International Journal of Management Science
Vol 14, No 2, November 2008

Solving A Quadratic Fractional Integer Programming Problem Using Linearization*

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(Received: April 29, 2008 / Revised: October 16, 2008 / Accepted: October 21, 2008)

ABSTRACT

This paper concentrates on reduction of a Quadratic Fractional Integer Programming Problem (QFIP) to a 0-1 Mixed Linear Programming Problem (0-1 MLP). The solution technique is based on converting the integer variables to binary variables and then the resulting Quadratic Fractional 0-1 Programming Problem is linearized to a 0-1 Mixed Linear Programming problem. It is illustrated with the help of a numerical example and is solved using the LINDO software.

Keywords: Quadratic Programming, Integer Programming, Fractional Programming, 0-1 Programming, Linearization

1. Introduction

Consider the Quadratic Fractional Integer Programming Problem (QFIP) with linear and/or quadratic constraints which is of the form

(QFIP) min $f(y_1, y_2, \cdots, y_k)$

subject to $g_i(y_1, y_2, \cdots, y_k) \geq b_i$, $i = 1, 2, \cdots, m$

$0 \leq y_j \leq u_j$ for $j = 1, 2, \cdots, k$

* The authors would like to thank the reviewers for their valuable comments and suggestions, which helped in improving the paper presentation.

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$$y = (y_1, y_2, \cdots, y_k) \text{ is an integer}$$

where the function f is a quadratic fractional function, $g_i(y_1, y_2, \cdots, y_k)$ $(1 \le i \le m)$ can be linear or quadratic.

Conversion of an integer formulation to a zero-one formulation is easily accomplished. Given the integer variable y_j has a finite upper bound u_j, the variable y_j can be expressed in terms of binary variables x_{jp}'s as follows:

$$y_j = \sum_{p=0}^{J} 2^p x_{jp}$$

$$y_j \le u_j$$

where $x_{jp} = 0$ or 1 for $0 \le p \le J$ and $j = 1, 2, \cdots, k$ and J is the smallest integer such that $u_j \le 2^{J+1} - 1$

Using the above transformation, the problem (QFIP) is reduced to the following 0-1 Quadratic Fractional Programming Problem (QFP1)

(QFP1) $\min F(x_1, x_2, \cdots, x_n)$

subject to $G_i(x_1, x_2, \cdots, x_n) \le b_i,\ i = 1, 2, \cdots, m$

$x = (x_1, x_2, \cdots, x_n) \in \{0, 1\}^n$

where the function F is a Quadratic Fractional function, $G_i(x_1, x_2, \cdots, x_n)$ can be linear or quadratic for $i = 1, 2, \cdots, m$.

The problem (QFP1) can be expressed in the form

(QFP1)

$$\min \frac{x^T A x + \alpha}{x^T B x + \beta}$$

subject to $Cx \ge c$

$x^T D^{\ell} x \ge \gamma_{\ell}$ for $\ell = 1, 2, \cdots, L$

$x = (x_1, x_2, \cdots, x_n) \in \{0, 1\}^n$

where $A = (a_{ij})$, $B = (b_{ij})$, $D^{\ell} = (d_{ij}^{\ell})$ for $\ell = 1, 2, \cdots, L$ are symmetric matrices of order $n \times n$, $C = (C_{ij})$ is an $(m-L) \times n$ matrix, $c \in \mathbb{R}^{m-L}$.

It is assumed that $x^T Bx + \beta > 0$ over the feasible set of the given problem.

Hence, the ensuing discussion will be directed at reducing zero-one quadratic fractional formulation to a mixed zero-one linear fractional formulation.

Systematic studies and applications of single-ratio fractional programs generally began to appear in the literature in the early 1960s. Since then, a rich body of work has been accomplished on the classification, theory, applications and solutions of these problems. An overview of this work is contained in the articles by Schaible [10], the monographs by Craven [6], Martos [9] and references therein. Integer programming and methods of solutions of integer programs has been studied by Balas [1], Balinski [2] and Beale [3] in detail.

The problem presented in this article can be handled in its present form, but it has been reduced to an equivalent 0-1 mixed linear program as it is easy to handle linear programs. The solution technique is based on branch and bound method and guarantees global optimal solution of the problem while the solution might not be global in nature when the problem is considered in its present form.

In general method for obtaining a global optimal solution of 0-1 Quadratic Fractional Programming problems has been proposed by Li [8] which was improved by Wu [11]. After that Chang [4, 5] proposed a model which required fewer auxilliary constraints to linearize the mixed 0-1 fractional programming problem than Li's and Wu's. Wanprach [12] presented a technique for solving quadratic programs. The problem considered in this paper would have been difficult to handle in its present form. This article presents a very simple method of solving quadratic fractional programs by linearization technique making the problem a LPP which can be handled easily.

The paper is organized as follows: Section 2 is divided into two subsections: Section 2.1 examines a special case of the problem considered in this paper. Solution 2.2 deals with the general case and section 3 explains the technique with the help of numerical examples.

2. Theoretical Development and Solution Technique

2.1 When the elements of A, B and D^ℓ $(1 \le \ell \le L)$ are non-negative

i.e. $a_{ij}, b_{ij}, d_{ij}^\ell \ge 0 \qquad (1 \le \ell \le L), x \in \{0, 1\}^n$

Define the feasible set $S = \{x \in \{0, 1\}^n \mid Cx \geq c, \quad x^T D^\ell x \geq \gamma_\ell \quad \{1 \leq \ell \leq L)\}$

Let $M_1 = \max_{1\leq i\leq n}\left(\sum_{j=1}^{n} \mid a_{ij} \mid\right) = \| A \|$; $\quad M_2 = \max_{1\leq i\leq n}\left(\sum_{j=1}^{n} \mid b_{ij} \mid\right) = \| B \|$

and $M_3^\ell = \max_{1\leq i\leq n}\left(\sum_{j=1}^{n} \mid d_{ij}^\ell \mid\right) = \| D^\ell \| \quad (1 \leq \ell \leq L)$

Define a 0–1 linear fractional programming problem (LFP1) as:

(LFP1) $\min_{S_1} \dfrac{\alpha + e^T s^1}{\beta + e^T t^1}$

where $S_1 = \{(x, s^0, s^1, t^0, t^1, r^{10}, r^{11}, \cdots, r^{L0}, r^{L1}) \mid x \in \{0, 1\}^n$;

$Cx \geq c$, $Ax = s^o + s^1$; $0 \leq s^o \leq M_1(e - x)$; $0 \leq s^1 \leq M_1 x$;

$Bx = t^0 + t^1$; $0 \leq t^o \leq M_2(e - x)$; $0 \leq t^1 \leq M_2 x$;

$D^\ell x \geq r^\ell$, $e^T r^\ell \geq \gamma_\ell$, $r^\ell \leq M_3^\ell x \quad (1 \leq \ell \leq L)\}$

Theorem 2.1: The problem (QFP1) has an optimal solution x^o iff $\exists$ $s^o(x^0)$, $s^1(x^0), t^0(x^0), t^1(x^0), r^{\ell 0}(x^0), r^{\ell 1}(x^o)(1 \leq \ell \leq L)$

s.t. $(x^o, s^o(x^o), s^1(x^o), t^o(x^o), t^1(x^o), r^{10}(x^0), r^{11}(x^0), \cdots, r^{L0}(x^0), r^{L1}(x^0))$

is an optimal solution of (LFPI).

Proof: Let x^o be an optimal solution of problem (QFP1). Corresponding to any feasible point x, feasible for the problem (QFP), define the vectors

$s^o(x)$, $s^1(x)$, $t^o(x)$, $t^1(x)$, $r^{10}(x)$, $r^{11}(x)$, $\cdots$, $r^{L0}(x^0), r^{L1}(x^0)$, as follows:

when $\quad x_i = 1$, $s_i^1(x) = (Ax)_i$, $t_i^1(x) = (Bx)_i$, $r_i^{\ell 1}(x) = (D^\ell x)_i \quad (1 \leq \ell \leq L)$,

$s_i^0(x) = t_i^0(x) = r_i^{\ell 0}(x) = 0 \quad (1 \leq \ell \leq L)$,

and when $x_i = 0$, $s_i^o(x) = (Ax)_i$, $t_i^0(x) = (Bx)_i$, $r_i^{\ell 0}(x) = (D^\ell x)_i$, $(1 \leq \ell \leq L)$

$s_i^1(x) = t_i^1(x) = r_i^{\ell 1}(x) = 0 \quad (1 \leq \ell \leq L)$,

Then it is clear that

$$x^T Ax = e^T s^1(x), \quad x^T Bx = e^T t^1(x), \quad x^T D^\ell x = e^T r^{\ell 1}(x)\ (1 \leq \ell \leq L)$$

It is clear that whenever $x \in S$,

$$(x, s^o(x), s^1(x), t^0(x), t^1(x), r^{10}(x), r^{11}(x), \cdots, r^{L0}(x), r^{L1}(x)) \in S_1$$

and $$\min_{S_1} \frac{\alpha + e^T s^1}{\beta + e^T t^1} = \min_{x \in S} \frac{\alpha + x^T Ax}{\beta + x^T Bx}$$

$$= \frac{\alpha + (x^o)^T Ax^o}{\beta + (x^o)^T Bx^o} = \frac{\alpha + e^T s^1(x^o)}{\beta + e^T t^1(x^o)}$$

Therefore, $(x^o, s^o(x^o), s^1(x^o), t^o(x^o), t^1(x^o), r^{10}(x^o), r^{11}(x^o), \cdots, r^{L0}(x^o), r^{L1}(x^o))$ is an optimal solution of (LFPI). The converse part of the theorem can be proved in a similar way.

By using Charnes and Cooper transformation $q = \frac{1}{\beta + e^T t^1}$, the 0-1 linear fractional programming problem (LFPI) is reduced to an equivalent linear programming problem.

It is to be observed that $q > 0$ (since $\beta + e^T t^1 > 0$). Also $q \le 1/\beta$ (since $e^T t^1 \ge 0$).

Next we define the vectors $S^0, S^1, T^0, T^1, R^{\ell 0}, R^{\ell 1} \in \mathbb{R}^{n \times 1}$ $(1 \le \ell \le L)$ as follows:

$$S^o = qs^0, \quad S^1 = qs^1, \quad T^0 = qt^0, \quad T^1 = qt^1,$$
$$R^{\ell 0} = qr^{\ell 0}, \quad R^{\ell 1} = qr^{\ell 1} \quad (1 \le \ell \le L), \quad K = qx$$

Now, consider a 0 - 1 mixed linear programming problem (MLP1) as

(MLP1) $\min_{S_2} \{\alpha q + e^T S^1\}$

where $S_2 = \{(x, S^o, S^1, T^o, T^1, R^{10}, R^{11}, \cdots, R^{L0}, R^{L1}, K, q)\}$

$CK \ge cq, \; AK = S^o + S^1, 0 \le S^o \le M_1(eq - K), \; 0 \le S^1 \le M_1 K, BK = T^o + T^1,$

$0 \le T^0 \le M_2(eq - K), \; 0 \le T^1 \le M_2 K, \; \beta q + e^T T^1 = 1, \; K \ge qe + \frac{1}{\beta}(x - e),$

$K \le qe, \; K \le \left(\frac{1}{\beta}\right) x, \; q \le \frac{1}{\beta}, \; D^\ell K - R^\ell \ge 0, \; e^T R^\ell \ge \gamma_\ell q,$

$R^\ell \le M_3^\ell K \; (1 \le \ell \le L), \; x \in \{0, 1\}^n, \; q \in R^+\}$

It is to be observed that the constraints

$$K \ge qe + \frac{1}{\beta}(x - e), \qquad K \le qe, \; K \le \left(\frac{1}{\beta}\right) x$$

imply $K_i = q$ when $x_i = 1$ and $K_i = 0$, when $x_i = 0$

All the facts stated above conclude that solving (QFP1) means to solve the 0-1 mixed linear programming problem (MLP1).

2.2 When some elements of A, B and D^ℓ $(1 \le \ell \le L)$ may be negative

Define the matrices $\bar{A}$, $\bar{B}$ and $\bar{D}^\ell$ $(1 \le \ell \le L)$ as follows:

$$\bar{A}x = Ax + M_1 e\,;\;\; \bar{B}x = Bx + M_2 e\,;\;\; \bar{D}^\ell x = D^\ell x + M_3^\ell e \;\; (1 \le \ell \le L)$$

Clearly, the vectors $\bar{A}x$, $\bar{B}x$, $\bar{D}^\ell x$ $(1 \le \ell \le L)$ are non-negative n-vectors. Thus the problem (QFP1) can be written as

$$\textbf{(QFP2)} \quad \min_{x \in \bar{S}_1} \frac{\alpha + x^T \bar{A} x - M_1 e^T x}{\beta + x^T \bar{B} x - M_2 e^T x}$$

where $\bar{S}_1 = \{x \in \{0, 1\}^n \mid Cx \ge c,\; x^T D^\ell x \ge \gamma_\ell \;\; (1 \le \ell \le L)\}$. Since $\beta + x^T\beta x > 0$, therefore $\beta + x^T \bar{B} x - M_2 e^T x > 0$ for all $x \in \bar{S}_1$.

Consider the following 0–1 linear fractional programming problem (LFP2) defined as

$$\textbf{(LFP2)} \quad \min_{x \in \bar{S}_2} \frac{\alpha + x^T s^1 - M_1 e^T x}{\beta + x^T t^1 - M_2 e^T x}$$

where $\bar{S}_2 = \{(x, s^o, s^1, t^o, t^1, r^{10}, r^{11}, \cdots, r^{L0}, r^{L1})\}$

$Cx \ge c,\; \bar{A}x = s^0 + s^1,\; 0 \le s^0 \le 2M_1(e - x),\; 0 \le s^1 \le 2M_1 x,\; \bar{B}x = t^0 + t^1,$

$0 \le t^0 \le 2M_2(e - x),\; 0 \le t^1 \le 2M_2 x,\; \bar{D}^\ell x = r^{\ell 0} + r^{\ell 1},\; 0 \le r^{\ell 0} \le 2M_3^\ell (e - x),$

$0 \le r^{\ell 1} \le 2M_3^\ell x\; (1 \le \ell \le L),\; x \in \{0, 1\}^n\}$

Since $\bar{A}x$, $\bar{B}x$ and $\bar{D}^\ell x$ $(1 \le \ell \le L)$ are non-negative in nature, therefore, we can choose non-negative vector s^o, s^1, t^o, t^1, $r^{\ell 0}, r^{\ell 1}$ $(1 \le \ell \le L)$ s.t.

$$\bar{A}x = s^0 + s^1 \quad \text{where} \quad s^1 \le \|\bar{A}\| x,\; s^0 \le \|\bar{A}\|(e - x)$$

$$\bar{B}x = t^0 + t^1 \quad \text{where} \quad t^1 \le \|\bar{B}\| x,\; t^0 \le \|\bar{B}\|(e - x)$$

$\bar{D}^\ell x = r^{\ell 0} + r^{\ell 1}$ $(1 \le \ell \le L)$ where $\| r^{\ell 1} \| \le \| \bar{D}^\ell \| x$ $(1 \le \ell \le L)$,

$\| r^{\ell 0} \| \le \| \bar{D}^\ell \| (e - x)$ $(1 \le \ell \le L)$

Also, $\| \bar{A} \| = \max_{x:\|x\|=1} \| Ax \|$

But $\| \bar{A}x \| = \| Ax + M_1 e \| \le \| Ax \| + M_1 \| e \|$

$\le \| A \| + M_1 \qquad (\because \| x \| < 1, \| e \| = 1)$

$= 2M_1$

$\Rightarrow \| \bar{A}x \| \le 2M_1 \qquad \forall x : \| x \| < 1$

$\Rightarrow \qquad \max_{x:\|x\|<1} \| \bar{A}x \| \le 2M_1$

$\Rightarrow \qquad \| \bar{A} \| \le 2M_1$

Similarly it can be shown that $\| \bar{B} \| \le 2M_2$ and $\| \bar{D}^\ell \| \le 2M_3^\ell$ $(1 \le \ell \le L)$

We shall now prove the equivalence of 0-1 quadratic fractional programming problem (QFP2) and hence of (QFP1) and the 0-1 linear fractional programming problem (LFP2).

Theorem 2.2: The problem (QFP2) and hence (QFP1) has an optimal solution x^o iff $\exists$ $n \times 1$ vectors $s^0(x^0)$, $s^1(x^0)$, $t^0(x^0)$, $t^1(x^0)$, $r^{10}(x^0)$, $r^{11}(x^0)$, $\cdots$, $r^{L0}(x^0)$, $r^{L1}(x^0)$ s.t. $(x^0, s^0(x^0), s^1(x^0), t^0(x^0), t^1(x^0), r^{10}(x^0), r^{11}(x^0), \cdots, r^{L0}(x^0), r^{L1}(x^0))$ is an optimal solution of the problem $(LFP2)_i$.

Proof: Let x^0 be an optimal solution of the problem (QFP1) and hence of (QFP2). Corresponding to any feasible point of the problem (LFP2) (or of (QFP1), define the vectors $s^0, s^1, t^0, t^1, r^{10}, r^{11}, \cdots, r^{L0}, r^{L1}$ as follows:

when $x_i = 1$: $s^0(x) = t^0(x) = r^{10}(x) = \text{---} = r^{L0}(x) = 0$,

$s^1(x) = (\bar{A}x)_i$, $t^1(x) = (\bar{B}x)_i$, $r^\ell(x) = (\bar{D}^\ell x)_i$, $1 \le \ell \le L$ and

when $x_i = 0$: $s^0(x) = (\bar{A}x)_i$, $t^0(x) = (\bar{B}x)_i$, $r^{\ell 0}(x) = (\bar{D}^\ell x)_i$, $(1 \le \ell \le L)$

$s^1(x) = t^1(x) = r^{11}(x) = \cdots = r^{L1}(x) = 0$

Clearly, whenever $x \in \bar{S}_1, (x, s^0, s^1, t^0, t^1, r^{10}, r^{11}, \cdots, r^{L0}, r^{L1}) \in \bar{S}_2$ we also see that

$$x^T \bar{A} x = e^T s^1, \quad x^T \bar{B} x = e^T t^1 \text{ and } x^T \bar{D}^\ell x = e^T r^{\ell 1} \quad (1 \le \ell \le L)$$

Now, consider $\min_{S_2} \frac{\alpha + e^T s^1 - M_1 e^T x}{\beta + e^T t^1 - M_2 e^T x} = \min_{S_1} \frac{\alpha + x^T \bar{A} x - M_1 e^T x}{\beta + x^T \bar{B} x - M_2 e^T x}$

$$= \frac{\alpha + (x^o)^T \bar{A} x^o - M_1 e^T x^o}{\beta + (x^o)^T \bar{B} x^o - M_2 e^T x^o} = \frac{\alpha + e^T s^1(x^o) - M_1 e^T x^o}{\beta + e^T t^1(x^o) - M_2 e^T x^o}$$

$$\Rightarrow (x^0, s^0(x^0), s^1(x^0), t^0(x^0), t^1(x^0), r^{10}(x^0), r^{11}(x^0), \cdots, r^{L0}(x^0), r^{L1}(x^0))$$

is an optimal solution of the problem (LFP2). The proof of the necessary part of the theorem is completed. The sufficient part of the theorem can also be proved in the same way. Hence we conclude that solving the 0-1 quadratic fractional programming problem (QFP1) is equivalent to solving the 0-1 linear fractional program (LFP2) which can be transformed to a 0-1 mixed linear programming problem by Charnes-Cooper transformation method as used before. Thus the problem (QFP2) is equivalent to the following 0-1 mixed linear programming problem (MLP2).

(MLP2) $\min_{\bar{\bar{S}}_2} \alpha q + e^T S^1 - M_1 e^T K$

where

$$\bar{\bar{S}}_2 = \{(x, S^0, S^1, T^0, T^1, R^{10}, R^{11}, \cdots, R^{L0}, R^{L1}, K, q) \mid CK \geq cq,\ \bar{A}K = S^0 + S^1,$$
$$0 \leq S^0 \leq 2M_1(eq - K),\ 0 \leq S^1 \leq 2M_1 K,\ \bar{B}K = T^0 + T^1,$$
$$0 \leq T^0 \leq 2M_2(eq - K),\ 0 \leq T^1 \leq 2M_2 K,\ \beta q + e^T T^1 = 1,$$
$$\bar{D}^\ell K - R^\ell \geq 0,\ e^T R^\ell \geq \gamma_\ell q,\ R^\ell \leq 2M_3^\ell K (1 \leq \ell \leq L),$$
$$K \geq qe + \left(\frac{1}{\beta}\right)(x - e),\ K \leq qe,\ K \leq \left(\frac{1}{\beta}\right)x,\ q \leq \frac{1}{\beta},\ x \in \{0, 1\}^n,\ q \in \mathbb{R}^+\}$$

where $q = \frac{1}{\beta + x^T \bar{B} x - M_2 e^T x}$.

3. Numerical Examples

3.1 Consider the following Quadratic Fractional Integer Programming Problem (QFIP)

(QFIP): $\min \frac{y^T A^1 y + \alpha}{y^T B^1 y + \beta} = \frac{2y_1^2 + 2y_1 y_2 + y_2^2 + 7}{y_1^2 + 4y_2^2 + 1}$

subject to $2y_1 + y_2 \geq 2$ (3.1)

$y_1 + 2y_2 \geq 3$ (3.2)

$y_1^2 + y_2^2 \geq 1$ (3.3)

$y_1 y_2 \geq 1$ (3.4)

$0 \leq y_1 \leq 3$ (3.5)

$0 \leq y_2 \leq 4$ (3.6)

where y_1, y_2 are integers.

It is assumed that $y^T B^1 y + \beta > 0$ over the feasible set.

The above mentioned problem is reduced to an equivalent 0-1 formulation using the transformation

$$y_1 = x_1 + 2x_2 \text{ and } y_2 = x_3 + 2x_4 + 4x_5$$

The resultant problem is then defined as

P(3.1) $$\min \frac{x^T A x + \alpha}{x^T B x + \beta} =$$

$$\min \frac{\begin{array}{c}(2x_1^2 + 8x_2^2 + x_3^2 + 4x_4^2 + 16x_5^2 + 8x_1x_2 + 2x_1x_3 + 4x_1x_4 + 8x_1x_5 \\ +4x_2x_3 + 8x_2x_4 + 16x_2x_5 + 4x_3x_4 + 8x_3x_5 + 16x_4x_5 + 7)\end{array}}{x_1^2 + 4x_2^2 + 4x_3^2 + 16x_4^2 + 64x_5^2 + 4x_1x_2 + 16x_3x_4 + 32x_3x_5 + 64x_4x_5 + 1}$$

subject to $2x_1 + 4x_2 + x_3 + 2x_4 + 4x_5 \geq 2$ (3.7)

$x_1 + 2x_2 + 2x_3 + 4x_4 + 8x_5 \geq 3$ (3.8)

$x_1 + 2x_2 \leq 3$ (3.9)

$x_3 + 2x_4 + 4x_5 \leq 4$ (3.10)

$x_1^2 + 4x_2^2 + 4x_1x_2 + x_3^2 + 4x_4^2 + 16x_5^2 + 4x_3x_4 + 16x_4x_5 + 8x_3x_5 \geq 1$ (3.11)

$x_1x_3 + 2x_1x_4 + 4x_1x_5 + 2x_2x_3 + 4x_2x_4 + 8x_2x_5 \geq 1$ (3.12)

where

$$A = \begin{bmatrix} 2 & 4 & 1 & 2 & 4 \\ 4 & 8 & 2 & 4 & 8 \\ 1 & 2 & 1 & 2 & 4 \\ 2 & 4 & 2 & 4 & 8 \\ 4 & 8 & 4 & 8 & 16 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 8 & 16 \\ 0 & 0 & 8 & 16 & 32 \\ 0 & 0 & 16 & 32 & 64 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 4 & 1 & 2 & 4 \\ 1 & 2 & 2 & 4 & 8 \end{bmatrix}, \quad c = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$D^1 = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 2 & 4 & 8 \\ 0 & 0 & 4 & 8 & 16 \end{bmatrix}, \quad D^2 = \begin{bmatrix} 0 & 0 & 1/2 & 1 & 2 \\ 0 & 0 & 1 & 2 & 4 \\ 1/2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 \end{bmatrix}$$

$(\gamma_1, \gamma_2) = (1, 1)$, $x = (x_1, x_2, x_3, x_4, x_5) \in \{0, 1\}^n$.

Here, $M_1 = \|A\| = 40$, $M_2 = \|B\| = 112$, $M_3' = \|D^1\| = 28$, $M_3^2 = \|D^2\| = 7$

It is to be observed that $\beta + x^T Bx > 0 \quad \forall\, x \in \{0, 1\}^n$

Step 1: Linearizing the terms x^TAx, x^TBx: For this include the following constraints in (P 3.1)

$Ax = s^o + s^1$, $s^1 \le M_1x$, $s^o \le M_1(e - x)$;

$Bx = t^o + t^1$, $t^1 \le M_2x$, $t^o \le M_2(e - x)$;

$$Ax = s^o + s^1 \Rightarrow 2x_1 + 4x_2 + 3x_3 + 6x_4 + 12x_5 = s_1^0 + s_1^1 \tag{3.13}$$

$$4x_1 + 8x_2 + 6x_3 + 12x_4 + 24x_5 = s_2^0 + s_2^1 \tag{3.14}$$

$$3x_1 + 6x_2 + x_3 + 2x_4 + 4x_5 = s_3^0 + s_3^1 \tag{3.15}$$

$$6x_1 + 12x_2 + 2x_3 + 4x_4 + 8x_5 = s_4^0 + s_4^1 \tag{3.16}$$

$$12x_1 + 24x_2 + 4x_3 + 8x_4 + 16x_5 = s_5^0 + s_5^1 \tag{3.17}$$

$$s^1 \le M_1x \Rightarrow s_i^1 \le 40x_i \quad \text{for } i = 1, 2, \cdots, 5 \tag{3.18}$$

$$s^o \le M_1(e - x) \Rightarrow s_i^o \le 40(1 - x_i) \quad \text{for } i = 1, 2, \cdots, 5 \tag{3.19}$$

$$Bx = t^o + t^1 \Rightarrow x_1 + 2x_2 = t_1^o + t_1^1 \tag{3.20}$$

$$2x_1 + 4x_2 = t_2^o + t_2^1 \tag{3.21}$$

$$4x_3 + 8x_4 + 16x_5 = t_3^o + t_3^1 \tag{3.22}$$

$$8x_3 + 16x_4 + 32x_5 = t_4^o + t_4^1 \tag{3.23}$$

$$16x_3 + 32x_4 + 64x_5 = t_5^o + t_5^1 \tag{3.24}$$

$$t^1 \le M_2x \Rightarrow t_i^1 \le 112x_i \quad \text{for } i = 1, 2, \cdots, 5 \tag{3.25}$$

$$t^o \le M_2(e - x) \Rightarrow t_i^o \le 112(1 - x_i), \quad \text{for } i = 1, 2, \cdots, 5 \tag{3.26}$$

Step 2: Linearization of Quadratic terms appeared in the constraint set: For this include the following constraints in (P3.1)

$D^1x \geq r^0$; $e^Tr^0 \geq \gamma_1$; $r^0 \leq M_3^1x$; $D^2x \geq r^1$; $e^Tr^1 \geq \gamma_2$; $r^1 \leq M_3^2x$;

$$D^1x \geq r^o \Rightarrow \quad x_1 + 2x_2 \geq r_1^o \tag{3.27}$$

$$2x_1 + 4x_2 \geq r_2^o \tag{3.28}$$

$$x_3 + 2x_4 + 4x_5 \geq r_3^o \tag{3.29}$$

$$2x_3 + 4x_4 + 8x_5 \geq r_4^o \tag{3.30}$$

$$4x_3 + 8x_4 + 16x_5 \geq r_5^o \tag{3.31}$$

$$e^Tr^o \geq \gamma_1 \Rightarrow r_1^o + r_2^o + r_3^o + r_4^o + r_5^o \geq 1 \tag{3.32}$$

$$r^o \leq M_3^1x \Rightarrow r_i^o \leq 28x_i \quad \text{for } i = 1, 2, \cdots, 5 \tag{3.33}$$

$$D^2x \geq r^1 \Rightarrow \quad x_3 + x_4 + 2x_5 \geq r_1^1 \tag{3.34}$$

$$x_3 + 2x_4 + 4x_5 \geq r_2^1 \tag{3.35}$$

$$x_1 + x_2 \geq r_3^1 \tag{3.36}$$

$$x_1 + 2x_2 \geq r_4^1 \tag{3.37}$$

$$2x_1 + 4x_2 \geq r_5^1 \tag{3.38}$$

$$e^Tr^1 \geq \gamma_2 \Rightarrow \quad r_1^1 + r_2^1 + r_3^1 + r_4^1 + r_5^1 \geq 1 \tag{3.39}$$

$$r^1 \leq M_3^2x \Rightarrow \quad r_i^1 \leq 7x_i \quad \textbf{for i = 1, 2, } \cdots\textbf{, 5} \tag{3.40}$$

Observe that $e = (1, 1, 1, 1, 1)^T \in R^5$; $s^o = (s_1^o, s_2^o, s_3^o, s_4^o, s_5^o)^T$;

$s^1 = (s_1^1, s_2^1, s_3^1, s_4^1, s_5^1)^T$; $t^o = (t_1^o, t_2^o, t_3^o, t_4^o, t_5^o)^T$;

$t^1 = (t_1^1, t_2^1, t_3^1, t_4^1, t_5^1)^T$; $r^o = (r_1^o, r_2^o, r_3^o, r_4^o, r_5^o)^T$;

$r^1 = (r_1^1, r_2^1, r_3^1, r_4^1, r_5^1)^T$; $\gamma = (\gamma_1, \gamma_2)$;

$s_i^o, s_i^1, r_i^o, r_i^1, t_i^o, t_i^1 \geq 0$ for i = 1, 2, 3, 4, 5.

Hence, the problem (P3.1) reduces to

(P3.2) $$\min \frac{7 + e^Ts^1}{1 + e^Tt^1}$$

subject to (3.7)-(3.10), (3.13)-(3.40)

where $x \in \{0, 1\}^5$; $s_i^0, s_i^1, r_i^0, r_i^1, t_i^0, t_i^1, K_i \geq 0$ for i = 1, 2, 3, 4, 5.

Step 3: Transformation of objective into non-fractional form:

For this transformation, include the following constraint in P(3.2)

$$q = \frac{1}{\beta + e^T t^1} \qquad \text{where } q > 0$$

i.e. $$q(1 + t_1^1 + t_2^1 + t_3^1 + t_4^1 + t_5^1) = 1 \tag{3.41}$$

Step 4: Substitute $S^o = qs^o$, $S^1 = qs^1$ $R^o = qr^o$ $R^1 = qr^1$, $K = qx$, $T^o = qt^o$, $T^1 = qt^1$ where S^o, S^1, R^o, R^1, T^o, T^1, $K \in R^5$ and are non-negative.

The substitution $K = qx$ must imply that $K_i = 0$ when $x_i = 0$ and $K_i = q$ when $x_i = 1$, which can be put in the form of the following constraints:

$K \le M_4 x$ where M_4 = upper bound on q,

$K \ge q - M_4(e - x)$ and $\qquad K \le qe$

Here $M_4 = 1$

$$K \le M_4 x \Rightarrow \qquad K_i \le x_i \, (1 \le i \le 5) \tag{3.42}$$

$$K \ge q + M_4(x - e) \Rightarrow \qquad K_i \ge q + (x_i - 1)\ (1 \le i \le 5) \tag{3.43}$$

$$K \le qe \Rightarrow \qquad K_i \le q\ (1 \le i \le 5) \tag{3.44}$$

Multiplying the constraints (3.7)-(3.10), (3.13)-(3.40) in the problem (P. 3.2) by q where q satisfies the constraint (3.41) and substituting as mentioned above, we get

(P3.3) $\min\ (7q + e^T S^1)$

subject to

$$2K_1 + 4K_2 + K_3 + 2K_4 + 4K_5 \ge 2q \tag{3.45}$$

$$K_1 + 2K_2 + 2K_3 + 4K_4 + 8K_5 \ge 3q \tag{3.46}$$

$$K_1 + 2K_2 \le 3q \tag{3.47}$$

$$K_3 + 2K_4 + 4K_5 \le 4q \tag{3.48}$$

$$2K_1 + 4K_2 + 3K_3 + 6K_4 + 12K_5 = S_1^o + S_1^1 \tag{3.49}$$

$$4K_1 + 8K_2 + 6K_3 + 12K_4 + 24K_5 = S_2^o + S_2^1 \tag{3.50}$$

$$3K_1 + 6K_2 + K_3 + 2K_4 + 4K_5 = S_3^o + S_3^1 \tag{3.51}$$

$$6K_1 + 12K_2 + 2K_3 + 4K_4 + 8K_5 = S_4^o + S_4^1 \tag{3.52}$$

$$12K_1 + 24K_2 + 4K_3 + 8K_4 + 16K_5 = S_5^o + S_5^1 \tag{3.53}$$

$$s_i^1 \le 40K_i \quad \text{for } i = 1, 2, 3, 4, 5 \tag{3.54}$$

$$s_i^o \le 40(q - K_i) \quad \text{for } i = 1, 2, 3, 4, 5 \tag{3.55}$$

$$K_1 + 2K_2 = T_1^o + T_1^1 \quad (3.56)$$
$$2K_1 + 4K_2 = T_2^o + T_2^1 \quad (3.57)$$
$$4K_3 + 8K_4 + 16K_5 = T_3^o + T_3^1 \quad (3.58)$$
$$8K_3 + 16K_4 + 32K_5 = T_4^o + T_4^1 \quad (3.59)$$
$$16K_3 + 32K_4 + 64K_5 = T_5^o + T_5^1 \quad (3.60)$$
$$T_i^1 \le 112K_i \quad \text{for } i = 1, 2, 3, 4, 5 \quad (3.61)$$
$$T_i^o \le 112\,(q - K_i) \quad \text{for } i = 1, 2, 3, 4, 5 \quad (3.62)$$
$$K_1 + 2K_2 \ge R_1^o \quad (3.63)$$
$$2K_1 + 4K_2 \ge R_2^o \quad (3.64)$$
$$K_3 + 2K_4 + 4K_5 \ge R_3^o \quad (3.65)$$
$$2K_3 + 4K_4 + 8K_5 \ge R_4^o \quad (3.66)$$
$$4K_3 + 8K_4 + 16K_5 \ge R_5^o \quad (3.67)$$
$$R_1^o + R_2^o + R_3^o + R_4^o + R_5^o \ge q \quad (3.68)$$
$$R_i^o \le 28K_i \quad \text{for } i = 1, 2, 3, 4, 5 \quad (3.69)$$
$$0.5K_3 + K_4 + 2K_5 \ge R_1^1 \quad (3.70)$$
$$K_3 + 2K_4 + 4K_5 \ge R_2^1 \quad (3.71)$$
$$0.5K_1 + K_2 \ge R_3^1 \quad (3.72)$$
$$K_1 + 2K_2 \ge R_4^1 \quad (3.73)$$
$$2K_1 + 4K_2 \ge R_5^1 \quad (3.74)$$
$$R_1^1 + R_2^1 + R_3^1 + R_4^1 + R_5^1 \ge q \quad (3.75)$$
$$R_i^1 \le 7K_i \quad \text{for } i = 1, 2, 3, 4, 5 \quad (3.76)$$
$$q + T_1^1 + T_2^1 + T_3^1 + T_4^1 + T_5^1 = 1 \quad (3.77)$$

Constraint (3.42)-(3.44)

$$q \le 1 \quad (3.78)$$

where $x = (x_1, x_2, x_3, x_4, x_5) \in \{0, 1\}^5$; K_i, S_1^o, S_i^1, R_i^o, R_i^1, T_i^o, $T_i^1 \ge 0$ for $i = 1, 2, 3, 4, 5$; $q > 0$

We see that the above problem is a mixed integer linear program and can be solved easily.

The solution of the problem, using Lindo software, is $x_1 = 1$, $x_2 = 0$,

$x_3 = 0$, $x_4 = 1$, $x_5 = 1$ and the objective function value is 0.390411.

Therefore, $y_1 = 1$, $y_2 = 6$ is the solution of the original problem and the value of objective function is 0.390411.

3.2 General Case

(P3.4) $$\min \frac{y^T A^1 y + \alpha}{y^T B^1 y + \beta} = \frac{4 + y_1^2 - y_2^2 - 6y_1y_2}{2y_1^2 - 4y_1y_2 - y_2^2 + 21}$$

subject to

$$y_1 + y_2 \le 4 \quad (3.79)$$

$$y_1 - 2y_2 \le 1 \quad (3.80)$$

$$y_1^2 + y_2^2 \ge 1/4 \quad (3.81)$$

$$y_1y_2 \ge 1 \quad (3.82)$$

$$0 \le y_1 \le 2 \quad (3.83)$$

$$0 \le y_2 \le 3 \quad (3.84)$$

where y_1, y_2 are integers.

It is assumed that $y^T B^1 y + \beta > 0$ over the feasible set.

This problem is reduced to an equivalent 0–1 formulation using the transformation $y_1 = x_1 + 2x_2$ and $y_2 = x_3 + 2x_4$

The resultant problem is then defined as

(P3.5) $$\min \frac{x^T A^1 x + \alpha}{x^T B^1 x + \beta} = \frac{(4 + x_1^2 + 4x_2^2 - x_3^2 - 4x_4^2 + 4x_1x_2 - 6x_1x_3 - 12x_1x_4 - 12x_2x_3 - 24x_2x_4 - 4x_3x_4)}{(21 + 2x_1^2 + 8x_2^2 - x_3^2 - 4x_4^2 + 8x_1x_2 - 4x_1x_3 - 8x_1x_4 - 8x_2x_3 - 16x_2x_4 - 4x_3x_4)}$$

s.t.

$$x_1 + 2x_2 + x_3 + 2x_4 \le 4 \quad (3.85)$$

$$x_1 + 2x_2 - 2x_3 - 4x_4 \le 1 \quad (3.86)$$

$$x_1 + 2x_2 \le 2 \quad (3.87)$$

$$x_3 + 2x_4 \le 3 \quad (3.88)$$

$$x_1^2 + 4x_2^2 + 4x_1x_2 + x_3^2 + 4x_4^2 + 4x_3x_4 \ge 1/4 \quad (3.89)$$

$$x_1x_3 + 2x_1x_4 + 2x_2x_3 + 4x_2x_4 \ge 1 \quad (3.90)$$

where $A = \begin{bmatrix} 1 & 2 & -3 & -6 \\ 2 & 4 & -6 & -12 \\ -3 & -6 & -1 & -2 \\ -6 & -12 & -2 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 & -2 & -4 \\ 4 & 8 & -4 & -8 \\ -2 & -4 & -1 & -2 \\ -4 & -8 & -2 & -4 \end{bmatrix}$,

$$C=\begin{bmatrix}1 & 2 & 1 & 2\\ 1 & 2 & -2 & -4\end{bmatrix}, \qquad c=\begin{bmatrix}4\\ 1\end{bmatrix},$$

$$D^1=\begin{bmatrix}1 & 2 & 0 & 0\\ 2 & 4 & 0 & 0\\ 0 & 0 & 1 & 2\\ 0 & 0 & 2 & 4\end{bmatrix}, \qquad D^2=\begin{bmatrix}0 & 0 & 0.5 & 1\\ 0 & 0 & 1 & 2\\ 0.5 & 1 & 0 & 0\\ 1 & 2 & 0 & 0\end{bmatrix},$$

$$\gamma=\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}=\begin{pmatrix}1/4\\ 1\end{pmatrix}$$

Here, $\beta+x^TBx>0 \quad \forall x\in\{0,1\}^4$

Also, $M_1=\|A\|=\max\limits_{1\le i\le 4}\sum\limits_{j=1}^{4}|a_{ij}|=24,\ \ M_2=\|B\|=\max\limits_{1\le i\le 4}\sum\limits_{j=1}^{4}|b_{ij}|=24,$

$M_3^1=\|D^1\|=\max\limits_{1\le i\le 4}\sum\limits_{j=1}^{4}|d_{ij}^1|=6,\ \ M_3^2=\|D^2\|=\max\limits_{1\le i\le 4}\sum\limits_{j=1}^{4}|d_{ij}^2|=3.$

Consider two matrices $\bar{A}$ and $\bar{B}$ s.t.

$$\bar{A}x=Ax+M_1e=\begin{bmatrix}x_1+2x_1-3x_3-6x_4+24\\ 2x_1+4x_2-6x_3-12x_4+24\\ -3x_1-6x_2-x_3-2x_4+24\\ -6x_1-12x_2-2x_3-4x_4+24\end{bmatrix}$$

$$\bar{B}x=Bx+M_2e=\begin{bmatrix}2x_1+4x_1-2x_3-4x_4+24\\ 4x_1+8x_2-4x_3-8x_4+24\\ -2x_1-4x_2-x_3-2x_4+24\\ -4x_1-8x_2-2x_3-4x_4+24\end{bmatrix}$$

Then $\bar{A}x$ and $\bar{B}x$ are non-negative 4-vectors and the objective function assumes the form

$$\min\frac{4+x^T(Ax+M_1e-M_1e)}{21+x^T(Bx+M_2e-M_2e)} = \min\frac{4+x^T\bar{A}x-M_1x^Te}{21+x^T\bar{B}x-M_2x^Te}$$

Step 1: Linearizing $x^T\bar{A}x$ and $x^T\bar{B}x$:

Since $\bar{A}x$ and $\bar{B}x$ are non-negative vectors, therefore we can find 4-vectors s^o, s^1, t^o, t^1 s.t.

$$\bar{A}x = s^0 + s^1 \text{ where } s^1 \le \|\bar{A}\| x,\ s^o \le \|\bar{A}\|(e - x)$$

$$\bar{B}x = t^o + t^1 \text{ where } t^1 \le \|\bar{B}\| x,\ t^o \le \|\bar{B}\|(e - x)$$

$$\text{Now, } \|\bar{A}\| \le 2M_1 = 48 \text{ and, } \|\bar{A}\| \le 2M_1 = 48$$

Step 2: Linearizing quadratic terms appeared in the constraints:

$$\bar{D}^{\ell}x - r^{\ell-1} + M_3^{\ell}e \ge 0,\ e^T r^{\ell-1} - M_3^{\ell}e^T x \ge \gamma_{\ell}$$

$$\text{and} \quad r^{\ell-1} \le 2M_3^{\ell}x \quad \text{for } 1 \le \ell \le 2$$

where $x \in \{0, 1\}$

Hence the problem (P 3.5) on including the constraints mentioned in Step 1 and Step 2, can be written as

(P3.6) $$\min \frac{4 + e^T s^1 - 24e^T x}{21 + e^T t^1 - 24e^T x}$$

subject to (3.85)-(3.88) and

$$x_1 + 2x_2 - 3x_3 - 6x_4 = s_1^o + s_1^1 - 24 \quad (3.91)$$

$$2x_1 + 4x_2 - 6x_3 - 12x_4 = s_2^o + s_2^1 - 24 \quad (3.92)$$

$$-3x_1 - 6x_2 - x_3 - 2x_4 = s_3^o + s_3^1 - 24 \quad (3.93)$$

$$-6x_1 - 12x_2 - 2x_3 - 4x_4 = s_4^o + s_4^1 - 24 \quad (3.94)$$

$$s_i^1 \le 48x_i \quad \text{for } i = 1, 2, 3, 4 \quad (3.95)$$

$$s_i^o \le 48(1 - x_i) \quad \text{for } i = 1,2,3,4 \quad (3.96)$$

$$2x_1 + 4x_2 - 2x_3 - 4x_4 = t_1^o + t_1^1 - 24 \quad (3.97)$$

$$4x_1 + 8x_2 - 4x_3 - 8x_4 = t_2^o + t_2^1 - 24 \quad (3.98)$$

$$-2x_1 - 4x_2 - x_3 - 2x_4 = t_3^o + t_3^1 - 24 \quad (3.99)$$

$$-4x_1 - 8x_2 - 2x_3 - 4x_4 = t_4^o + t_4^1 - 24 \quad (3.100)$$

$$t_i^1 \le 48x_i \quad \text{for } i = 1, 2, 3, 4 \quad (3.101)$$

$$t_i^o \le 48(1 - x_i) \quad \text{for } i = 1,2,3,4 \quad (3.102)$$

$$x_1 + 2x_2 - r_1^o + 6 \ge 0 \quad (3.103)$$

$$2x_2 + 4x_2 - r_2^o + 6 \ge 0 \quad (3.104)$$

$$x_3 + 2x_4 - r_3^o + 6 \ge 0 \quad (3.105)$$

$$2x_3 + 4x_4 - r_4^o + 6 \ge 0 \quad (3.106)$$

$$r_i^o - 6x_i \geq 0.25 \quad \text{for } i = 1, 2, 3, 4 \tag{1.107}$$

$$r_i^o \leq 12x_i \quad \text{for } i = 1, 2, 3, 4 \tag{3.108}$$

$$0.5x_3 + x_4 - r_1^1 + 3 \geq 0 \tag{3.109}$$

$$x_3 + 2x_4 - r_2^1 + 3 \geq 0 \tag{3.110}$$

$$0.5x_1 + x_2 - r_3^1 + 3 \geq 0 \tag{3.111}$$

$$x_1 + 2x_2 - r_4^1 + 3 \geq 0 \tag{3.112}$$

$$r_i^1 - 3x_i \geq 1 \quad \text{for } i = 1, 2, 3, 4 \tag{3.113}$$

$$r_i^1 \leq 6x_i \quad \text{for } i = 1, 2, 3, 4 \tag{3.114}$$

where $x \in \{0, 1\}^4$; s_i^o, s_i^1, r_i^o, r_i^1, t_i^o, $t_i^1 \geq 0$ for $i = 1, 2, 3, 4$.

Hence, the above problem is a 0-1 linear fractional programming problem

Step 3: Reduction to a non-fractional problem:

Choose a positive real number q s.t.

$$q(21 + e^T t^1 - 24e^T x) = 1 \tag{3.115}$$

As seen earlier $(21 + e^T t^1 - 24e^T x) > 0 \; \forall \; x \in \{0, 1\}^4$, therefore, $\exists$ an $\in > 0$ s.t.

$$q = \frac{1}{21 + e^T t^1 - 24e^T x} \leq \frac{1}{\in} \tag{3.116}$$

where $\in$ is a very small real number to be suitably chosen. Since all the entries of B and β are integers, so we can take $\in = 1$.

Include the constraints (3.115) and (3.116) in (P 3.6) to obtain

(P3.7) $\min 4q + e^T s^1 q - 4e^T x q$

s.t. (3.85)-(3.88) and (3.91)-(3.116)

and $q > 0$

Step 4 : Reduction to a mixed 0-1 linear program

Multiply the constraints (3.85)-(3.88) and (3.91)-(3.115) by q along with the following substitution.

$$S^0 = qs^0,\ S^1 = qs^1,\ R^0 = qr^0,\ R^1 = qr^1,\ T^0 = qt^0,\ T^1 = qt^1 \text{ and } K = qx$$

where $S_i^0,\ S_i^1,\ R_i^0,\ R_i^1,\ T_i^0,\ T_i^1,\ K_i \geq 0$ for i = 1, 2, 3, 4.

Also, qx = K must imply that $K_i = 0$ when $x_i = 0$ and $K_i = q$ when $x_i = 1$ for which the following constraints should be included in the problem

$$K \geq q + \frac{1}{\in}(x-e),\quad K \leq qe,\quad K \leq \frac{1}{\in}x \tag{3.117}$$

Thus the problem considered is reduced to the following mixed 0-1 linear programming problem:

(P3.8) $\min 4q + e^T S^1 - 24e^T K$

s.t.

$$K_1 + 2K_2 + K_3 + 2K_4 \leq 4q \tag{3.118}$$

$$K_1 + 2K_2 - 2K_3 + 4K_4 \leq q \tag{3.119}$$

$$K_1 + 2K_2 \leq 2q \tag{3.120}$$

$$K_3 + 2K_4 \leq 3q \tag{3.121}$$

$$K_1 + 2K_2 - 3K_3 - 6K_4 = S_1^o + S_1^1 - 24q \tag{3.122}$$

$$2K_1 + 4K_2 - 6K_3 - 12K_4 = S_2^o + S_2^1 - 24q \tag{3.123}$$

$$-3K_1 - 6K_2 - K_3 - 2K_4 = S_3^o + S_3^1 - 24q \tag{3.124}$$

$$-6K_1 - 12K_2 - 2K_3 - 4K_4 = S_4^o + S_4^1 - 24q \tag{3.125}$$

$$S_i^1 \leq 48K_i \quad \text{for } i = 1, 2, 3, 4. \tag{3.126}$$

$$S_i^o \leq 48(q - K_i) \quad \text{for } i = 1, 2, 3, 4. \tag{3.127}$$

$$2K_1 + 4K_2 - 2K_3 - 4K_4 = T_1^o + T_1^1 - 24q \tag{3.128}$$

$$4K_1 + 8K_2 - 4K_3 - 8K_4 = T_2^o + T_2^1 - 24q \tag{3.129}$$

$$-2K_1 - 4K_2 - K_3 - 2K_4 = T_3^o + T_3^1 - 24q \tag{3.130}$$

$$-4K_1 - 8K_2 - 2K_3 - 4K_4 = T_4^o + T_4^1 - 24q \tag{3.131}$$

$$T_i^1 \leq 48K_i \quad \text{for } i = 1, 2, 3, 4 \tag{3.132}$$

$$T_i^o \leq 48(q - K_i) \quad \text{for } i = 1, 2, 3, 4 \tag{3.133}$$

$$K_1 + 2K_2 - R_1^o + 6q \geq 0 \tag{3.134}$$

$$2K_1 + 4K_2 - R_2^o + 6q \geq 0 \quad (3.135)$$

$$K_3 + 2K_4 - R_3^o + 6q \geq 0 \quad (3.136)$$

$$2K_3 + 4K_4 - R_4^o + 6q \geq 0 \quad (3.137)$$

$$R_i^o - 6K_i \geq 0.25q \quad \text{for } i = 1, 2, 3, 4. \quad (3.138)$$

$$R_i^o \leq 12K_i \quad \text{for } i = 1, 2, 3, 4. \quad (3.139)$$

$$0.5K_3 + K_4 - R_1^1 + 3q \geq 0 \quad (3.140)$$

$$K_3 + 2K_4 - R_2^1 + 3q \geq 0 \quad (3.141)$$

$$0.5K_1 + K_2 - R_3^1 + 3q \geq 0 \quad (3.142)$$

$$K_1 + 2K_2 - R_4^1 + 3q \geq 0 \quad (3.143)$$

$$R_i^1 - 3K_i \geq q \quad \text{for } i = 1, 2, 3, 4. \quad (3.144)$$

$$R_i^1 \leq 6K_i \quad \text{for } i = 1, 2, 3, 4. \quad (3.145)$$

$$21q + T_1^1 + T_2^1 + T_3^1 + T_4^1 - 24K_1 - 24K_2 - 24K_3 - 24K_4 = 1 \quad (3.146)$$

Constraint (3.116), (3.117)

$$x \in \{0, 1\}^4$$

$$S_i^o,\ S_i^1,\ R_i^o,\ R_i^1,\ T_i^o,\ T_i^1,\ K_i \geq 0 \quad \text{for } i = 1, 2, 3, 4;\ q > 0$$

The optimal solution of this problem is –68 at $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$ using LINDO software.

Hence, the optimal solution of the original problem is, $y_1 = 1$, $y_2 = 1$ and objective function value is $z = -2/18 = -0.111111$.

Conclusions: This paper proposes a technique for solving integer quadratic fractional programming problems. The algorithm is based on the linearization approach in which the quadratic terms appearing in the objective as well as the constraint set are linearized resulting in a linear fractional programming problem. The problem is then solved using the Charnes and Cooper method.

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