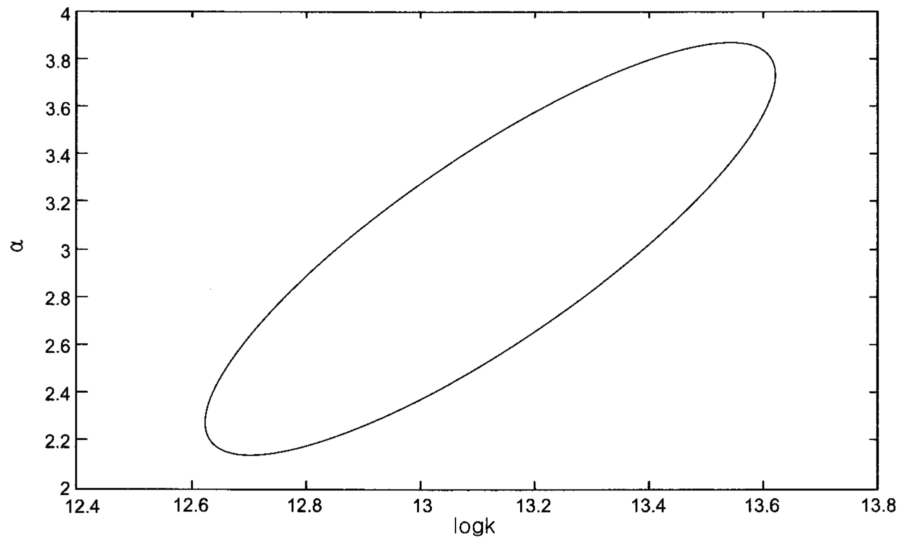

Figure 1. The ellipsoidal representation F

It is obvious that $r_1 + r_2 (<0)$ is decreasing in P and it goes to negative infinity as P goes to infinity. This means that $r_1 - r_2$ is increasing in P and it converges to zero as P goes to infinity. Thus, $E(P)/\overline{E(P)} = e^{r_1 - r_2}$ is increasing and converges to one as P goes to infinity. ■

It is easy to show that $\overline{E(P)}$ is monotone decreasing, and $E(P)$ is increasing over the range (0, $e^{-g/2f}$] and decreasing over [$e^{-g/2f}$, ∞), where $e^{-g/2f}$ is the point at which the extreme value of $E(P)$ is attained. We should note that the approximation error grows significantly as P approaches zero. But, it can be found that the error is reasonably bounded when P is greater than the extreme point of $E(P)$. At that point, the ratio $E(P)/\overline{E(P)}$ is

$$E(e^{-g/2f}) / \overline{E(e^{-g/2f})} = e^{-\frac{\det(M)^2}{f\sqrt{-g^2/4f+h}}} \tag{11}$$

4. Numerical Example

To illustrate the proposed model and the solution method, an example problem will be solved in this section. For this problem, we choose the values of parameters as a = 50, i = 0.1, r = 5, and b = 0.01. We assume that the ellipsoidal representation for the uncertain parameter vector (log$k$, α)′ is given by

$$\begin{pmatrix} \log k \\ \alpha \end{pmatrix} \in F = \left\{ \begin{pmatrix} \log(500000) \\ 3 \end{pmatrix} + \begin{pmatrix} 0.4 & 0.3 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \mid \| w \|_2 \le 1 \right\} \tag{12}$$

Thus, $\bar{k} = 500000$, $f = 0.73$, $g = -0.72$, $h = 0.25$, and $\det(M) = 0.23$. Figure 1 shows the above ellipsoid. In this uncertainty set, it can be seen that the variations of α and k are significant.

The profit maximization problem (P) is realized as

$$\text{(P)} \quad \max_{P,Q} 500000 P^{-3} e^{-\left\| M^T \begin{pmatrix} -1 \\ \log P \end{pmatrix} \right\|_2} (P - 50Q^{-1} - 5Q^{-0.01}) - 0.25Q^{0.99} \tag{13}$$

The exponential function $E(P) = e^{-\left\| M^T \begin{pmatrix} -1 \\ \log P \end{pmatrix} \right\|_2}$ is approximated by $\overline{E(P)} = 1.524015P^{-0.8544}$. In order to show the accuracy of the approximation, the graphs of the two functions are created and superimposed in Figure 2. Figure 3 plots $E(P)/\overline{E(P)}$ and shows how the approximation quality is improved as P increases. The extreme point of $E(P)$ is

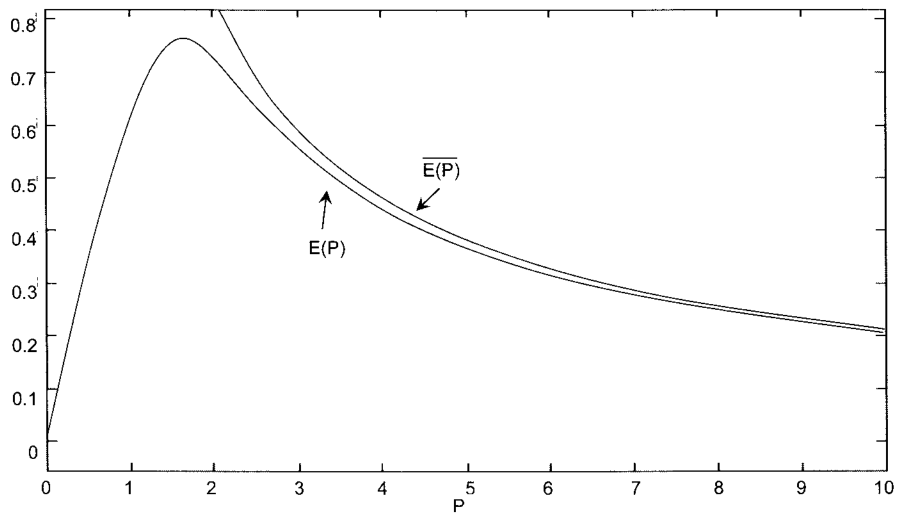

Figure 2. Graphs of $E(P)$ and $\overline{E(P)}$

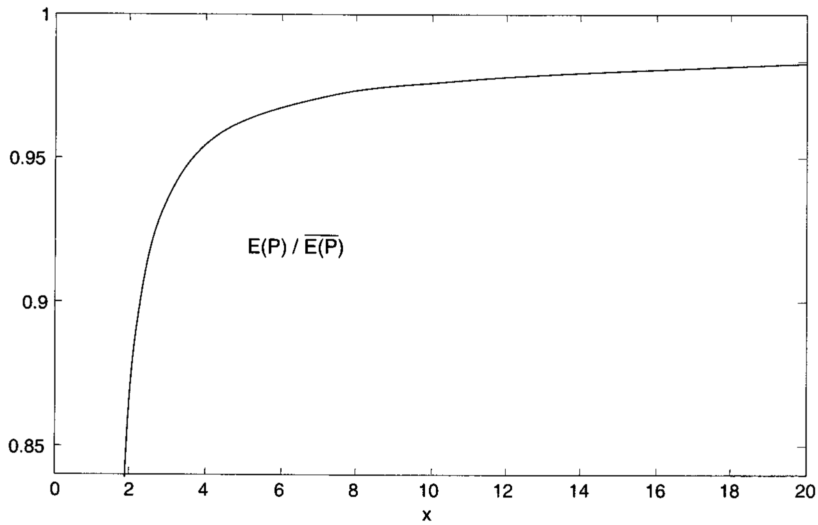

Figure 3. Approximation error

1.6375, and at this point $E(P)/\overline{E(P)} = 0.764$.

We finally arrive at the following GP:

$$\text{(TP)} \quad \begin{array}{ll} \min & z^{-1} \\ s.t. & 0.0000013zP^{2.8544} + 50P^{-1}Q^{-1} + 0.0000003281P^{2.8544}Q^{0.99} + 5P^{-1}Q^{-0.01} \le 1. \end{array}$$

In order to solve this problem, we used *cvx*[5] which is a Matlab-based modeling and solution system for convex optimization and has the capability of solving GPs.

The optimal solution for (TP) is obtained as follows:

$$P^* = 6.5253,\ Q^* = 404.4884,\ z^* = 839.7685.$$

At the optimal point, $E(P)/\overline{E(P)} = 0.9701$. This indicates $\overline{E(P)}$ is a reasonably good approximation of $E(P)$ around the optimal point.

5. Concluding Remarks

We have discussed the pricing/lot-sizing problem of determining the robust optimal order quantity and selling price that maximize the profit for a retailer. Differently from other previous studies, we explicitly incorporated the uncertainty of parameters into the problem. The exponential term in the resulting problem was approximated by a power function, by which the problem of interest was transformed into a GP. We claimed that the approximation error is reasonably bounded. The transformed GP could be solved efficiently and reliably using interior-point methods to obtain the robust optimal order quantity and selling price.

When there is significant uncertainty in demand function, the proposed model can provide a useful way to find out robust decisions on order quantity and selling price.

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