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A Study on the Stochastic Finite Element Method for Dynamic Problem of Nonlinear Continuum

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Abstract

The main idea of this paper introduce stochastic structural parameters and random dynamic excitation directly into the dynamic functional variational formulations, and developed the nonlinear dynamic analysis of a stochastic variational principle and the corresponding stochastic finite element method via the weighted residual method and the small parameter perturbation technique. An interpolation method was adopted, which is based on representing the random field in terms of an interpolation rule involving a set of deterministic shape functions. Direct integration Wilson-θ Method was adopted to solve finite element equations. Numerical examples are compared with Monte-Carlo simulation method to show that the approaches proposed herein are accurate and effective for the nonlinear dynamic analysis of structures with random parameters.
Keywords: stochastic variational principle, nonlinear response, stochastic finite element method, perturbation technique

1 Introduction

The analysis of structural systems with uncertain properties modeled by random fields has been the subject of extensive research in the past two decades, over these years , the majority of the research work has focused on developing various stochastic finite element method (SFEM) for the numerical solution of the stochastic partial differential equations involved in such problem, i.e. stochastic spectral approaches(Ghanem and Spanos 2003), a variety of Monte-Carlo simulation techniques (Ditlevsen and Madsen 1996)(Shreider 1966) as well as many numerical realizations of the perturbation technique (Kleiber and Hien 1992).

Since nonlinear analysis of structures with stochastic parameters is of considerable importance as many of the buildings, offshore structures, ships, etc. Some dynamic excitations that are excited by nature's actions which exhibit randomly fluctuating character cannot be treated as deterministic systems. It is essential to consider nonlinearity arising from geometrical and/or material properties in random structure dynamics.

A classical perturbation-based stochastic method is applied only to linear elastic systems. However, the linear model may not be always enough to predict the response of realities, Therefore, material and/or nonlinearity must be included in the analyses. Liu and Kiureghian (Liu and Kiureghian 1988), Haldar and Zhou (Haldar and Zhou 1992) all studied the solutions of two-dimensional nonlinear SFEM via the method of partial differentiation. Hisada and Noguchi (Hisada and Noguchi 1989) proposed perturbation SFEM to solve material nonlinear problems. Liu, et al. (Liu, et al. 1986) apply the First-order perturbation technique to solve the nonlinear dynamic response of random structure. Zhao L. (Zhao and Chen 2000) used minimum potential energy principle and perturbation technique to solve nonlinear dynamic problem. Ioannis D. addressed the perturbation-based stochastic finite element analysis to study deformation processes of inelastic solids (Loannis and Zhan 2006).

In the present paper, the primary focus is placed on nonlinear stochastic variational principle of dynamic analysis and incremental SFEM for random structures under dynamic excitation. The weighted residual method was considered based on the primary equations of the incremental problems and then more reasonable formulations of incremental SFEM are developed to use for solving nonlinear problems of random structural dynamics. Nonlinearities due to material and geometrical effects have been included in this paper.

2 Nonlinear stochastic variational formula of random structural dynamics

In update Lagrangian descriptions (Bathe 1966) the last known configuration was adopted as the reference state, the region taken up by the body at this instant will be denoted by ${}^t\Omega$, and the primary equations describing the incremental problem (Kleiber and Hien 1992) may be presented in the time interval $[t, t+\Delta t]$ as follows:

①incremental equations of motion

$$\Delta_t S_{ij,j}^{(I)} + \rho\Delta_t f_i = \rho\Delta_t \ddot{u}_i \tag{1}$$

②stress-type boundary conditions

$$\Delta_t S_{ij}^{(I)} n_j = \Delta_t \hat{t}_i \qquad x_k \in \partial^t S_\delta \tag{2}$$

③linearized incremental strain-displacement relations

$$\Delta_t \varepsilon_{ij} \doteq \frac{1}{2}\left(\Delta_t u_{i,j} + \Delta_t u_{j,i}\right) \tag{3}$$

④ linearized incremental constitutive equation

$$\Delta_t S_{ij}^{(II)} = {}_t C_{ijkl} \Delta_t \varepsilon_{ij} \tag{4}$$

⑤ kinematic boundary conditions

$$\Delta u_i = \Delta \bar{u}_i \qquad x_k \in \partial\Omega_u^t \tag{5}$$

in above equation, the constitutive modulus ${}_t C_{ijkl}$ are assumed to possibly be functions of 'initial' fields such as stresses and/or internal variables; $\Delta_t S_{ij}^{(I)}$ and $\Delta_t S_{ij}^{(II)}$ are the first and second Piola-Kirchhoff stress tensor based on the current configuration, respectively, and are related by the equation:

$$\Delta_t S_{ij}^{(I)} = \Delta_t S_{ij}^{(II)} + \Delta_t u_{i,k} {}^t\tau_{kj} \tag{6}$$

where, ${}^t\tau_{ij}$ is Cauchy stress tensor in the configuration at time t.

Let us now illustrate the approach by considering the weighted residual method. We assumed that the kinematics conditions, i.e. Eqs.(3)—(5) are satisfied, and then relaxing the kinetic conditions, i.e. Eqs(1)—(2), choosing $\delta(\Delta_t u_i)$ as the weighted function, so the weighted residual formulation is:

$$\int_{\Omega}\left(\Delta_t S_{ij,j}^{(I)} + \rho\Delta_t f_i - \rho\Delta_t \ddot{u}_i\right)\delta(\Delta_t u_i)d\Omega + \int_{S_\sigma}\left(\Delta_t \hat{t}_i - \Delta_t S_{ij}^{(I)} n_j\right)\delta(\Delta_t u_i)dS = 0 \tag{7}$$

Integral by parts on the first part of left hand side has:

$$\int_{\Omega}\Delta_t S_{ij,j}^{(I)}\delta(\Delta_t u_i)d\Omega = \int_{\Omega}\left[\Delta_t S_{ij}^{(I)}\delta(\Delta_t u_i)\right]_{,j} d\Omega - \int_{\Omega}\Delta_t S_{ij}^{(I)}\delta(\Delta_t u_{i,j})d\Omega \tag{8}$$

by employing Gauss-Ostrogradski theorem:

$$\int_{\Omega} A_{,j} d\Omega = \int_{S} A n_j dS \tag{9}$$

the Eq.(8) becomes:

$$\int_{\Omega}\Delta_t S_{ij,j}^{(I)}\delta(\Delta_t u_i)d\Omega = \int_{S}\Delta_t S_{ij}^{(I)}\delta(\Delta_t u_i)n_j dS - \int_{\Omega}\Delta_t S_{ij}^{(I)}\delta(\Delta_t u_{i,j})d\Omega \tag{10}$$

where, $S = S_u \cup S_\sigma$, from Eq.(5) and (4) we can deduce:

$$\delta(\Delta u_i) = \delta(\Delta \bar{u}_i) = 0 \qquad x_k \in S_u^t \tag{11}$$

$$\delta(\Delta_t \varepsilon_{ij}) \doteq \delta\left(\frac{1}{2}(\Delta_t u_{i,j} + \Delta_t u_{j,i})\right) = \delta(\Delta_t u_{i,j}) \tag{12}$$

Substitute Eqs.(11) and (12) into Eq.(10), we have:

$$\int_{\Omega}\Delta_t S_{ij,j}^{(I)}\delta(\Delta_t u_i)d\Omega = \int_{S_\sigma}\Delta_t S_{ij}^{(I)}\delta(\Delta_t u_i)n_j dS_\sigma - \int_{\Omega}\Delta_t S_{ij}^{(I)}\delta(\Delta_t \varepsilon_{ij})d\Omega \tag{13}$$

by introducing Eq.(13) into Eq.(7), the weighted residual formulation of (7) becomes:

$$\int_{\Omega}\Delta_t S_{ij}^{(I)}\delta(\Delta_t \varepsilon_{ij})d\Omega + \int_{S_\sigma}\left(\rho\Delta_t \ddot{u}_i - \rho\Delta_t f_i - \Delta_t \hat{t}_i\right)\delta(\Delta_t u_i)dS_\sigma = 0 \tag{14}$$

expressing first Piola-Kirchhoff stress tensor in terms of second Piola-Kirchhoff stress tensor by Eq.(6) and employing Eq.(4), we have:

$$\begin{aligned}&\int_{\Omega} {}_tC_{ijkl}\Delta_t \varepsilon_{ij}\delta(\Delta_t \varepsilon_{kl})d\Omega + \int_{\Omega} {}^t\tau_{kl}\Delta_t u_{i,k}\delta(\Delta_t u_{i,l})d\Omega + \int_{\Omega}(\rho\Delta_t \ddot{u}_i)\delta(\Delta_t u_i)d\Omega \\ &= \int_{\Omega}\rho\Delta_t f_i\delta(\Delta_t u_i)d\Omega + \int_{S_\sigma}\Delta_t \hat{t}_i\delta(\Delta_t u_i)dS_\sigma\end{aligned} \tag{15}$$

where, ${}_tC_{ijkl}$ is tangent stress-strain tensor at time t.

Furthermore, let us assumed the damping effects is the nature of the body and is proportional to the velocities of the body particles, so Eq.(15) becomes:

$$\begin{aligned}&\int_{\Omega}\rho\Delta_t \ddot{u}_i\delta(\Delta_t u_i)d\Omega + \int_{\Omega}\alpha\rho\Delta_t \dot{u}_i\delta(\Delta_t u_i)d\Omega + \int_{\Omega} {}_tC_{ijkl}\Delta_t \varepsilon_{ij}\delta(\Delta_t \varepsilon_{kl})d\Omega \\ &= \int_{\Omega}\rho\Delta_t f_i\delta(\Delta_t u_i)d\Omega + \int_{S_\sigma}\Delta_t \hat{t}_i\delta(\Delta_t u_i)dS_\sigma - \int_{\Omega} {}^t\tau_{kl}\Delta_t u_{i,k}\delta(\Delta_t u_{i,l})d\Omega\end{aligned} \tag{16}$$

where, the constant α is proportional damping factor.

Now let us consider a nonlinear random structure under stochastic dynamic load and define

$$\mathbf{b}(x_k) = \{b_1(x_k) \quad b_2(x_k) \quad \cdots \quad b_R(x_k)\}, k = 1,2,3 \tag{17}$$

as a R-dimensional discretiziation random vector which can represent randomness in the cross-sectional area and length of truss, beam and frame members, thickness of plate and shell elements, Young's modulus and mass density of the material, etc., as well as time-invariant randomness in the external dynamic load. In addition, the variance of random field for the structural system should be very small.

$$\Delta u_i[b(x_k);x_k] = \Delta u_i[b^0(x_k);x_k] + \varepsilon \Delta u_i^{;r}[b^0(x_k);x_k]\Delta b_r(x_k) + \frac{1}{2}\varepsilon^2 \Delta u_i^{;rs}[b^0(x_k);x_k]\Delta b_r(x_k)\Delta b_s(x_k) \tag{18}$$

where

$$\varepsilon \Delta b_r(x_k) = \delta b(x_k) = \varepsilon[b_r(x_k) - b_r^0(x_k)] \tag{19}$$

$$\varepsilon^2 \Delta b_r(x_k)\Delta b_s(x_k) = \delta b_r(x_k)\delta b_s(x_k) = \varepsilon^2[b_r(x_k) - b_r^0(x_k)][b_s(x_k) - b_s^0(x_k)] \tag{20}$$

for stochastic functions $C_{ijkl}, \Delta e_{ij}, \tau_{ij}, \Delta f_i, \Delta p_i, \xi, \rho$, we can also obtain the second-order expanding forms similar to Eq. (18). By introducing these expanding forms into the Eq.(16), we obtain the following zeroth-, first- and second-order variational formulations via equating terms of equal orders for small parameter ε (for simplicity, the left subscript 't' is dropped):

Zeroth-order variational principle (ε^0 terms)

$$\int_\Omega C_{ijrs}^0 \Delta \varepsilon_{rs}^0 \delta(\Delta \varepsilon_{ij}) d\Omega + \int_\Omega \left(\rho^0 \Delta \ddot{u}_i^0 + \eta^0 \Delta \dot{u}_i^0\right) \delta(\Delta u_i) d\Omega = \int_\Omega \Delta f_i^0 \delta(\Delta u_i) d\Omega + \int_{S_f} \Delta \hat{t}_i^0 \delta(\Delta u_i) dS \tag{21}$$

First-order variational principle (ε^1 terms)

$$\begin{aligned} &\int_\Omega (C_{ijrs}^0 \Delta \varepsilon_{rs}^{;k} + C_{ijrs}^{;k} \Delta \varepsilon_{rs}^0) \delta(\Delta \varepsilon_{ij}) dV + \int_\Omega (\rho^0 \Delta \ddot{u}_i^{;k} + \rho^{;k} \Delta \ddot{u}_i^0) \delta(\Delta u_i) dV + \int_\Omega (\eta^0 \Delta \dot{u}_i^{;k} + \eta^{;k} \Delta \dot{u}_i^0) \delta(\Delta u_i) d\Omega \\ &= \int_\Omega \Delta f_i^{;k} \delta(\Delta u_i) dV + \int_{\partial\Omega_\sigma} \Delta p_i^{;k} \delta(\Delta u_i) dS \end{aligned} \tag{22}$$

Second-order variational principle (ε^2 terms)

$$\begin{aligned} &\int_\Omega (C_{ijrs}^0 \Delta e_{rs}^{;kl} + C_{ijrs}^{;kl} \Delta e_{rs}^0 + C_{ijrs}^{;k} \Delta e_{rs}^{;l}) \delta(\Delta e_{ij}) d\Omega + \int_\Omega (\rho^0 \Delta \ddot{u}_i^{;kl} + \rho^{;kl} \Delta \ddot{u}_i^0 + \rho^{;k} \Delta \ddot{u}_i^{;l}) \delta(\Delta u_i) d\Omega \\ &+ \int_\Omega (\xi^0 \Delta \dot{u}_i^{;kl} + \xi^{;kl} \Delta \dot{u}_i^0 + \xi^{;k} \Delta \dot{u}_i^{;l}) \delta(\Delta u_i) d\Omega = \int_\Omega \Delta f_i^{;kl} \delta(\Delta u_i) d\Omega + \int_{S_f} \Delta p_i^{;kl} \delta(\Delta u_i) dS \end{aligned} \tag{23}$$

3 Formulation of incremental stochastic finite element method for random structural dynamic analysis

In the framework of the FEM philosophy the fields $\mathbf{b}_r(x_k)$ have to be represented by a set of basic random variables. Thus, it is necessary to discretize $\mathbf{b}^0(x_k)$ by expressing them in terms of nodal values of the appropriate means and covariances. Some methods have been proposed, i.e. the local averages method (Vanmarcke 1983). In the present paper, we adopted interpolation method (Liu, Belytschko and Mani 1988) ,which is based on representing the random field in terms of an interpolation rule involving a set of deterministic shape functions and the random nodal values of the field:

$$\mathbf{b}^{(e)}(x_k) = \tilde{\boldsymbol{\varphi}}^{(e)}(x_k)\hat{\mathbf{b}}^{(e)}(x_k) \tag{24}$$

where,

$$\mathbf{b}^{(e)}(x_k) = \left\{ b_1^{(e)}(x_k) \quad b_2^{(e)}(x_k) \quad \cdots \quad b_R^{(e)}(x_k) \right\}^T \tag{25}$$

$$\hat{\mathbf{b}}^{(e)}(x_k) = \left\{ \hat{b}_1^{(e)}(x_k) \quad \hat{b}_2^{(e)}(x_k) \quad \cdots \quad \hat{b}_{\hat{n}}^{(e)}(x_k) \right\}^T \tag{26}$$

where, $\hat{n} = R \times \overline{n}$, $\overline{n}$ is the number of element node; $\mathbf{b}^{(e)}(x_k)$ is element random variable vector; $\hat{\mathbf{b}}^{(e)}(x_k)$ is element nodal random variable vector; $\tilde{\boldsymbol{\varphi}}^{(e)}$ is element random variable interpolation matrix:

$$\tilde{\boldsymbol{\varphi}}^{(e)} = \begin{bmatrix} \varphi_1^{(e)} & \mathbf{0} & \varphi_2^{(e)} & \mathbf{0} & \cdots & \\ \mathbf{0} & \varphi_1^{(e)} & \mathbf{0} & \varphi_2^{(e)} & \mathbf{0} & \cdots \\ & & \cdots & & & \end{bmatrix}_{R \times \hat{n}} \tag{27}$$

we can relate element random vector $\mathbf{b}^{(e)}(x_k)$ to total nodal random vector $\hat{\mathbf{b}}(x_k)$ by:

$$\mathbf{b}^{(e)}(x_k) = \tilde{\boldsymbol{\Phi}}^{(e)}(x_k)\hat{\mathbf{b}}(x_k) \tag{28}$$

where,

$$\tilde{\boldsymbol{\Phi}}^{(e)}(x_k) = \tilde{\boldsymbol{\varphi}}^{(e)}\mathbf{A}_{\hat{n}\times\hat{N}}^{(e)} \tag{29}$$

$$\hat{\mathbf{b}}(x_k) = \left\{ \hat{b}_1(x_k) \quad \hat{b}_2(x_k) \quad \cdots \quad \hat{b}_{\hat{N}}(x_k) \right\}^T \tag{30}$$

$$\hat{N} = R \times \overline{N}$$

$\hat{\mathbf{b}}(x_k)$ is total nodal random variable vector, $\overline{N}$ is the total number of model node. $\mathbf{A}_{\hat{n}\times\hat{N}}^{(e)}$ is element Boolean matrix.

The expectation and covariance of random fields $\mathbf{b}^{(e)}(x_k)$ can be written as in matrix form:

$$E\left[\mathbf{b}^{(e)}(x_k)\right] = \mathbf{b}^{(e)0}(x_k) = \tilde{\boldsymbol{\Phi}}^{(e)}(x_k)\hat{\mathbf{b}}^0 \tag{31}$$

$$Cov\left(b_r(x_k), b_s(x_k)\right) \equiv \mathbf{S}_b^{(e)} = \tilde{\boldsymbol{\Phi}}^{(e)}\hat{\mathbf{S}}_{\hat{\mathbf{b}}}\tilde{\boldsymbol{\Phi}}^{(e)T} \tag{32}$$

where

$$\hat{\mathbf{S}}_{\hat{\mathbf{b}}} = \begin{bmatrix} Var(\hat{b}_1) & Cov(\hat{b}_1, \hat{b}_2) & \cdots & Cov(\hat{b}_1, \hat{b}_{\hat{N}}) \\ & Var(\hat{b}_2) & \cdots & Cov(\hat{b}_2, \hat{b}_{\hat{N}}) \\ \vdots & \vdots & \ddots & \vdots \\ & sym & \cdots & Var(\hat{b}_{\hat{N}}) \end{bmatrix} \tag{33}$$

and

$$\Delta\mathbf{b}^{(e)}(x_k) = \tilde{\boldsymbol{\Phi}}^{(e)}(x_k)\Delta\hat{\mathbf{b}} \tag{34}$$

where

$$\Delta\hat{\mathbf{b}} = \hat{\mathbf{b}} - \hat{\mathbf{b}}^0 \tag{35}$$

where, $\hat{\mathbf{b}}^0$ and $\hat{\mathbf{S}}_{\hat{\mathbf{b}}}$ stand for mean value vector and the covariance matrix of the nodal random variable $\hat{\mathbf{b}}$, respectively. The remaining random field variables in the problem considered of element e, i.e. the tangent elastic modulus $D_{ijkl}^{(e)}[\mathbf{b}^{(e)}(x_k);x_k]$,mass density $\rho^{(e)}[\mathbf{b}^{(e)}(x_k);x_k]$,incremental body force $\Delta f_i^{(e)}[\mathbf{b}^{(e)}(x_k);x_k]$, incremental boundary traction $\Delta\hat{t}_i^{(e)}[\mathbf{b}^{(e)}(x_k);x_k]$, incremental displacement $\Delta u_i^{(e)}[\mathbf{b}^{(e)}(x_k);x_k]$, Cauchy stresses tensor $\tau_{ij}^{(e)}[\mathbf{b}^{(e)}(x_k);x_k]$ at time *t*, incremental strains tensor $\Delta\varepsilon_{ij}^{(e)}[\mathbf{b}(x_k);x_k]$, proportional damping coefficient $\alpha^{(e)}[\mathbf{b}^{(e)}(x_k);x_k]$ (the left subscript '*t*' was dropped for simplicity), are expanded as:

$$\begin{aligned}
&\mathbf{D}^{(e)}[\mathbf{b}^{(e)}(x_k);x_k] = \tilde{\mathbf{\Phi}}^{(e)}(x_k)\mathbf{D}^{(e)}[\hat{\mathbf{b}}(x_k);x_k]\\
&\boldsymbol{\rho}^{(e)}[\mathbf{b}^{(e)}(x_k);x_k] = \tilde{\mathbf{\Phi}}^{(e)}(x_k)\boldsymbol{\rho}^{(e)}[\hat{\mathbf{b}}(x_k);x_k]\\
&\Delta\mathbf{f}^{(e)}[\mathbf{b}^{(e)}(x_k);x_k] = \tilde{\mathbf{\Phi}}^{(e)}(x_k)\Delta\mathbf{f}^{(e)}[\hat{\mathbf{b}}(x_k);x_k]\\
&\Delta\hat{\mathbf{t}}^{(e)}[\mathbf{b}^{(e)}(x_k);x_k] = \tilde{\mathbf{\Phi}}^{(e)}(x_k)\Delta\hat{\mathbf{t}}^{(e)}[\hat{\mathbf{b}}(x_k);x_k]\\
&\Delta\mathbf{u}^{(e)}[\mathbf{b}^{(e)}(x_k);x_k] = \mathbf{\Phi}^{(e)}(x_k)\Delta\hat{\mathbf{u}}^{(e)}[\hat{\mathbf{b}}(x_k);x_k]\\
&\boldsymbol{\tau}^{(e)}[\mathbf{b}^{(e)}(x_k);x_k] = \tilde{\mathbf{\Phi}}^{(e)}(x_k)\boldsymbol{\tau}^{(e)}[\hat{\mathbf{b}}(x_k);x_k]\\
&\Delta\mathbf{e}^{(e)}[\mathbf{b}^{(e)}(x_k);x_k] = \tilde{\mathbf{\Phi}}^{(e)}(x_k)\Delta\mathbf{e}^{(e)}[\hat{\mathbf{b}}(x_k);x_k]\\
&\boldsymbol{\alpha}^{(e)}[\mathbf{b}^{(e)}(x_k);x_k] = \tilde{\mathbf{\Phi}}^{(e)}(x_k)\boldsymbol{\alpha}^{(e)}[\hat{\mathbf{b}}(x_k);x_k]
\end{aligned} \tag{36}$$

Substitute Eqs.(36) into Eq.(16), and collecting terms of equal order with respect to the small parameter ε,we obtain the zreoth-, first- and second-order finite element equations for the stochastic, nonlinear dynamic problem in the following form:

Zeroth-order incremental equation (ε^0 terms):

$$\mathbf{M}^0(\hat{\mathbf{b}}^0)\Delta\ddot{\hat{\mathbf{u}}}(\hat{\mathbf{b}}_\rho^0) + \mathbf{C}^0(\hat{\mathbf{b}}^0)\Delta\dot{\hat{\mathbf{u}}}(\hat{\mathbf{b}}^0) + \mathbf{K}^{(T)0}(\hat{\mathbf{b}}^0)\Delta\hat{\mathbf{u}}(\hat{\mathbf{b}}^0) = \Delta\mathbf{Q}^0(\hat{\mathbf{b}}^0) \tag{37}$$

First-order incremental equation (ε^1 terms):

$$\begin{aligned}
&\mathbf{M}^0(\hat{\mathbf{b}}_\rho^0)\Delta\ddot{\hat{\mathbf{u}}}^{,\rho}(\hat{\mathbf{b}}_\rho^0) + \mathbf{C}^0(\hat{\mathbf{b}}_\rho^0)\Delta\dot{\hat{\mathbf{u}}}^{,\rho}(\hat{\mathbf{b}}_\rho^0) + \mathbf{K}^{(T)0}(\hat{\mathbf{b}}_\rho^0)\Delta\hat{\mathbf{u}}^{,\rho}(\hat{\mathbf{b}}_\rho^0) = \Delta\mathbf{Q}^{,\rho}(\hat{\mathbf{b}}_\rho^0)\\
&-\left[\mathbf{M}^{,\rho}(\hat{\mathbf{b}}_\rho^0)\Delta\ddot{\hat{\mathbf{u}}}^0(\hat{\mathbf{b}}_\rho^0) + \mathbf{C}^{,\rho}(\hat{\mathbf{b}}_\rho^0)\Delta\dot{\hat{\mathbf{u}}}^0(\hat{\mathbf{b}}_\rho^0) + \mathbf{K}^{(T),\rho}(\hat{\mathbf{b}}_\rho^0)\Delta\hat{\mathbf{u}}^0(\hat{\mathbf{b}}_\rho^0)\right]
\end{aligned} \tag{38}$$

Second-order incremental equation (ε^2 terms):

$$\begin{aligned}
&\mathbf{M}^0(\hat{\mathbf{b}}_\rho^0)\Delta\ddot{\hat{\mathbf{u}}}^{,\rho\sigma}(\hat{\mathbf{b}}_\rho^0) + \mathbf{C}^0(\hat{\mathbf{b}}_\rho^0)\Delta\dot{\hat{\mathbf{u}}}^{,\rho\sigma}(\hat{\mathbf{b}}_\rho^0) + \mathbf{K}^{(T)0}(\hat{\mathbf{b}}_\rho^0)\Delta\hat{\mathbf{u}}^{,\rho\sigma}(\hat{\mathbf{b}}_\rho^0) = \Delta\mathbf{Q}^{,\rho\sigma}(\hat{\mathbf{b}}_\rho^0)\\
&-\left[\mathbf{M}^{,\rho}(\hat{\mathbf{b}}_\rho^0)\Delta\ddot{\hat{\mathbf{u}}}^{,\sigma}(\hat{\mathbf{b}}_\rho^0) + \mathbf{C}^{,\rho}(\hat{\mathbf{b}}_\rho^0)\Delta\dot{\hat{\mathbf{u}}}^{,\sigma}(\hat{\mathbf{b}}_\rho^0) + \mathbf{K}^{(T),\rho}(\hat{\mathbf{b}}_\rho^0)\Delta\hat{\mathbf{u}}^{,\sigma}(\hat{\mathbf{b}}_\rho^0)\right]\\
&-\left[\mathbf{M}^{,\rho\sigma}(\hat{\mathbf{b}}_\rho^0)\Delta\ddot{\hat{\mathbf{u}}}^0(\hat{\mathbf{b}}_\rho^0) + \mathbf{C}^{,\rho\sigma}(\hat{\mathbf{b}}_\rho^0)\Delta\dot{\hat{\mathbf{u}}}^0(\hat{\mathbf{b}}_\rho^0) + \mathbf{K}^{(T),\rho\sigma}(\hat{\mathbf{b}}_\rho^0)\Delta\hat{\mathbf{u}}^0(\hat{\mathbf{b}}_\rho^0)\right]
\end{aligned} \tag{39}$$

The formulations of $\mathbf{M}^0, \mathbf{M}^{,\rho}, \mathbf{M}^{,\rho\sigma}, \mathbf{K}^0\ldots$ referring to Appendix A.

Once $\Delta\hat{\mathbf{u}}^0$, $\Delta\hat{\mathbf{u}}^{,\rho}$, $\Delta\hat{\mathbf{u}}^{,\rho\sigma}$ have been solved by using Eqs.(37)—(39),the probabilistic distribution expressions for the expectation of incremental and total displacement are obtained as

$$E[\Delta\hat{\mathbf{u}}] = \Delta\hat{\mathbf{u}}^0 + \frac{1}{2}\hat{\mathbf{S}}_{\hat{\mathbf{b}}}^T\Delta\hat{\mathbf{u}}^{,\rho\sigma} E[\Delta\hat{\mathbf{u}}(t+\Delta t)] = E[\hat{\mathbf{u}}(t)] + E[\Delta\hat{\mathbf{u}}] \tag{40}$$

cross-variances evaluated at two space-time points $\zeta_1 = (x_m^1, t_1)$ and $\zeta_1 = (x_m^2, t_2)$ are:

$$\mathbf{Cov}\left(\hat{u}_i(t_1), \hat{u}_j(t_2)\right) = \left(\hat{\mathbf{u}}^{,\rho}\right)^T \hat{\mathbf{S}}_{\mathbf{b}} \hat{\mathbf{u}}^{,\sigma} \tag{41}$$

the second-order accurate expectation of $\mathbf{e}$ at time t are,

$$\begin{aligned} E\left[\Delta\mathbf{e}[\mathbf{b}(x_k); x_k, t]\right] &= \Delta\mathbf{e}^0 + \frac{1}{2}\hat{\mathbf{S}}_{\mathbf{b}} \Delta\mathbf{e}^{,\rho\sigma} \\ E\left[\mathbf{e}[\mathbf{b}(x_k); x_k, t + \Delta t]\right] &= E\left[\mathbf{e}(t)\right] + E\left[\Delta\mathbf{e}\right] \end{aligned} \tag{42}$$

the stain cross-variance at any two space-time points can be presented as,

$$\mathbf{Cov}\left(e_{ij}[x_k^{(1)}, t_1], e_{kl}[x_k^{(2)}, t_2]\right) = \left(\mathbf{e}(t_1)^{,\rho}\right)^T \hat{\mathbf{S}}_{\mathbf{b}} \mathbf{e}(t_2)^{,\sigma} \tag{43}$$

the second-order accurate expectation of $\boldsymbol{\sigma}$ at time t are,

$$\begin{aligned} E\left[\Delta\boldsymbol{\sigma}[\mathbf{b}(x_k); x_k, t]\right] &= \Delta\boldsymbol{\sigma}^0 + \frac{1}{2}\hat{\mathbf{S}}_{\mathbf{b}} \Delta\boldsymbol{\sigma}^{,\rho\sigma} \\ E\left[\boldsymbol{\sigma}[\mathbf{b}(x_k); x_k, t + \Delta t]\right] &= E\left[\boldsymbol{\sigma}(t)\right] + E\left[\Delta\boldsymbol{\sigma}\right] \end{aligned} \tag{44}$$

the stress cross-variance at any two space-time points can be presented as,

$$\mathbf{Cov}\left(\sigma_{ij}[x_k^{(1)}, t_1], \sigma_{kl}[x_k^{(2)}, t_2]\right) = \left(\boldsymbol{\sigma}(t_1)^{,\rho}\right)^T \hat{\mathbf{S}}_{\mathbf{b}} \boldsymbol{\sigma}(t_2)^{,\sigma} \tag{45}$$

Finding the statistical moments for the internal state variables follows the same pattern.

4 Direct integration Wilson-θ Method

The Wilson-θ method is essentially an extension of the linear acceleration method, in which a linear variation of acceleration from time t to $t+\Delta t$ is assumed. Referring to Figure 1, in the Wilson-θ method the acceleration is assumed to be linear form from t to time $t+\Delta t$, where $\theta \geq 1.0$ (generally, we employ $\theta = 1.4$). When $\theta = 1.0$, the method reduces to the linear acceleration scheme.

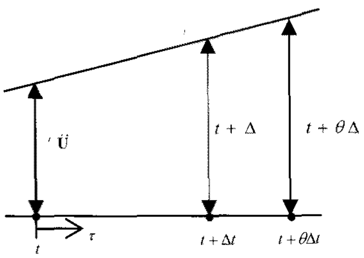

Figure 1: Linear acceleration assumption of Wilson-θ method

Let τ denote the increase in time, where $0 \leq \tau \leq \theta\Delta t$; then for the time interval t to $t+\theta\Delta t$, it is assumed that

$$ {}^{t+\tau}\ddot{\mathbf{U}}={}^{t}\ddot{\mathbf{U}}+\frac{\tau}{\theta\Delta t}\left({}^{t+\theta\Delta t}\ddot{\mathbf{U}}-{}^{t}\ddot{\mathbf{U}}\right) \tag{46} $$

integrating formulation (42), we obtain:

$$ {}^{t+\tau}\dot{\mathbf{U}}={}^{t}\dot{\mathbf{U}}+{}^{t}\ddot{\mathbf{U}}\tau+\frac{\tau^2}{2\theta\Delta t}\left({}^{t+\theta\Delta t}\ddot{\mathbf{U}}-{}^{t}\ddot{\mathbf{U}}\right) \tag{47} $$

$$ {}^{t+\tau}\mathbf{U}={}^{t}\mathbf{U}+{}^{t}\dot{\mathbf{U}}\tau+\frac{1}{2}{}^{t}\ddot{\mathbf{U}}\tau^2+\frac{1}{6\theta\Delta t}\tau^3\left({}^{t+\theta\Delta t}\ddot{\mathbf{U}}-{}^{t}\ddot{\mathbf{U}}\right) \tag{48} $$

Using formulations (43) and (44), we have at time $t+\theta\Delta t$,

$$ {}^{t+\tau}\dot{\mathbf{U}}={}^{t}\dot{\mathbf{U}}+\frac{\theta\Delta t}{2}\left({}^{t+\theta\Delta t}\ddot{\mathbf{U}}+{}^{t}\ddot{\mathbf{U}}\right) \tag{49} $$

$$ {}^{t+\theta\Delta t}\mathbf{U}={}^{t}\mathbf{U}+\theta\Delta t\,{}^{t}\dot{\mathbf{U}}+\frac{\theta^2\Delta t^2}{6}\left({}^{t+\theta\Delta t}\ddot{\mathbf{U}}+2\,{}^{t}\ddot{\mathbf{U}}\right) \tag{50} $$

from which we can solve for ${}^{t+\theta\Delta t}\ddot{\mathbf{U}}$ and ${}^{t+\theta\Delta t}\dot{\mathbf{U}}$ in terms of ${}^{t+\theta\Delta t}\mathbf{U}$:

$$ {}^{t+\theta\Delta t}\ddot{\mathbf{U}}=\frac{6}{\theta^2\Delta t^2}\left({}^{t+\theta\Delta t}\mathbf{U}-{}^{t}\mathbf{U}\right)-\frac{6}{\theta\Delta t}{}^{t}\dot{\mathbf{U}}-2\,{}^{t}\ddot{\mathbf{U}} \tag{51} $$

$$ {}^{t+\theta\Delta t}\dot{\mathbf{U}}=\frac{3}{\theta\Delta t}\left({}^{t+\theta\Delta t}\mathbf{U}-{}^{t}\mathbf{U}\right)-2\,{}^{t}\dot{\mathbf{U}}-\frac{\theta\Delta t}{2}{}^{t}\ddot{\mathbf{U}} \tag{52} $$

By introducing Eq.s (51) and (52) into the incremental equation of motion specified at the time instant $t+\theta\Delta t$ we obtain the algebraic system for $\Delta^{t+\theta\Delta t}\hat{\mathbf{u}}$ as:

$$ \mathbf{K}^{(eff)}\Delta^{t+\theta\Delta t}\hat{\mathbf{u}}=\Delta\mathbf{Q}^{(eff)} \tag{53} $$

where

$$ \mathbf{K}^{(eff)}=\frac{6}{(\theta\Delta t)^2}\mathbf{M}+\frac{3}{\theta\Delta t}\mathbf{C}+\mathbf{K}^{(T)} \tag{54} $$

$$ \Delta\mathbf{Q}^{(eff)}=\theta\Delta\mathbf{Q}+M\left[\frac{6}{\theta\Delta t}{}^{t}\dot{\mathbf{U}}+2\,{}^{t}\ddot{\mathbf{U}}\right]+\mathbf{C}\left[2\,{}^{t}\dot{\mathbf{U}}+\frac{\theta\Delta t}{2}{}^{t}\ddot{\mathbf{U}}\right] \tag{55} $$

An additional comment on the application of the Wilson-θ method (and of the implicit integration approach as a whole) should be made at this point. The configuration at time $t+\Delta t$ (consequently, at time $t+\theta\Delta t$), for which the equilibrium conditions are established, is unknown. This usually makes it necessary to carry out additional iterations in order to find a more accurate solution within each time interval. Such a solution can be obtained by using the Newton-type iteration techniques, among which the Newton-Raphson scheme is defined as

$$ \mathbf{K}^{(eff)}(t+\theta\Delta t)\Delta^{t+\theta\Delta t}\mathbf{U}_{(m)}=\mathbf{Q}(t+\theta\Delta t)-\mathbf{F}^{(eff)}_{(m-1)}(t+\theta\Delta t) \tag{56} $$

$$ m=1,2,\ldots $$

where the vector $\mathbf{F}^{(eff)}_{(m-1)}$ stand for the internal nodal forces corresponding to the displacements

$$ {}^{t+\theta\Delta t}\mathbf{U}_{(m-1)} = {}^{t+\theta\Delta t}\mathbf{U}_{(m-2)} + \Delta^{t+\theta\Delta t}\mathbf{U}_{(m-1)} \tag{57}$$

$\Delta^{t+\theta\Delta t}\mathbf{U}_{(m)}$ is the m-th correction to the incremental displacement vector $\Delta^{t+\theta\Delta t}\mathbf{U}$ and m denotes the iteration number. Since the updating and factorizing of the effective 'stiffness' matrix take place anew at each iteration, the computation cost of the method may be high, it could, therefore, turn out better to use a modified iteration scheme given by

$$ \mathbf{K}^{(eff)}(t)\Delta^{t+\theta\Delta t}\mathbf{U}_{(m)} = \mathbf{Q}(t+\theta\Delta t) - \mathbf{F}^{(eff)}_{(m-1)}(t+\theta\Delta t) \tag{58}$$

where the effective 'stiffness' matrix has to be factorized only once at the beginning of each time interval.

5 Numerical Example

A model of part of deck frame of cargo tanker subjected to a time-dependent concentrate load is considered, cf. Figure 2. the load time function P(t) , which is shown in Figure 3, is applied at point A. Nonlinear material constitutive relation is shown in Figure 4. The element cross-section areas $A_k, k = 1,2,\cdots,100$, are assumed as random variables. The vertical and horizontal beams are discretized by 10 equal-length element each. The mean value, correlation function and coefficient of variation for the cross-sectional areas are assumed as follows:

$$E(A_i) = A^0 = 10.0$$

$$\mu(A_i, A_j) = \exp(-\frac{|x_i - x_j|}{\lambda})$$

$$\gamma_A = 0.05$$

While $x_1 = 0.0$, $x_2 = 0.01$, $x_3 = 0.02, \cdots, x_{50} = 0.5$, with $\lambda = 0.5$. The following deterministic data are assumed: length $L_1 = 20$, $L_2 = 20$, Young's modules $E = 2.0\times10^7$, $E_T = 2.0\times10^5$, Poisson's ratio $\nu = 0.2$, mass density $\rho = 0.001$, and damping factor $\xi = 0.05$. Moment of inertial about x-axis $I_x = 20.3$.

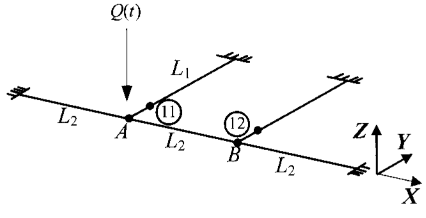

Figure 2: Model of frame structure

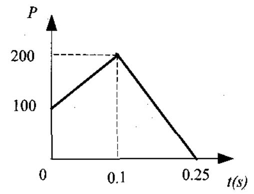

Figure 3: Dynamic load time function

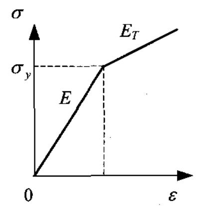

Figure 4: Nonlinear material constitutive relation

The mean values and variances of z-axis displacements at node point A are depicted in Figure 5 and Figure 6, respectively; the mean values and variances of stresses of element 11 are depicted in Figure 7 and Figure 8, respectively; similarly, covariances of displacements and stresses are depicted in Figure 9-10, as follows:

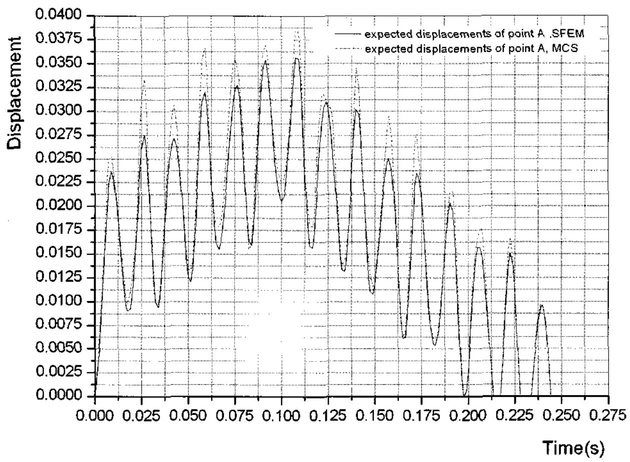

Figure 5: Mean values of z-axis displacements at node A

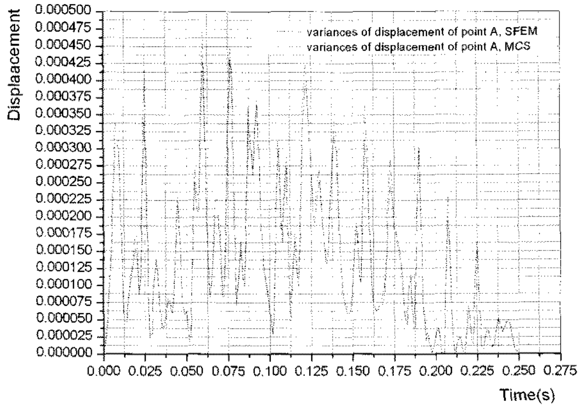

Figure 6: Variances of z-axis displacements at node A

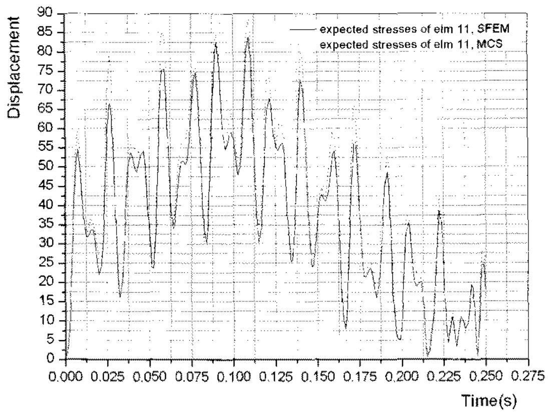

Figure 7: Mean values of Stresses of element 11

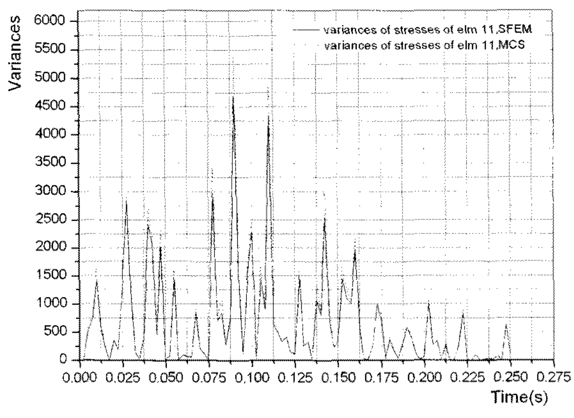

Figure 8: Variances of stresses of element 11

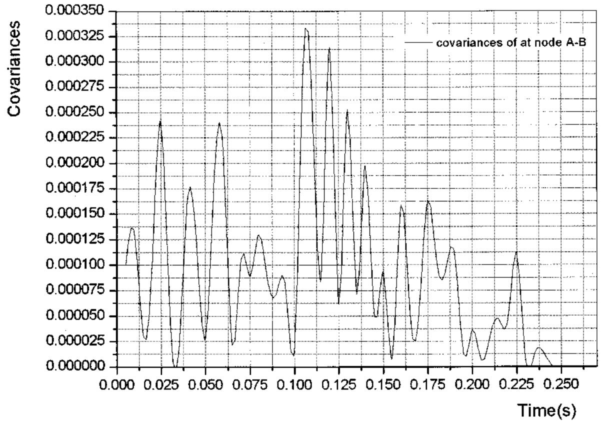

Figure 9: Covariances of z-axis displacements at node A-B

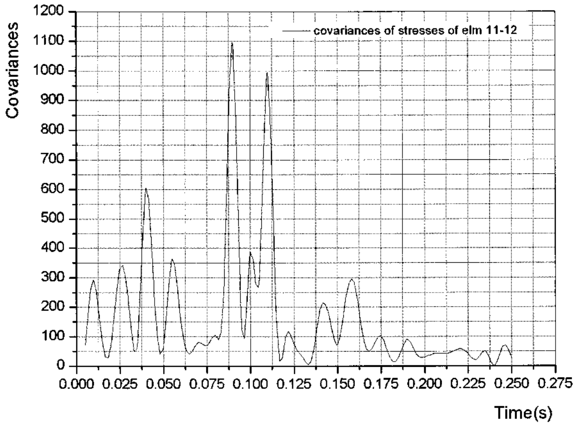

Figure 10: Covariances of stresses of element 11-12

6 Conclusions:

The stochastic variational principle is formulated to solve nonlinear structural dynamics via the weighted residual method and the small parameter perturbation technique. Base on incremental analysis of nonlinear problems, the corresponding formulation of incremental SFEM is developed. Applying Wilson-θ method to zeroth-, first- and second-order incremental recurrence equantions of motion of the structure, the transient nonlinear analysis of random structure under stochastic dynamic excitation may be obtained conveniently. In addition, the proposed method in this paper can be incorporated into widely used deterministic finite element programs in natural and concise manner. The results compared with MCS method verified that the method proposed in this paper may be used for nonlinear dynamic analysis of random structures effectively.

Appendix A: Formulation of Functions

• Zeroth-order functions

$$\mathbf{M}^0 = \sum_{e=1}^{\tilde{N}} \int_{V^{(e)}} \tilde{\mathbf{\Phi}}^{(e)0} \mathbf{\rho}^{(e)0} \left[\mathbf{\Phi}^{(e)0}\right]^T \mathbf{\Phi}^{(e)0} dV^{(e)}$$

$$\mathbf{C}^0 = \sum_{e=1}^{\tilde{N}} \int_{V^{(e)}} \tilde{\mathbf{\Phi}}^{(e)0} \mathbf{\alpha}^{(e)0} \tilde{\mathbf{\Phi}}^{(e)0} \mathbf{\rho}^{(e)0} \left[\mathbf{\Phi}^{(e)0}\right]^T \mathbf{\Phi}^{(e)0} dV^{(e)}$$

$$\mathbf{K}^{(T)0} = \mathbf{K}_L^{(T)0} + \mathbf{K}_{NL}^{(T)0}$$

$$\mathbf{K}_L^{(T)0} = \sum_{e=1}^{\tilde{N}} \int_{V^{(e)}} \left[\mathbf{B}_L^{(e)0}\right]^T \tilde{\mathbf{\Phi}}^{(e)0} \mathbf{D}^{(e)0} \mathbf{B}_L^{(m)0} dV^{(e)}$$

$$\mathbf{K}_{NL}^{(T)0} = \sum_{e=1}^{\tilde{N}} \int_{V^{(e)}} \left[\mathbf{B}_{NL}^{(e)0}\right]^T \tilde{\mathbf{\Phi}}^{(e)0} \mathbf{\tau}^{(e)0} \mathbf{B}_{NL}^{(e)0} dV^{(e)}$$

$$\Delta\mathbf{Q}^0 = \sum_{e=1}^{\tilde{N}} \int_{V^{(e)}} \left[\mathbf{\Phi}^{(e)0}\right]^T \tilde{\mathbf{\Phi}}^{(e)0} \mathbf{\rho}^{(e)0} \tilde{\mathbf{\Phi}}^{(e)0} \Delta\mathbf{f}^{(e)0} dV^{(e)} + \sum_{e=1}^{\tilde{N}} \int_{S^{(e)}} \left[\mathbf{\Phi}^{(e)0}\right]^T \tilde{\mathbf{\Phi}}^{(e)0} \Delta\hat{\mathbf{t}}^{(e)0} dS^{(e)}$$

where, $\mathbf{B}_L^{(e)} = \mathbf{L}\mathbf{\Phi}^{(e)}$ is linear strain-displacement relation matrix, $\mathbf{L}$ is differential matrix, $\mathbf{B}_{NL}^{(m)}$ is nonlinear strain-displacement relationship matrix (Bathe 1966), $\tilde{N}$ is total number of elements in structure;

• First-order functions

$$\mathbf{M}^{,\rho} = \sum_{m=1}^{\tilde{N}} \Big\{ \int_{V^{(e)}} \tilde{\mathbf{\Phi}}^{(e)0} \mathbf{\rho}^{(e),\rho} \left[\mathbf{\Phi}^{(e)0}\right]^T \mathbf{\Phi}^{(e)0} dV^{(e)} + \int_{V^{(e)}} \tilde{\mathbf{\Phi}}^{(e),\rho} \mathbf{\rho}^{(e)0} \left[\mathbf{\Phi}^{(e)0}\right]^T \mathbf{\Phi}^{(e)0} dV^{(e)} + \int_{V^{(e)}} \tilde{\mathbf{\Phi}}^{(e)0} \mathbf{\rho}^{(e)0} \left[\mathbf{\Phi}^{(m),\rho}\right]^T \mathbf{\Phi}^{(e)0} dV^{(e)} + \int_{V^{(e)}} \tilde{\mathbf{\Phi}}^{(e)0} \mathbf{\rho}^{(e)0} \left[\mathbf{\Phi}^{(m)0}\right]^T \mathbf{\Phi}^{(e),\rho} dV^{(e)} \Big\}$$

$$\begin{aligned}\mathbf{C}^{,\rho} = \sum_{e=1}^{\tilde{N}} \Big\{ &\int_{V^{(e)}} \tilde{\mathbf{\Phi}}^{(e)0} \mathbf{\alpha}^{(e),\rho} \tilde{\mathbf{\Phi}}^{(e)0} \mathbf{\rho}^{(e)0} \left[\mathbf{\Phi}^{(e)0}\right]^T \mathbf{\Phi}^{(e)0} dV^{(e)} \\ &+ \int_{V^{(e)}} \tilde{\mathbf{\Phi}}^{(e)0} \mathbf{\alpha}^{(e)0} \tilde{\mathbf{\Phi}}^{(e)0} \mathbf{\rho}^{(e),\rho} \left[\mathbf{\Phi}^{(e)0}\right]^T \mathbf{\Phi}^{(e)0} dV^{(e)} \\ &+ \int_{V^{(e)}} \tilde{\mathbf{\Phi}}^{(e),\rho} \mathbf{\alpha}^{(e)0} \tilde{\mathbf{\Phi}}^{(e)0} \mathbf{\rho}^{(e)0} \left[\mathbf{\Phi}^{(e)0}\right]^T \mathbf{\Phi}^{(e)0} dV^{(e)} \\ &+ \int_{V^{(e)}} \tilde{\mathbf{\Phi}}^{(e)0} \mathbf{\alpha}^{(e)0} \tilde{\mathbf{\Phi}}^{(e),\rho} \mathbf{\rho}^{(e)0} \left[\mathbf{\Phi}^{(e)0}\right]^T \mathbf{\Phi}^{(e)0} dV^{(e)} \\ &+ \int_{V^{(e)}} \tilde{\mathbf{\Phi}}^{(e)0} \mathbf{\alpha}^{(e)0} \tilde{\mathbf{\Phi}}^{(e)0} \mathbf{\rho}^{(e)0} \left[\mathbf{\Phi}^{(e),\rho}\right]^T \mathbf{\Phi}^{(e)0} dV^{(e)} \\ &+ \int_{V^{(e)}} \tilde{\mathbf{\Phi}}^{(e)0} \mathbf{\alpha}^{(e)0} \tilde{\mathbf{\Phi}}^{(e)0} \mathbf{\rho}^{(e)0} \left[\mathbf{\Phi}^{(e)0}\right]^T \mathbf{\Phi}^{(e),\rho} dV^{(e)} \Big\}\end{aligned}$$

$$\mathbf{K}^{(T),\rho} = \mathbf{K}_L^{(T),\rho} + \mathbf{K}_{NL}^{(T),\rho}$$

$$\begin{aligned}\mathbf{K}_L^{(T),\rho} = \sum_{e=1}^{\tilde{N}} \Big\{ &\int_{V^{(e)}} \left[\mathbf{B}_L^{(e)0}\right]^T \tilde{\mathbf{\Phi}}^{(e)0} \mathbf{C}^{(e),\rho} \mathbf{B}_L^{(e)0} dV^{(e)} \\ &+ \int_{V^{(e)}} \left[\mathbf{B}_L^{(e),\rho}\right]^T \tilde{\mathbf{\Phi}}^{(e)0} \mathbf{C}^{(e)0} \mathbf{B}_L^{(e)0} dV^{(e)} \\ &+ \int_{V^{(e)}} \left[\mathbf{B}_L^{(e)0}\right]^T \tilde{\mathbf{\Phi}}^{(e),\rho} \mathbf{C}^{(e)0} \mathbf{B}_L^{(e)0} dV^{(e)} \\ &+ \int_{V^{(e)}} \left[\mathbf{B}_L^{(e)0}\right]^T \tilde{\mathbf{\Phi}}^{(e)0} \mathbf{C}^{(e)0} \mathbf{B}_L^{(e),\rho} dV^{(e)} \Big\}\end{aligned}$$

$$
\begin{aligned}
\mathbf{K}_{NL}^{(T),\rho} = \sum_{e=1}^{\tilde{N}} \Big\{ & \int_{V^{(e)}} \left[\mathbf{B}_{NL}^{(e)0}\right]^T \tilde{\mathbf{\Phi}}^{(e)0} \boldsymbol{\tau}^{(e),\rho} \mathbf{B}_{NL}^{(e)0} dV^{(e)} \\
& + \int_{V^{(e)}} \left[\mathbf{B}_{NL}^{(e),\rho}\right]^T \tilde{\mathbf{\Phi}}^{(e)0} \boldsymbol{\tau}^{(e)0} \mathbf{B}_{NL}^{(e)0} dV^{(e)} \\
& + \int_{V^{(e)}} \left[\mathbf{B}_{NL}^{(e)0}\right]^T \tilde{\mathbf{\Phi}}^{(e),\rho} \boldsymbol{\tau}^{(e)0} \mathbf{B}_{NL}^{(e)0} dV^{(e)} \\
& + \int_{V^{(e)}} \left[\mathbf{B}_{NL}^{(e)0}\right]^T \tilde{\mathbf{\Phi}}^{(e)0} \boldsymbol{\tau}^{(e)0} \mathbf{B}_{NL}^{(e),\rho} dV^{(e)} \Big\}
\end{aligned}
$$

$$
\begin{aligned}
\Delta\mathbf{Q}^{,\rho} = \sum_{e=1}^{\tilde{N}} \Big\{ & \int_{V^{(e)}} \left[\mathbf{\Phi}^{(e)0}\right]^T \tilde{\mathbf{\Phi}}^{(e)0} \boldsymbol{\rho}^{(e),\rho} \tilde{\mathbf{\Phi}}^{(e)0} \Delta\mathbf{f}^{(e)0} dV^{(e)} \\
& + \int_{V^{(e)}} \left[\mathbf{\Phi}^{(e),\rho}\right]^T \tilde{\mathbf{\Phi}}^{(e)0} \boldsymbol{\rho}^{(e)0} \tilde{\mathbf{\Phi}}^{(e)0} \Delta\mathbf{f}^{(e)0} dV^{(e)} \\
& + \int_{V^{(e)}} \left[\mathbf{\Phi}^{(e)0}\right]^T \tilde{\mathbf{\Phi}}^{(e),\rho} \boldsymbol{\rho}^{(e)0} \tilde{\mathbf{\Phi}}^{(e)0} \Delta\mathbf{f}^{(e)0} dV^{(e)} \\
& + \int_{V^{(e)}} \left[\mathbf{\Phi}^{(e)0}\right]^T \tilde{\mathbf{\Phi}}^{(e)0} \boldsymbol{\rho}^{(e)0} \tilde{\mathbf{\Phi}}^{(e),\rho} \Delta\mathbf{f}^{(e)0} dV^{(e)} \\
& + \int_{V^{(e)}} \left[\mathbf{\Phi}^{(e)0}\right]^T \tilde{\mathbf{\Phi}}^{(e)0} \boldsymbol{\rho}^{(e)0} \tilde{\mathbf{\Phi}}^{(e)0} \Delta\mathbf{f}^{(e),\rho} dV^{(e)} \Big\}
\end{aligned}
$$

- Second-order functions

$$
\mathbf{M}^{,(2)} = \sum_{e=1}^{\tilde{N}} \int_{V^{(e)}} \left[\sum_{k=0}^{2} \sum_{j=0}^{k} \sum_{i=0}^{2-k} \binom{2}{k} \binom{k}{j} \binom{2-k}{i} \tilde{\mathbf{\Phi}}_{(e)}^{,(j)} \boldsymbol{\rho}_{(e)}^{,(k-j)} \left[\mathbf{\Phi}_{(e)}^{,(i)}\right]^T \mathbf{\Phi}_{(e)}^{,(2-k-i)} dV^{(e)} \right]
$$

$$
\begin{aligned}
\mathbf{C}^{,\rho\sigma} = \sum_{m=1}^{\tilde{N}} \int_{V^{(m)}} \Bigg[& \sum_{k=0}^{2} \sum_{i=0}^{k} \sum_{m=0}^{i} \sum_{j=0}^{2-k} \sum_{n=0}^{j} \binom{2}{k} \binom{k}{i} \binom{i}{m} \binom{2-k}{j} \binom{j}{n} \\
& \tilde{\mathbf{\Phi}}_{(e)}^{,(m)} \boldsymbol{\alpha}_{(e)}^{,(i-m)} \tilde{\mathbf{\Phi}}_{(e)}^{,(k-i)} \boldsymbol{\rho}_{(e)}^{,(n)} \left[\mathbf{\Phi}_{(e)}^{,(j-n)}\right]^T \mathbf{\Phi}_{(e)}^{,(2-k-j)} dV^{(m)} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\Delta\mathbf{Q}^{,\rho\sigma} = \sum_{e=1}^{\tilde{N}} \int_{V^{(e)}} \Bigg[& \sum_{k=0}^{2} \sum_{i=0}^{k} \sum_{m=0}^{i} \sum_{j=0}^{2-k} \binom{2}{k} \binom{k}{i} \binom{i}{m} \binom{2-k}{j} \\
& \left[\mathbf{\Phi}_{(e)}^{,(m)}\right]^T \tilde{\mathbf{\Phi}}_{(e)}^{,(i-m)} \boldsymbol{\rho}_{(e)}^{,(k-i)} \tilde{\mathbf{\Phi}}_{(e)}^{,(j)} \Delta\mathbf{f}_{(e)}^{,(2-k-j)} dV^{(e)} \Bigg]
\end{aligned}
$$

$$
\mathbf{K}^{(T),\rho\sigma} = \mathbf{K}_L^{(T),\rho\sigma} + \mathbf{K}_{NL}^{(T),\rho\sigma}
$$

$$
\mathbf{K}_L^{(T),\rho\sigma} = \sum_{e=1}^{\tilde{N}} \int_{V^{(e)}} \left[\sum_{k=0}^{2} \sum_{j=0}^{k} \sum_{i=0}^{2-k} \binom{2}{k} \binom{k}{j} \binom{2-k}{i} \left[\mathbf{B}_{L(e)}^{,(j)}\right]^T \tilde{\mathbf{\Phi}}_{(e)}^{,(k-j)} \mathbf{C}_{(e)}^{,(i)} \mathbf{B}_{L(e)}^{,(2-k-j)} dV^{(e)} \right]
$$

$$
\mathbf{K}_{NL}^{(T),\rho\sigma} = \sum_{e=1}^{\tilde{N}} \int_{V^{(e)}} \left[\sum_{k=0}^{2} \sum_{j=0}^{k} \sum_{i=0}^{2-k} \binom{2}{k} \binom{k}{j} \binom{2-k}{i} \left[\mathbf{B}_{NL(e)}^{,(j)}\right]^T \tilde{\mathbf{\Phi}}_{(e)}^{,(k-j)} \boldsymbol{\tau}_{(e)}^{,(i)} \mathbf{B}_{NL(e)}^{,(2-k-j)} dV^{(e)} \right]
$$

The $()^{,(i)}$ represents the i-th derivative of random variable.

All the functions are valuated at the expectations $\hat{\mathbf{b}}^0$ of the nodal random variables vector $\hat{\mathbf{b}}$.

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