J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 15, Number 4 (November 2008), Pages 377–385

FUZZY ABYSMS OF HILBERT ALGEBRAS

YOUNG BAE JUN [a], KYOUNG JA LEE [b] AND CHUL HWAN PARK [c,*]

ABSTRACT. The notion of fuzzy abysms in Hilbert algebras is introduced, and several properties are investigated. Relations between fuzzy subalgebra, fuzzy deductive systems, and fuzzy abysms are considered.

1. INTRODUCTION

Hilbert algebras are important tools for certain investigations in algebraic logic since they can be considered as fragments of any propositional logic containing a logical connective implication ($\rightarrow$) and the constant 1 which is considered as the logical value "true". Following the introduction of fuzzy sets by Zadeh [13], the fuzzy set theory developed by Zadeh himself and others has found many applications in the domain of mathematics and elsewhere. Dudek and Jun [7] discussed the fuzzy ideals of Hilbert algebras. Jun and Hong [11] considered the fuzzification of deductive systems of Hilbert algebras. Jun [8] treated the extension of fuzzy deductive systems in Hilbert algebras. Our present paper is concerned with the fuzzification of abysms in Hilbert algebras which exists between fuzzy subalgebras and fuzzy deductive systems. We introduce the notion of fuzzy abysms in Hilbert algebras, and investigate several properties. We consider relations between fuzzy subalgebras, fuzzy deductive systems, and fuzzy abysms.

2. PRELIMINARIES

A Hilbert algebra can be considered as a fragment of propositional logic containing only a logical connective implication "$\rightarrow$" and the constant 1 which is interpreted

Received by the editors January 20, 2008. Accepted October 28, 2008.
*Corresponding author.
2000 *Mathematics Subject Classification.* 06F35, 03G25, 94D05.
Key words and phrases. (commutative) Hilbert algebra, (fuzzy) subalgebra, (fuzzy) deductive system, (fuzzy) abysm.

as the value "true". In the following, the logical connective implication "$\to$" will be denoted by "$\cdot$".

An algebra $\mathcal{H} := (H; \cdot, 1)$ of type (2,0) is called a *Hilbert algebra* if it satisfies:

(H1) $(\forall a, b \in H)\ (a \cdot (b \cdot a) = 1)$.

(H2) $(\forall a, b, c \in H)\ ((a \cdot (b \cdot c)) \cdot ((a \cdot b) \cdot (a \cdot c)) = 1)$.

(H3) $(\forall a, b \in H)\ (a \cdot b = b \cdot a = 1 \Rightarrow a = b)$.

If $\mathcal{H} := (H; \cdot, 1)$ is a Hilbert algebra and we define a binary relation $\preceq$ in $\mathcal{H}$ by $a \preceq b$ if and only if $a \cdot b = 1$, then $\preceq$ is a partial order in $\mathcal{H} := (H; \cdot, 1)$. A Hilbert algebra $\mathcal{H} := (H; \cdot, 1)$ is said to be *commutative* if $(x \cdot y) \cdot y = (y \cdot x) \cdot x$ for all $x, y \in H$.

In a Hilbert algebra $\mathcal{H} := (H; \cdot, 1)$, we have the following assertions:

(a1) $x \preceq y \cdot x$.

(a2) $x \cdot 1 = 1,\ 1 \cdot x = x$.

(a3) $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$.

(a4) $x \preceq (x \cdot y) \cdot y$.

(a5) $x \cdot (y \cdot z) = y \cdot (x \cdot z)$.

(a6) $x \cdot y \preceq (y \cdot z) \cdot (x \cdot z)$.

(a7) $x \preceq y \Rightarrow z \cdot x \preceq z \cdot y,\ y \cdot z \preceq x \cdot z$.

A subset S of a Hilbert algebra $\mathcal{H}$ is called a *subalgebra* of $\mathcal{H}$ if $x \cdot y \in S$ for all $x, y \in S$. The concept of a deductive system on a Hilbert algebra $\mathcal{H} := (H; \cdot, 1)$ was also introduced by A. Diego [6] as a subset of H containing 1 and closed under a "deduction", i.e.:

Definition 2.1. A nonempty subset D of a Hilbert algebra $\mathcal{H} := (H; \cdot, 1)$ is called a *deductive system* of $\mathcal{H}$ if it satisfies:

(Di) $1 \in D$,

(Dii) $(\forall x \in D)\ (\forall y \in H)\ (x \cdot y \in D \Rightarrow y \in D)$.

Lemma 2.2 ([5]). *A deductive system D of a Hilbert algebra $\mathcal{H} := (H, \cdot, 1)$ has the following property:*

$$(\forall x \in D)(\forall y \in H)(x \preceq y \Rightarrow y \in D).$$

For any subsets X and Y of a Hilbert algebra $\mathcal{H}$, we define

$$X \cdot Y := \{x \cdot y \mid x \in X, y \in Y\}.$$

Lemma 2.3 ([12]). *If A is a subset of a Hilbert algebra $\mathcal{H} := (H; \cdot, 1)$ containing 1, then B is contained in $A \cdot B$ for every subset B of H.*

For more concepts/results on Hilbert algebras, the readers refer to [1, 2, 3, 4, 5, 6, 9, 10, 12].

We now review some fuzzy logic concepts. A *fuzzy set* in a set X is a function $\mu : X \to [0, 1]$, and the set

$$U(\mu; k) := \{x \in X \mid \mu(x) \geq k\},\ k \in [0, 1]$$

is called a *level set* of μ.

Definition 2.4 ([7]). A fuzzy set μ in a Hilbert algebra $\mathcal{H}$ is called a *fuzzy subalgebra* of $\mathcal{H}$ if it satisfies:

$$(\forall x, y \in H)\,(\mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\}).$$

Definition 2.5 ([7, 11]). A fuzzy set μ in a Hilbert algebra $\mathcal{H}$ is called a *fuzzy deductive system* of $\mathcal{H}$ if it satisfies:

(i) $(\forall x \in H)$ $(\mu(1) \geq \mu(x))$.
(ii) $(\forall x, y \in H)$ $(\mu(y) \geq \min\{\mu(x), \mu(x \cdot y)\})$.

Lemma 2.6 ([11]). *A fuzzy set μ in a Hilbert algebra $\mathcal{H}$ is a fuzzy subalgebra (resp. fuzzy deductive system) of $\mathcal{H}$ if and only if for every $m \in [0, 1]$, the nonempty level set $U(\mu; m)$ is a subalgebra (resp. deductive system) of $\mathcal{H}$.*

3. Fuzzy Abysms

In what follows, $\mathcal{H} := (H, \cdot, 1)$ or simply $\mathcal{H}$ will always denote a Hilbert algebra unless otherwise specified.

Definition 3.1 ([12]). Let $\mathcal{H} := (H; \cdot, 1)$ be a Hilbert algebra. If a nonempty subset A of $\mathcal{H}$ satisfies the following equality:

$$H \cdot A = A,$$

then we say that A is an *abysm* of $\mathcal{H}$.

Definition 3.2. A fuzzy set μ in $\mathcal{H}$ is called a *fuzzy abysm* of $\mathcal{H}$ if it satisfies:

$$(\forall k \in [0, 1])\,(U(\mu; k) \neq \emptyset \Rightarrow H \cdot U(\mu; k) = U(\mu; k)). \tag{3.1}$$

For a fuzzy set μ in $\mathcal{H}$ and $a \in H$, consider a set

$$\Omega_a := \{x \in H \mid \mu(a) \leq \mu(x)\}. \tag{3.2}$$

Then $\Omega_a = U(\mu; \mu(a))$, and so μ is a fuzzy abysm of $\mathcal{H}$ if and only if Ω_a is an abysm of $\mathcal{H}$ for all $a \in H$.

Example 3.3. Let μ be a fuzzy set in $\mathcal{H}$ given by

$$\mu(x) = \begin{cases} m & \text{if } x = 1, \\ n & \text{otherwise} \end{cases}$$

for all $x \in H$, where $m, n \in [0, 1]$ with $m > n$. Then μ is a fuzzy abysm of $\mathcal{H}$.

Example 3.4. Let $H = \{1, a, b, c, d\}$ be a set with the following Cayley table.

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	1	1	c	d
c	1	a	b	1	d
d	1	1	1	1	1

Then $\mathcal{H} := (H, \cdot, 1)$ is a Hilbert algebra.

(i) Let μ be a fuzzy set in H given by

$$\mu(x) = \begin{cases} m & \text{if } x \in \{1, a\}, \\ n & \text{otherwise} \end{cases}$$

for all $x \in H$, where $m, n \in [0, 1]$ with $m > n$. Then μ is a fuzzy abysm of $\mathcal{H}$.

(ii) Let γ be a fuzzy set in H given by

$$\gamma(x) = \begin{cases} m & \text{if } x \in \{1, a, b\}, \\ n & \text{otherwise} \end{cases}$$

for all $x \in H$, where $m, n \in [0, 1]$ with $m > n$. Then γ is a fuzzy abysm of $\mathcal{H}$.

(iii) Let σ be a fuzzy set in H given by

$$\sigma(x) = \begin{cases} m & \text{if } x \in \{1, a, c\}, \\ n & \text{otherwise} \end{cases}$$

for all $x \in H$, where $m, n \in [0, 1]$ with $m > n$. Then σ is a fuzzy abysm of $\mathcal{H}$.

Example 3.5. Let $H = \{1, a, b, c\}$ be a set with the following Cayley table.

$\cdot$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	a
c	1	1	1	1

Then $\mathcal{H} := (H, \cdot, 1)$ is a Hilbert algebra. Let μ be a fuzzy set in H given by

$$\mu(x) = \begin{cases} m & \text{if } x \in \{1, a, b\}, \\ n & \text{otherwise} \end{cases}$$

for all $x \in H$, where $m, n \in [0,1]$ with $m > n$. Then μ is a fuzzy abysm of $\mathcal{H}$. Now let γ be a fuzzy set in H given by

$$\gamma(x) = \begin{cases} 0.8 & \text{if } x \in \{1, c\}, \\ 0.3 & \text{otherwise} \end{cases}$$

for all $x \in H$. Since $H \cdot U(\gamma; 0.8) = H \cdot \{1, c\} = H \neq U(\gamma; 0.8)$, we know that γ is not a fuzzy abysm of $\mathcal{H}$.

Proposition 3.6. *Let Q be an abysm of a commutative Hilbert algebra $\mathcal{H}$. Then*

$$(\forall x \in H)\,(\forall a \in Q)\,(a \preceq x \;\Rightarrow\; x \in Q). \tag{3.3}$$

Proof. Let $x \in H$ and $a \in Q$ be such that $a \preceq x$. Then $a \cdot x = 1$, and so

$$x = 1 \cdot x = (a \cdot x) \cdot x = (x \cdot a) \cdot a \in H \cdot Q = Q.$$

This completes the proof. □

Theorem 3.7. *Let Q be an abysm of a commutative Hilbert algebra $\mathcal{H}$. For every $a \in H \setminus Q$, define a fuzzy set μ_a on H by*

$$\mu_a(x) = \begin{cases} m & \text{if } x \in \{y \in H \mid a \cdot y \in Q\}, \\ n & \textit{otherwise} \end{cases}$$

for all $x \in H$, where $m > n$ in $[0,1]$. Then μ_a is a fuzzy abysm of $\mathcal{H}$.

Proof. Let $k \in [0,1]$. If $k > m$, then $U(\mu_a; k) = \emptyset$. If $n < k \leq m$, then $U(\mu_a; k) = \{y \in H \mid a \cdot y \in Q\}$. Let $y \in U(\mu_a; k)$ and $x \in H$. Then $a \cdot y \preceq x \cdot (a \cdot y) = a \cdot (x \cdot y)$. Since $a \cdot y \in Q$ and Q is an abysm, it follows from Proposition 3.6 that $a \cdot (x \cdot y) \in Q$ so that $x \cdot y \in U(\mu_a; k)$. This shows that $H \cdot U(\mu_a; k) \subseteq U(\mu_a; k)$. The reverse inclusion follows from Lemma 2.3. Hence $H \cdot U(\mu_a; k) = U(\mu_a; k)$. If $k \leq n$, then $U(\mu_a; k) = H$ and thus $H \cdot U(\mu_a; k) = U(\mu_a; k)$. Therefore μ_a is a fuzzy abysm of $\mathcal{H}$. □

Theorem 3.8. *For any $a \in H$, let γ be a fuzzy set in H given by*

$$\gamma(x) = \begin{cases} m & \text{if } a \preceq x, \\ n & \textit{otherwise} \end{cases}$$

for all $x \in H$, where $m > n$ in $[0,1]$. Then γ is a fuzzy abysm of $\mathcal{H}$.

Proof. Let $k \in [0,1]$. If $k > m$, then $U(\gamma; k) = \emptyset$. If $n < k \leq m$, then $U(\gamma; k) = \{x \in H \mid a \preceq x\}$. Let $x \in U(\gamma; k)$ and $y \in H$. Then $a \preceq x$, and so $a \preceq y \cdot a \preceq y \cdot x$. Hence $y \cdot x \in U(\gamma; k)$, which shows that $H \cdot U(\gamma; k) \subseteq U(\gamma; k)$. The reverse inclusion follows from Lemma 2.3. Hence $H \cdot U(\gamma; k) = U(\gamma, k)$. If $k \leq n$, then clearly $H \cdot U(\gamma; k) = U(\gamma, k)$. Hence γ is a fuzzy abysm of $\mathcal{H}$. □

Proposition 3.9. *Every fuzzy abysm μ of $\mathcal{H}$ satisfies the following inequality.*

$$(\forall x \in H)\,(\mu(1) \geq \mu(x)).$$

Proof. If μ is a fuzzy abysm of $\mathcal{H}$, then $H \cdot U(\mu; m) = U(\mu; m)$ for all $m \in \mathrm{Im}(\mu)$. Since $U(\mu; m) \neq \emptyset$, there exists $x \in U(\mu; m)$. Hence

$$1 = x \cdot x \in H \cdot U(\mu; m) = U(\mu; m),$$

and so $\mu(1) \geq \mu(x)$ for all $x \in H$. □

Theorem 3.10. *Every fuzzy abysm of $\mathcal{H}$ is a fuzzy subalgebra of $\mathcal{H}$.*

Proof. Let μ be a fuzzy abysm of $\mathcal{H}$. Assume that $U(\mu; m) \neq \emptyset$ for all $m \in [0, 1]$. For any $x, y \in U(\mu; m)$, we have

$$x \cdot y \in U(\mu; m) \cdot U(\mu; m) \subseteq H \cdot U(\mu; m) = U(\mu; m).$$

Thus $U(\mu; m)$ is a subalgebra of $\mathcal{H}$, and so μ is a fuzzy subalgebra of $\mathcal{H}$ by Lemma 2.6. □

The following example shows that the converse of Theorem 3.10 is not true in general.

Example 3.11. The fuzzy set γ in Example 3.5 is a fuzzy subalgebra of $\mathcal{H}$ which is not a fuzzy abysm of $\mathcal{H}$.

Theorem 3.12. *Every fuzzy deductive system of $\mathcal{H}$ is a fuzzy abysm of $\mathcal{H}$.*

Proof. Let μ be a fuzzy deductive system of $\mathcal{H}$. Assume that $U(\mu; m) \neq \emptyset$ for all $m \in [0, 1]$. Let $x \in H$ and $y \in U(\mu; m)$. Since $y \preceq x \cdot y$ and $U(\mu; m)$ is a deductive system, it follows from Lemma 2.2 that $x \cdot y \in U(\mu; m)$. This shows that

$$H \cdot U(\mu; m) \subseteq U(\mu; m). \tag{3.4}$$

Combining (3.4) and Lemma 2.3, we have $H \cdot U(\mu; m) = U(\mu; m)$. Hence μ is a fuzzy abysm of $\mathcal{H}$. □

The converse of Theorem 3.12 may not be true as seen in the following example.

Example 3.13. The fuzzy set μ in Example 3.5 is a fuzzy abysm of $\mathcal{H}$. Note that

$$\mu(c) = n < m = \min\{\mu(a), \mu(a \cdot c)\}.$$

Hence μ is not a fuzzy deductive system of $\mathcal{H}$.

Lemma 3.14 ([12]). *For any subsets A, B and E of $\mathcal{H}$, we have*

(i) $A \subseteq B \Rightarrow A \cdot E \subseteq B \cdot E,\ E \cdot A \subseteq E \cdot B.$

(ii) $(A \cap B) \cdot E \subseteq A \cdot E \cap B \cdot E$.
(iii) $E \cdot (A \cap B) \subseteq E \cdot A \cap E \cdot B$.
(iv) $(A \cup B) \cdot E = A \cdot E \cup B \cdot E$.
(v) $E \cdot (A \cup B) = E \cdot A \cup E \cdot B$.

Theorem 3.15. *Let μ and γ be fuzzy abysms of $\mathcal{H}$. Then $\mu \wedge \gamma$ and $\mu \vee \gamma$ are fuzzy abysms of $\mathcal{H}$.*

Proof. Let $m \in [0,1]$ be such that $U(\mu\wedge\gamma; m) \neq \emptyset$. Then there exists $x \in U(\mu\wedge\gamma; m)$, and so

$$m \leq (\mu \wedge \gamma)(x) = \min\{\mu(x), \gamma(x)\}.$$

It follows that $x \in U(\mu; m)$ and $x \in U(\gamma; m)$ so that $H \cdot U(\mu; m) = U(\mu; m)$ and $H \cdot U(\gamma; m) = U(\gamma; m)$. Using Lemma 3.14(iii), we have

$$\begin{aligned} H \cdot U(\mu \wedge \gamma; m) &= H \cdot (U(\mu; m) \cap U(\gamma; m)) \\ &\subseteq H \cdot U(\mu; m) \cap H \cdot U(\gamma; m) \\ &= U(\mu; m) \cap U(\gamma; m) = U(\mu \wedge \gamma; m). \end{aligned}$$

The reverse inclusion follows from Lemma 2.3. Hence

$$H \cdot U(\mu \wedge \gamma; m) = U(\mu \wedge \gamma; m),$$

i.e., $\mu \wedge \gamma$ is a fuzzy abysm of $\mathcal{H}$. Now assume that $U(\mu \vee \gamma; n) \neq \emptyset$ for all $n \in [0,1]$. Then there exists $y \in U(\mu \vee \gamma; n)$, and so

$$n \leq (\mu \vee \gamma)(y) = \max\{\mu(y), \gamma(y)\}.$$

Hence $\mu(y) \geq n$ or $\gamma(y) \geq n$. We may assume that $\mu(y) \geq n$ without loss of generality. Then $y \in U(\mu; n)$, and thus $H \cdot U(\mu; n) = U(\mu; n)$. If $U(\gamma; n) = \emptyset$, then

$$\begin{aligned} H \cdot U(\mu \vee \gamma; n) &= H \cdot (U(\mu; n) \cup U(\gamma; n)) \\ &= H \cdot U(\mu; n) = U(\mu; n) \\ &= U(\mu; n) \cup U(\gamma; n) \\ &= U(\mu \vee \gamma; n). \end{aligned}$$

If $U(\gamma; n) \neq \emptyset$, then $H \cdot U(\gamma; n) = U(\gamma; n)$. Hence

$$\begin{aligned} H \cdot U(\mu \vee \gamma; n) &= H \cdot (U(\mu; n) \cup U(\gamma; n)) \\ &= H \cdot U(\mu; n) \cup H \cdot U(\gamma; n) \\ &= U(\mu; n) \cup U(\gamma; n) = U(\mu \vee \gamma; n). \end{aligned}$$

Therefore $\mu \vee \gamma$ is a fuzzy abysm of $\mathcal{H}$. □

In general, the union of two fuzzy deductive systems of $\mathcal{H}$ may not be a fuzzy deductive system of $\mathcal{H}$ as seen in the following example.

Example 3.16. Let $H = \{1, a, b, c, d\}$ be a set with the following Cayley table.

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	b	d
b	1	a	1	a	d
c	1	1	1	1	d
d	1	1	b	b	1

Then $\mathcal{H} := (H, \cdot, 1)$ is a Hilbert algebra. Let μ and γ be fuzzy sets in $\mathcal{H}$ given by

$$\mu(x) = \begin{cases} m & \text{if } x \in \{1, a\}, \\ n & \text{otherwise} \end{cases}$$

and

$$\gamma(x) = \begin{cases} m & \text{if } x \in \{1, b\}, \\ n & \text{otherwise} \end{cases}$$

respectively, for all $x \in H$ where $m, n \in [0, 1]$ with $m > n$. Then μ and γ are fuzzy deductive systems of $\mathcal{H}$. But $\mu \vee \gamma$ is not a fuzzy deductive system of $\mathcal{H}$ since

$$(\mu \vee \gamma)(c) < \min\{(\mu \vee \gamma)(a), (\mu \vee \gamma)(a \cdot c)\}.$$

But we know that Theorem 3.12 and Theorem 3.15 induce the following corollary.

Corollary 3.17. *The union and intersection of two fuzzy deductive systems of H are fuzzy abysms of $\mathcal{H}$.*

REFERENCES

1. D. Busneag: A note on deductive systems of a Hilbert algebra. *Kobe J. Math.* **2** (1985), 29–35.
2. D. Busnea: Hilbert algebras of fractions and maximal Hilbert algebras of quotients. *Kobe J. Math.* **5** (1988), 161–172.
3. ______ : Hertz algebras of fractions and maximal Hertz algebras of quotients. *Math. Japon.* **39** (1993), 461–469.
4. I. Chajda: The lattice of deductive systems on Hilbert algebras. *SEA Bull. Math.* **26** (2002), 21–26.
5. I. Chajda, R. Halaš & Y.B. Jun: Annihilators and deductive systems in commutative Hilbert algebras. *Comment. Math. Univ. Carolinae* **43** (2002), no. 3, 407–417.
6. A. Diego: Sur les algébres de Hilbert. *Collection de Logigue Math. Ser. A* (Ed. Hermann, Paris) **21** (1966), 1–52.
7. W.A. Dudek & Y.B. Jun: On fuzzy ideals in Hilbert algebras. *Novi Sad J. Math.* **29** (1999), no. 2, 193–207.

8. Y.B. Jun: Extensions of fuzzy deductive systems in Hilbert algebras. *Fuzzy Sets and Systems* **79** (1996), 263–265.
9. ______ : Deductive systems of Hilbert algebras. *Math. Jpn.* **43** (1996), no. 1, 51–54.
10. ______ : Commutative Hilbert algebras. *Soochow J. Math.* **22** (1996), no. 4, 477–484.
11. Y.B. Jun & S.M. Hong, Fuzzy deductive systems of Hilbert algerbas. *Indian J. Pure Appl. Math.* **27** (1996), no. 2, 141–151.
12. Y.B. Jun, S.Y. Kim & E.H. Roh: The abysm of a Hilbert algerba. *Sci. Math. Jpn.* **65** (2007), no. 1, 135–140; **e2006**, 1055–1060.
13. L.A. Zadeh: Fuzzy sets. *Inform. and Control* **8** (1965), 338–353.

[a]DEPARTMENT OF MATHEMATICS EDUCATION (AND RINS), GYEONGSANG NATIONAL UNIVERSITY, JINJU 660-701, KOREA
Email address: `skywine@gmail.com`

[b]DEPARTMENT OF MATHEMATICS EDUCATION, HANNAM UNIVERSITY, DAEJEON 306-791, KOREA
Email address: `kjlee@hnu.kr`

[c]DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ULSAN, ULSAN 680-749, KOREA
Email address: `skyrosemary@gmail.com`