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HILBERT-SCHMIDT INTERPOLATION ON $AX = Y$ IN A TRIDIAGONAL ALGEBRA ALG$\mathcal{L}$

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ABSTRACT. Given operators X and Y acting on a separable complex Hilbert space $\mathcal{H}$, an interpolating operator is a bounded operator A such that $AX = Y$. In this article, we investigate Hilbert-Schmidt interpolation problems for operators in a tridiagonal algebra and we get the following : Let $\mathcal{L}$ be a subspace lattice acting on a separable complex Hilbert space $\mathcal{H}$ and let $X = (x_{ij})$ and $Y = (y_{ij})$ be operators acting on $\mathcal{H}$. Then the following are equivalent:

(1) There exists a Hilbert-Schmidt operator $A = (a_{ij})$ in Alg$\mathcal{L}$ such that $AX = Y$.

(2) There is a bounded sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ and

$$y_{1i} = \alpha_1 x_{1i} + \alpha_2 x_{2i}$$
$$y_{2k\ i} = \alpha_{4k-1} x_{2k\ i}$$
$$y_{2k+1\ i} = \alpha_{4k} x_{2k\ i} + \alpha_{4k+1} x_{2k+1\ i} + \alpha_{4k+2} x_{2k+2\ i} \text{ for all } i, k \text{ in } \mathbb{N}.$$

1. INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A}$ be a subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of all operators acting on $\mathcal{H}$. Suppose that X and Y are specified, not necessarily in the algebra. Under what conditions can we expect there to be a solution of the operator equation $AX = Y$, where the operator A is required to lie in $\mathcal{A}$? We refer to such a question as an interpolation problem. The 'n-operator interpolation problem', asks for an operator A such that $AX_i = Y_i$ for fixed finite collections $\{X_1, X_2, \cdots, X_n\}$ and $\{Y_1, Y_2, \cdots, Y_n\}$. The n-operator interpolation problem was considered for a C^*-algebra $\mathcal{U}$ by Kadison [8]. In case $\mathcal{U}$ is a nest algebra, the (one-operator) interpolation problem was solved by Lance [9]: his result was extended by Hopenwasser [2] to the case that $\mathcal{U}$ is a CSL-algebra. Munch [10] obtained conditions for interpolation in case A is required to lie in the ideal of Hilbert-Schmidt operators

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in a nest algebra. Hopenwasser [3] once again extended the interpolation condition to the ideal of Hilbert-Schmidt operators in a CSL-algebra. Hopenwasser's paper also contains a sufficient condition for interpolation n-vectors, although necessity was not proved in that paper.

We establish some notations and conventions. A commutative subspace lattice $\mathcal{L}$, or CSL $\mathcal{L}$ is a strongly closed lattice of pairwise-commuting projections acting on a Hilbert space $\mathcal{H}$. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. The symbol Alg$\mathcal{L}$ is the algebra of all bounded operators on $\mathcal{H}$ that leave invariant all the projections in $\mathcal{L}$. If $\mathcal{L}$ is CSL, Alg$\mathcal{L}$ is called a CSL-algebra. Let x and y be two vectors in a Hilbert space $\mathcal{H}$. Then $\langle x, y\rangle$ means the inner product of the vectors x and y. Let M be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{M}$ means the closure of M and $\overline{M}^{\perp}$ the orthogonal complement of $\overline{M}$. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers.

2. Results

Let $\mathcal{H}$ be a separable complex Hilbert space with a fixed orthonormal basis $\{e_1, e_2, \cdots\}$. Let $x_1, x_2, \cdots, x_n$ be vectors in $\mathcal{H}$. Then $[x_1, x_2, \cdots, x_n]$ means the closed subspace generated by the vectors $x_1, x_2, \cdots, x_n$. Let $\mathcal{L}$ be the subspace lattice generated by the subspaces $[e_{2k-1}]$, $[e_{2k-1}, e_{2k}, e_{2k+1}]$ $(k = 1, 2, \cdots)$. Then the algebra Alg$\mathcal{L}$ is called a tridiagonal algebra which was introduced by F. Gilfeather and D. Larson [1]. These algebras have been found to be useful counterexample to a number of plausible conjectures.

Let $\mathcal{A}$ be the algebra consisting of all bounded operators acting on $\mathcal{H}$ of the form

$$\begin{pmatrix} * & * & & & \\ & * & & & \\ & * & * & * & \\ & & & * & \\ & & & * & \ddots \end{pmatrix}$$

relative to the orthonormal basis $\{e_1, e_2, \cdots\}$, where all non-starred entries are zero. It is easy to see that Alg$\mathcal{L}$=$\mathcal{A}$.

We consider interpolation problems for the above tridiagonal algebra Alg$\mathcal{L}$.

Theorem 2.1. *Let $Alg\mathcal{L}$ be the tridiagonal algebra on a separable complex Hilbert space $\mathcal{H}$ and let $X = (x_{ij})$ and $Y = (y_{ij})$ be operators acting on $\mathcal{H}$. Then the following are equivalent:*

(1) *There exists a Hilbert-Schmidt operator $A = (a_{ij})$ in $Alg\mathcal{L}$ such that $AX = Y$.*

(2) *There is a bounded sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ and*

$$y_{1i} = \alpha_1 x_{1i} + \alpha_2 x_{2i}$$
$$y_{2k\ i} = \alpha_{4k-1} x_{2k\ i}$$
$$y_{2k+1\ i} = \alpha_{4k} x_{2k\ i} + \alpha_{4k+1} x_{2k+1\ i} + \alpha_{4k+2} x_{2k+2\ i} \text{ for all } i, k \text{ in } \mathbb{N}.$$

Proof. Suppose that A is a Hilbert-Schmidt operator $A = (a_{ij})$ in Alg$\mathcal{L}$ such that $AX = Y$. Let $\alpha_n = a_{ij}$ for $n = i + j - 1$ and $\{e_n\}$ the standard orthonormal basis for $\mathcal{H}$. Since A is Hilbert-Schmidt, $\sum_i \|Ae_i\|^2 < \infty$. Hence

$$\begin{aligned}
\sum_i \|Ae_i\|^2 &= \sum_i \sum_j |\langle Ae_i, e_j\rangle|^2 \\
&= \sum_{k=1}^{\infty} \langle Ae_{2k-1}, e_{2k-1}\rangle + \sum_{k=1}^{\infty} \langle Ae_{2k}, (e_{2k-1} + e_{2k} + e_{2k+1})\rangle \\
&= \sum_{k=1}^{\infty} |\alpha_{4k-3}|^2 + \sum_{k=1}^{\infty} (|\alpha_{4k-2}|^2 + |\alpha_{4k-1}|^2 + |\alpha_{4k}|^2) \\
&= \sum_{k=1}^{\infty} |\alpha_k|^2 < \infty.
\end{aligned}$$

Since $AX = Y$, for all i, k in $\mathbb{N}$,

$$y_{1i} = \alpha_1 x_{1i} + \alpha_2 x_{2i}$$
$$y_{2k\ i} = \alpha_{4k-1} x_{2k\ i}$$
$$y_{2k+1\ i} = \alpha_{4k} x_{2k\ i} + \alpha_{4k+1} x_{2k+1\ i} + \alpha_{4k+2} x_{2k+2\ i}.$$

Conversely, assume that there is a bounded sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that

$$\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$$

and for all i, k in $\mathbb{N}$,

$$y_{1i} = \alpha_1 x_{1i} + \alpha_2 x_{2i}$$
$$y_{2k\ i} = \alpha_{4k-1} x_{2k\ i}$$

$$y_{2k+1\ i} = \alpha_{4k}x_{2k\ i} + \alpha_{4k+1}x_{2k+1\ i} + \alpha_{4k+2}x_{2k+2\ i}.$$

Let A be a matrix with $a_{ij} = \alpha_n$ for $i+j-1 = n$. Then A is a Hilbert-Schmidt operator. Since for all i, k in $\mathbb{N}$,

$$\begin{aligned} y_{1i} &= \alpha_1 x_{1i} + \alpha_2 x_{2i} \\ y_{2k\ i} &= \alpha_{4k-1}x_{2k\ i} \\ y_{2k+1\ i} &= \alpha_{4k}x_{2k\ i} + \alpha_{4k+1}x_{2k+1\ i} + \alpha_{4k+2}x_{2k+2\ i}, \\ AX &= Y. \end{aligned}$$

□

Theorem 2.2. *Let n be a fixed natural number ($n \geq 2$). Let Alg$\mathcal{L}$ be the tridiagonal algebra on a separable complex Hilbert space $\mathcal{H}$ and let $X_i = (x_{jk}^{(i)})$ and $Y_i = (y_{jk}^{(i)})$ be operators acting on $\mathcal{H}$ for $i = 1, 2, \cdots, n$. Then the following are equivalent:*

(1) *There exists a Hilbert-Schmidt operator $A = (a_{jk})$ in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for all $i = 1, 2, \cdots, n$.*

(2) *There is a bounded sequence $\{\alpha_m\}$ in $\mathbb{C}$ such that $\sum_{m=1}^{\infty} |\alpha_m|^2 < \infty$ and*

$$\begin{aligned} y_{1j}^{(i)} &= \alpha_1 x_{1j}^{(i)} + \alpha_2 x_{2j}^{(i)} \\ y_{2k\ j}^{(i)} &= \alpha_{4k-1}x_{2k\ j}^{(i)} \\ y_{2k+1\ j}^{(i)} &= \alpha_{4k}x_{2k\ j}^{(i)} + \alpha_{4k+1}x_{2k+1\ j}^{(i)} + \alpha_{4k+2}x_{2k+2\ j}^{(i)} \end{aligned}$$
for $k \in \mathbb{N}$ and $i = 1, 2, \cdots, n$.

Proof. Suppose that A is a Hilbert-Schmidt operator $A = (a_{jk})$ in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for all $i = 1, 2, \cdots, n$. Let $\alpha_m = a_{jk}$ for $m = j+k-1$ and $\{e_m\}$ is the standard orthonormal basis for $\mathcal{H}$. Since A is Hilbert-Schmidt, $\sum_m \|Ae_m\|^2 < \infty$. Hence

$$\begin{aligned} \sum_m \|Ae_m\|^2 &= \sum_m \sum_j |\langle Ae_m, e_j\rangle|^2 \\ &= \sum_{l=1}^{\infty} \langle Ae_{2l-1}, e_{2l-1}\rangle + \sum_{l=1}^{\infty} \langle Ae_{2l}, (e_{2l-1} + e_{2l} + e_{2l+1})\rangle \\ &= \sum_{l=1}^{\infty} |\alpha_{4l-3}|^2 + \sum_{l=1}^{\infty} (|\alpha_{4l-2}|^2 + |\alpha_{4l-1}|^2 + |\alpha_{4l}|^2) \\ &= \sum_{l=1}^{\infty} |\alpha_l|^2 < \infty. \end{aligned}$$

Since $AX_i = Y_i$ for all $i = 1, 2, \cdots, n$,

$$\begin{aligned} y_{1j}^{(i)} &= \alpha_1 x_{1j}^{(i)} + \alpha_2 x_{2j}^{(i)} \\ y_{2k\ j}^{(i)} &= \alpha_{4k-1} x_{2k\ j}^{(i)} \\ y_{2k+1\ j}^{(i)} &= \alpha_{4k} x_{2k\ j}^{(i)} + \alpha_{4k+1} x_{2k+1\ j}^{(i)} + \alpha_{4k+2} x_{2k+2\ j}^{(i)} \text{ for } j,k \in \mathbb{N}, \end{aligned}$$

for all $i = 1, 2, \cdots, n$.

Conversely, assume that there is a bounded sequence $\{\alpha_m\}$ in $\mathbb{C}$ such that $\sum_{m=1}^{\infty} |\alpha_m|^2 < \infty$ and

$$\begin{aligned} y_{1j}^{(i)} &= \alpha_1 x_{1j}^{(i)} + \alpha_2 x_{2j}^{(i)} \\ y_{2k\ j}^{(i)} &= \alpha_{4k-1} x_{2k\ j}^{(i)} \\ y_{2k+1\ j}^{(i)} &= \alpha_{4k} x_{2k\ j}^{(i)} + \alpha_{4k+1} x_{2k+1\ j}^{(i)} + \alpha_{4k+2} x_{2k+2\ j}^{(i)} \text{ for } j,k \in \mathbb{N}, \end{aligned}$$

for all $i = 1, 2, \cdots, n$. Let A be a matrix with $a_{jk} = \alpha_n$ for $j + k - 1 = n$. Then A is a Hilbert-Schmidt operator. Since

$$\begin{aligned} y_{j1}^{(i)} &= \alpha_1 x_{j1}^{(i)} + \alpha_2 x_{j2}^{(i)} \\ y_{j\ 2k}^{(i)} &= \alpha_{4k-1} x_{j\ 2k}^{(i)} \\ y_{j\ 2k+1}^{(i)} &= \alpha_{4k} x_{j\ 2k}^{(i)} + \alpha_{4k+1} x_{j\ 2k+1}^{(i)} + \alpha_{4k+2} x_{j\ 2k+2}^{(i)} \text{ for } k \in \mathbb{N}, \end{aligned}$$

for all $i = 1, 2, \cdots, n$, $AX_i = Y_i$. □

By the similar way with the above, we have the following.

Theorem 2.3. *Let $Alg\mathcal{L}$ be the tridiagonal algebra on a separable complex Hilbert space $\mathcal{H}$ and let $X_i = (x_{jk}^{(i)})$ and $Y_i = (y_{jk}^{(i)})$ be operators acting on $\mathcal{H}$ for $i = 1, 2, \cdots$. Then the following are equivalent:*

(1) *There exists a Hilbert-Schmidt operator $A = (a_{jk})$ in $Alg\mathcal{L}$ such that $AX_i = Y_i$ for all $i = 1, 2, \cdots$.*

(2) *There is a sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that $\sum_{n1}^{\infty} |\alpha_n|^2 < \infty$ and for all i, j, k in $\mathbb{N}$,*

$$\begin{aligned} y_{1j}^{(i)} &= \alpha_1 x_{1j}^{(i)} + \alpha_2 x_{2j}^{(i)} \\ y_{2k\ j}^{(i)} &= \alpha_{4k-1} x_{2k\ j}^{(i)} \\ y_{2k+1\ j}^{(i)} &= \alpha_{4k} x_{2k\ j}^{(i)} + \alpha_{4k+1} x_{2k+1\ j}^{(i)} + \alpha_{4k+2} x_{2k+2\ j}^{(i)}. \end{aligned}$$

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