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POSITIVE SOLUTIONS OF SELF-ADJOINT BOUNDARY VALUE PROBLEM WITH INTEGRAL BOUNDARY CONDITIONS AT RESONANCE*

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ABSTRACT. In this paper, we study the self-adjoint second order boundary value problem with integral boundary conditions:

$$(p(t)x'(t))' + f(t, x(t)) = 0,\ t \in (0,1),$$
$$x'(0) = 0,\ x(1) = \int_0^1 x(s)g(s)ds.$$

A new result on the existence of positive solutions is obtained. The interesting points are: the first, we employ a new tool–the recent Leggett-Williams norm-type theorem for coincidences; the second, the boundary value problem is involved in integral condition; the third, the solutions obtained are positive.

1. INTRODUCTION

This paper is concerned with the existence of positive solutions to the following boundary value problem at resonance:

$$(p(t)x'(t))' + f(t, x(t)) = 0,\ t \in (0,1), \tag{1.1}$$

$$x'(0) = 0,\ x(1) = \int_0^1 x(s)g(s)ds, \tag{1.2}$$

Throughout, we assume that

(A1) $p \in C[0,1] \cap C^1(0,1),\ p(t) > t(2-t)$ on $[0,1]$,

$$\int_0^1 \frac{1}{p(t)}dt \le e \text{ and } \int_0^1 \int_s^1 \frac{\tau}{p(\tau)}d\tau g(s)ds > 0;$$

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(A2) $g \in L^1[0,1]$ with $g(t) \geq 0$ on $[0,1]$,

$$\int_0^1 g(s)ds = 1, \quad g(t) \geq \frac{\int_0^1 \left(\int_s^1 \frac{\tau}{p(\tau)} d\tau\right) g(s)ds}{p(t)\int_0^1 \frac{\tau}{p(\tau)} d\tau \int_t^1 \frac{d\tau}{p(\tau)}}.$$

Although the existing literature on solutions of BVPs is quite wide, to the best of our knowledge, only few papers deal with the existence of positive solutions to multi-point BVPs at resonance. In particular, there has no work done for the BVP (1.1)-(1.2). Moreover, Our main approach is different from the ones existed and our main ingredient is the Leggett-Williams norm-type theorem for coincidences obtained by O'Regan and Zima [4].

2. Related Lemmas

For the convenience of the reader, we review some standard facts on Fredholm operators and cones in Banach spaces. Let X, Y be real Banach spaces. Consider a linear mapping $L : \mathrm{dom}L \subset X \to Y$ and a nonlinear operator $N : X \to Y$. Assume that

1° L is a Fredholm operator of index zero, i.e. $\mathrm{Im}L$ is closed and $\mathrm{dimKer}L = \mathrm{codim}\ \mathrm{Im}L < \infty$.

The assumption 1° implies that there exist continuous projections $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Ker}Q = \mathrm{Im}L$. Moreover, since $\dim \mathrm{Im}Q = \mathrm{codim}\ \mathrm{Im}L$, there exists an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$. Denote by L_p the restriction of L to $\mathrm{Ker}P \cap \mathrm{dom}L$. Clearly, L_p is an isomorphism from $\mathrm{Ker}P \cap \mathrm{dom}L$ to $\mathrm{Im}L$, we denote its inverse by $K_p : \mathrm{Im}L \to \mathrm{Ker}P \cap \mathrm{dom}L$. It is known (see [3]) that the coincidence equation $Lx = Nx$ is equivalent to

$$x = (P + JQN)x + K_P(I - Q)Nx.$$

Let C be a cone in X such that

(i) $\mu x \in C$ for all $x \in C$ and $\mu \geq 0$,

(ii) $x, -x \in C$ implies $x = \theta$.

It is well known that C induces a partial order in X by

$$x \preceq y \text{ if and only if } y - x \in C.$$

The following property is valid for every cone in a Banach space.

Lemma 2.1 ([3]). *Let C be a cone in X. Then for every $u \in C \setminus \{0\}$ there exists a positive number $\sigma(u)$ such that*

$$||x+u|| \geq \sigma(u)||u|| \text{ for all } x \in C.$$

Let $\gamma : X \to C$ be a retraction, that is, a continuous mapping such that $\gamma(x) = x$ for all $x \in C$. Set

$$\Psi := P + JQN + K_p(I-Q)N \quad \text{and} \quad \Psi_\gamma := \Psi \circ \gamma.$$

We make use of the following result due to O'Regan and Zima.

Theorem 2.1 ([4]). *Let C be a cone in X and let Ω_1, Ω_2 be open bounded subsets of X with $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Assume that 1° and the following conditions hold.*

- 2° *$QN : X \to Y$ is continuous and bounded and $K_p(I-Q)N : X \to X$ is compact on every bounded subset of X,*
- 3° *$Lx \neq \lambda Nx$ for all $x \in C \cap \partial\Omega_2 \cap \mathrm{Im}L$ and $\lambda \in (0,1)$,*
- 4° *γ maps subsets of $\overline{\Omega}_2$ into bounded subsets of C,*
- 5° $\deg\{[I-(P+JQN)\gamma]|_{\mathrm{Ker}L}, \mathrm{Ker}L \cap \Omega_2, 0\} \neq 0$,
- 6° *there exists $u_0 \in C \setminus \{0\}$ such that $||x|| \leq \sigma(u_0)||\Psi x||$ for $x \in C(u_0) \cap \partial\Omega_1$, where $C(u_0) = \{x \in C : \mu u_0 \preceq x \text{ for some } \mu > 0\}$ and $\sigma(u_0)$ such that $||x+u_0|| \geq \sigma(u_0)||x||$ for every $x \in C$,*
- 7° *$(P+JQN)\gamma(\partial\Omega_2) \subset C$,*
- 8° *$\Psi_\gamma(\overline{\Omega}_2 \setminus \Omega_1) \subset C$.*

Then the equation $Lx = Nx$ has a solution in the set $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

For simplicity of notation, we set

$$\omega := \int_0^1 \left(\int_s^1 \frac{\tau}{p(\tau)} d\tau \right) g(s) ds,$$

$$l(s) := \int_s^1 \left(\int_\tau^1 \frac{1}{p(r)} dr \right) g(\tau) d\tau + \int_s^1 \frac{1}{p(\tau)} d\tau \int_0^s g(\tau) d\tau, \tag{2.1}$$

$$k(t,s) := \begin{cases} \displaystyle\int_t^1 \frac{1}{p(\tau)} d\tau - \int_s^1 \frac{\tau}{p(\tau)} d\tau, & 0 \leq s \leq t \leq 1, \\ \displaystyle\int_s^1 \frac{1-\tau}{p(\tau)} d\tau, & 0 \leq t < s \leq 1, \end{cases} \tag{2.2}$$

and

(2.3)

$$G(t,s)=\begin{cases} \frac{1}{\omega}l(s)\Big(1-\int_t^1\frac{\tau}{p(\tau)}d\tau+\int_0^1\frac{\tau^2}{p(\tau)}d\tau\Big) \\ -\int_s^1\frac{\tau}{p(\tau)}d\tau+\int_t^1\frac{1}{p(\tau)}d\tau, & 0\le s<t\le 1,\\ \frac{1}{\omega}l(s)\Big(1-\int_t^1\frac{\tau}{p(\tau)}d\tau+\int_0^1\frac{\tau^2}{p(\tau)}d\tau\Big)+\int_s^1\frac{1-\tau}{p(\tau)}d\tau, & 0\le t\le s\le 1.\end{cases}$$

In view of (A1) and (A2), one gets that $G(t,s)\ge 0$ for $t,s\in[0,1]$. Note that

$$1-\frac{\kappa}{\omega}l(s)\ge 0,\ \ s\in[0,1] \tag{2.4}$$

for every

$$\kappa\in\left(0,\ \frac{\int_0^1\Big(\int_s^1\frac{\tau}{p(\tau)}d\tau\Big)g(s)ds}{\int_0^1\Big(\int_s^1\frac{1}{p(\tau)}d\tau\Big)g(s)ds}\right].$$

Set

$$\kappa:=\min\left\{\frac{\int_0^1\Big(\int_s^1\frac{\tau}{p(\tau)}d\tau\Big)g(s)ds}{\int_0^1\Big(\int_s^1\frac{1}{p(\tau)}d\tau\Big)g(s)ds},\ \frac{1}{\max\limits_{t,s\in[0,1]}G(t,s)}\right\}.$$

Note that $\kappa\le 1$.

3. MAIN RESULT

In order to prove the existence result, we present here a definition.

Definition 3.1. We say that the function $f:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ satisfies the *Carathéodory* with respect to $L^1[0,\infty)$, if the following conditions hold:

- (A1) for each $u\in\mathbb{R}$, the mapping $t\mapsto f(t,u)$ is Lebesgue measurable,
- (A2) for a.e. $t\in[0,\infty)$, the mapping $u\mapsto f(t,u)$ is continuous on $\mathbb{R}$,
- (A3) for each $r>0$, there exists $\alpha_r\in L^1[0,\infty)$ such that for a.e. $t\in[0,\infty)$ and every u such that $|u|\le r$, we have $|f(t,u)|\le\alpha_r(t)$.

Now, we state our result on the existence of positive solutions for the BVP (1.1)-(1.2).

Theorem 3.1. *Assume that*

(H1) $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ *satisfies the* L^1*-Carathéodory conditions,* (H2) *there exists* $B > 0$ *and* $\kappa \in (0,1]$ *such that* $f(t,B) < 0$ *for* $t \in [0,1]$ *and*

$$-\kappa x < f(t,x) \quad \text{for} \quad t \in [0,1], \ x \in [0,B],$$

(H3) *there exist* $b \in (0,B)$, $t_0 \in [0,1]$, $\rho \in (0,1]$, $\delta \in (0,1)$ *and* $q \in L^1[0,1]$, $q(t) \geq 0$ *on* $t \in [0,1]$, $h \in C([0,1] \times (0,b], \mathbb{R}^+)$ *such that* $f(t,x) \geq q(t)h(t,x)$ *for* $t \in [0,1]$ *and* $x \in (0,b]$. *For each* $t \in [0,1]$, $\frac{h(t,x)}{x^\rho}$ *is non-increasing on* $x \in (0,b]$ *with*

$$\int_0^1 G(t_0,s)q(s)\frac{h(s,b)}{b}ds \geq \frac{1-\delta}{\delta^\rho}. \tag{3.1}$$

Then the BVP (1.1)-(1.2) *has at least one positive solution on* $[0,1]$.

Proof. Consider the Banach spaces $X = C[0,1]$ with the sup norm $\|x\| = \max_{t\in[0,1]} |x(t)|$ and $Y = L^1[0,1]$ with the usual integral norm $\|y\| = \int_0^1 |y(t)|dt$.

Define $L : \text{dom}L \subset X \to Y$ and $N : X \to Y$ with

$$\text{dom}L = \Big\{x \in X : x'(0) = 0, \ x(1) = \int_0^1 x(s)g(s)ds, \\ x', \ px' \in AC[0,1], \ (px')' \in L^1[0,1]\Big\}$$

by $Lx(t) = -(p(t)x'(t))'$ and $Nx(t) = f(t,x(t))$, $t \in [0,1]$, respectively. Then

$$\text{Ker}L = \big\{x \in \text{dom}L : x(t) \equiv c \text{ on } [0,1]\big\}$$

and

$$\text{Im}L = \Big\{y \in Y : \int_0^1 l(s)y(s)ds = 0\Big\}.$$

Next, define the projections $P : X \to X$ by

$$(Px)(t) = \int_0^1 x(s)ds$$

and $Q : Y \to Y$ by

$$(Qy)(t) = \frac{1}{\omega}\int_0^1 l(s)y(s)ds. \tag{3.2}$$

Clearly, $\text{Im}P = \text{Ker}L$ and $\text{Ker}Q = \text{Im}L$. So $\dim \text{Ker}L = 1 = \dim \text{Im}Q = \text{codim Im}L$. Notice that $\text{Im}L$ is closed, L is Fredholm operator of index zero, i.e. 1° holds.

Note that for $y \in \text{Im}L$ the inverse $K_p : \text{Im}L \to \text{dom}L \cap \text{Ker}P$ of L_p is given by

$$(K_p y)(t) = \int_0^1 k(t,s)y(s)ds.$$

Since f satisfies the L^1-Carathéodory conditions, 2° holds.

Consider the cone

$$C = \{x \in X : x(t) \geq 0 \text{ on } [0,1]\}.$$

Let

$$\Omega_1 = \{x \in X : \delta||x|| < |x(t)| < b \text{ on } [0,1]\}$$

and

$$\Omega_2 = \{x \in X : ||x|| < B\}.$$

Clearly, Ω_1 and Ω_2 are bounded and open sets and

$$\overline{\Omega}_1 = \{x \in X : \delta||x|| \leq |x(t)| \leq b \text{ on } [0,1]\} \subset \Omega_2$$

(see [4]). Moreover, $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Let $J = I$ and $(\gamma x)(t) = |x(t)|$ for $x \in X$. Then γ is a retraction and maps subsets of $\overline{\Omega}_2$ into bounded subsets of C, which means that 4° holds.

In order to prove 3°, suppose that there exist $x_0 \in \partial\Omega_2 \cap C \cap \text{dom} L$ and $\lambda_0 \in (0,1)$ such that $Lx_0 = \lambda_0 N x_0$, then $(p(t)x'(t))' + \lambda_0 f(t, x_0(t)) = 0$ for all $t \in [0,1]$. Let $t_1 \in [0,1)$ such that $x_0(t_1) = B$, $x_0'(t_1) = 0$, then

$$-\lambda_0 f(t_1, B) = (p(t_1)x_0'(t_1))' = p'(t_1)x_0'(t_1) + p(t_1)x_0''(t_1) = p(t_1)x_0''(t_1) \leq 0,$$

which contradicts (H2).

To prove 5°, consider $x \in \text{Ker} L \cap \overline{\Omega}_2$. Then $x(t) \equiv c$ on $[0,1]$. Let

$$H(c,\lambda) = c - \lambda|c| - \frac{\lambda}{\omega}\int_0^1 l(s)f(s,|c|)ds$$

for $c \in [-B,B]$ and $\lambda \in [0,1]$. It is easy to show that $0 = H(c,\lambda)$ implies $c \geq 0$. Suppose $0 = H(B,\lambda)$ for some $\lambda \in (0,1]$. Then

$$0 \leq B(1-\lambda) = \frac{\lambda}{\omega}\int_0^1 l(s)f(s,B)ds,$$

which contradicts (H2). In addition, if $\lambda = 0$, then $B = 0$, which is impossible. Thus, $H(x,\lambda) \neq 0$ for $x \in \text{Ker} L \cap \partial\Omega_2$, $\lambda \in [0,1]$. Therefore,

$$\deg\{H(\cdot,1), \text{Ker} L \cap \Omega_2, \theta\} = \deg\{H(\cdot,0), \text{Ker} L \cap \Omega_2, \theta\}.$$

However,

$$\deg\{H(\cdot,0), \text{Ker} L \cap \Omega_2, \theta\} = \deg\{I, \text{Ker} L \cap \Omega_2, \theta\} = 1.$$

Then

$$\deg\{[I - (P + JQN)\gamma]_{\text{Ker} L}, \text{Ker} L \cap \Omega_2, \theta\} = \deg\{H(\cdot,1), \text{Ker} L \cap \Omega_2, \theta\} \neq 0.$$

Next, we prove 8°. Let $x \in \overline{\Omega}_2 \setminus \Omega_1$ and $t \in [0,1]$,

$$\begin{aligned}\Psi_\gamma x &= \int_0^1 |x(s)|ds + \frac{1}{\omega}\int_0^1 l(s)f(s,|x(s)|)ds \\ &\quad + \int_0^1 k(t,s)[f(s,|x(s)|) - \frac{1}{\omega}\int_0^1 l(\tau)f(\tau,|x(\tau)|)d\tau]ds \\ &= \int_0^1 |x(s)|ds + \int_0^1 G(t,s)f(s,|x(s)|)ds \\ &\geq \int_0^1 (1-\kappa G(t,s))|x(s)|ds \geq 0.\end{aligned}$$

Hence, $\Psi_\gamma(\overline{\Omega}_2 \setminus \Omega_1) \subset C$, i.e. 8° holds.

Since for $x \in \partial\Omega_2$,

$$\begin{aligned}(P+JQN)\gamma x &= \int_0^1 |x(s)|ds + \frac{1}{\omega}\int_0^1 l(s)f(s,|x(s)|)ds \\ &\geq \int_0^1 (1-\frac{\kappa}{\omega}l(s))|x(s)|ds \geq 0.\end{aligned}$$

Thus, $(P+JQN)\gamma x \subset C$ for $x \in \partial\Omega_2$, 7° holds.

It remains to verify 6°. Let $u_0(t) \equiv 1$ on $[0,1]$. Then $u_0 \in C \setminus \{0\}$, $C(u_0) = \{x \in C : x(t) > 0 \text{ on } [0,1]\}$ and we can take $\sigma(u_0) = 1$. Let $x \in C(u_0) \cap \partial\Omega_1$. Then $x(t) > 0$ on $[0,1]$, $0 < ||x|| \leq b$ and $x(t) \geq \delta||x||$ on $[0,1]$. By (H3), for every $x \in C(u_0) \cap \partial\Omega_1$,

$$\begin{aligned}(\Psi x)(t_0) &= \int_0^1 x(s)ds + \int_0^1 G(t_0,s)f(s,x(s))ds \\ &\geq \delta||x|| + \int_0^1 G(t_0,s)q(s)h(s,x(s))ds \\ &= \delta||x|| + \int_0^1 G(t_0,s)q(s)\frac{h(s,x(s))}{x^\rho(s)}x^\rho(s)ds \\ &\geq \delta||x|| + \delta^\rho||x||^\rho \int_0^1 G(t_0,s)q(s)\frac{h(s,b)}{b||x||^{\rho-1}}ds \\ &\geq ||x||.\end{aligned}$$

Thus, $||x|| \leq \sigma(u_0)||\Psi x||$ for all $x \in C(u_0) \cap \partial\Omega_1$.

By Theorem 2.1, the BVP (1.1)-(1.2) has a positive solution x^* on $[0,1]$ with $b \leq ||x^*|| \leq B$. This completes the proof of Theorem 3.1. □

Remark 3.1. Note that with the projection $P(x) = x(0)$, condition 7° and 8° of Theorem 2.1 are no longer satisfied.

Example 3.1. Consider the following BVP

$$
(3.3) \qquad \begin{cases} (e^t(x'(t)))' + f(t,x(t)) = 0, \ t \in (0,1), \\ x'(0) = 0, \ x(1) = \displaystyle\int_0^1 2tx(t)dt. \end{cases}
$$

Corresponding to BVP (1.1), we have $p(t) = e^t$, $g(t) = 2t$, $f(t,x) = (t - t^2 + 1)(x^2 - 4x + 3)\sqrt{(x-2)^2+1}$. When $\kappa = \frac{2(3e-8)}{2e-5}$, we may choose $B = \frac{6}{5}$, $b = \frac{1}{2}$, $t_0 = 0$, $\rho = 1$, $\delta = \frac{1}{2}$, $g(t) = t - t^2 + 1$ and $h(t,x) = \sqrt{(x-2)^2+1}$. We can check that conditions (H1)-(H3) are all satisfied. Therefore, BVP (3.3) has at least one positive solution on $[0,1]$.

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