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FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIVE TYPE MAPPINGS IN SYMMETRIC SPACES

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Abstract. In this paper, we give a generalized contractive condition for four self mappings of symmetric spaces and give some results on fixed point theorems in symmetric spaces.

1. Introduction

In 1986, Jungck[5] introduced the concept of compatible mappings in metric spaces and proved some fixed point theorems. This notion of compatible mappings was frequently used to proved the existence of common fixed points. However, the study of the existence of common fixed points for noncompatible mappings is, also, very interesting. Pant[7] initially proved some common fixed point theorems for noncompatible mappings in metric spaces. In [1], the authors gave a notion (E-A) which generalizes the concept of noncompatible mappings in metric spaces, and they proved some common fixed point theorems for noncompatible mappings under strict contractive conditions. In [8], the authors proved some common fixed point theorems for strict contractive noncompatibile mappings in metric spaces. Recently, in [4] the authors extended the results of [1] and [8] to symmetric(semi-metric) spaces under tight conditions. In [2], the author gave a common fixed point theorem for noncompatible self mappings in a symmetric spaces under a contractive condition of integral type. Also, in [3] the au-

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thors proved some common fixed point theorems for mappings satisfy property (E-A) in symmetric spaces.

In order to obtain common fixed point theorems in symmetric spaces, some axioms are needed. In [4], the authors assumed axiom (W3) and in [2] the author assumed axioms (W3), (W4) and (H.E). And in [3], the authors assumed axiom (H.E) and (C.C), and they studied relationships between these axioms.

In this paper we give a generalized contractive condition for four self mappings of symmetric spaces and give some common fixed point theorems for four mappings in symmetric spaces. Especially, we give a generalization of theorem 1 of [2] without the condition (W3). And we give some examples which justfies the necessity of axioms.

2. Preliminaries

A *symmetric* on a set X is a function $d : X \times X \to [0, \infty)$ satisfying the following conditions:
(i) $d(x, y) = 0$ if and only if $x = y$ for $x, y \in X$,
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

Let d be a symmetric on a set X. For $x \in X$ and $\epsilon > 0$, let $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$. A topology $\tau(d)$ on X defined as follows: $U \in \tau(d)$ if and only if for each $x \in U$, there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. A subset S of X is a neighbourhood of $x \in X$ if there exists $U \in \tau(d)$ such that $x \in U \subset S$. A symetric d is a *semi-metric* if for each $x \in X$ and each $\epsilon > 0$, $B(x, \epsilon)$ is a neighbourhood of x in the topology $\tau(d)$.

A *symmetric*(resp., *semi-metric*) *space* (X, d) is a topological space whose topology $\tau(d)$ on X is induced by symmetric(resp., semi-metric) d.

The difference of a symmetric and a metric comes from the triangle inequality. Actually a symmetirc space need not be Housdorff. In order to obtain fixed point theorems on a symmetric space (X, d), we need some additional axioms. The following axioms can be found in [9].

(W3) For a sequence $\{x_n\}$ in $X, x, y \in X$, $lim_{n\to\infty} d(x_n, x) = 0$ and $lim_{n\to\infty} d(x_n, y) = 0$ imply $x = y$.

(W4) For sequences $\{x_n\}, \{y_n\}$ in X and $x \in X$, $lim_{n\to\infty} d(x_n, x) = 0$

and $lim_{n\to\infty}d(y_n, x_n) = 0$ imply $lim_{n\to\infty}d(y_n, x) = 0$.

Also the following axiom can be found in [2].
(H.E) For sequences $\{x_n\}, \{y_n\}$ in X and $x \in X$, $lim_{n\to\infty}d(x_n, x) = 0$ and $lim_{n\to\infty}d(y_n, x) = 0$ imply $lim_{n\to\infty}d(x_n, y_n) = 0$.

The next axiom which is related to the continuity of the symmetric d can be found in [3].

(C.C) For sequence $\{x_n\}$ in X and $x, y \in X$, $lim_{n\to\infty}d(x_n, x) = 0$ implies $lim_{n\to\infty}d(x_n, y) = d(x, y)$.

Note that if d is a metric, then (W3), (W4), (H.E) and (C.C) are automatically satisfied. And if $\tau(d)$ is Hausdorff, then (W3) is satisfied.

Lemma 2.1[3]. *For axioms in symmetric space* (X, d), *we have*
(1) *(W4)* $\Longrightarrow$ *(W3),*
(2) *(C.C)* $\Longrightarrow$ *(W3).*

Note that other relationships in Lemma 2.1 do not hold(see [3]).

Let (X, d) be a symmetric(or semi-metric) space and let f, g be self mappings of X. Then we say that the pair (f, g) satisfies property (E-A)[1] if there exist a sequence $\{x_n\}$ in X and a point $t \in X$ such that $lim_{n\to\infty}d(fx_n, t) = lim_{n\to\infty}d(gx_n, t) = 0$.

A subset S of a symmetric space (X, d) is said to be *d-closed* if for a sequence $\{x_n\}$ in S and a point $x \in X$, $lim_{n\to\infty}d(x_n, x) = 0$ implies $x \in S$. For a symmetric space (X, d), d-closedness implies $\tau(d)$-closedness, and if d is a semi-metric, the converse is also true.

From now on, we denote Λ by the class of nondecreasing continuous function $\alpha : [0, \infty) \to [0, \infty)$ such that
(α1) $\alpha(0) = 0$,
(α2) $\alpha(s) > 0$ for all $s > 0$.

Note that if $\alpha(s) = \int_0^s \varphi(t)dt$, then $\alpha \in \Lambda$ where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable, nonnegative and $\int_0^u \varphi(t)dt > 0$ for each $u > 0$.

And we denote Φ by the class of nondecreasing right upper semi-continuous function $\phi : [0, \infty) \to [0, \infty)$ satisfying:
(ϕ1) $\phi(t) < t$ for all $t > 0$,

($\phi 2$) for each $t > 0$, $lim_{n\to\infty}\phi^n(t) = 0$.

Note that $\phi(0) = 0$.

3. Main Theorems

For the existence of a common fixed point of four self mappings of a symmetric space, we need a condition, so called weak compatibility.

Recall that for self mappings f and g of a set, the pair (f, g) is said to be *weakly compatible*[6] if $fgx = gfx$, whenever $fx = gx$. Obviously, if f and g are commuting, the pair (f, g) is weakly compatible.

THEOREM 3.1. *Let (X, d) be a symmetric(semi-metric) space that satisfies (H.E) and (C.C) and let A, B, S and T be self mappings of X and $\alpha \in \Lambda$ and $\phi \in \Phi$ satisfying*
(1) *$AX \subset TX$ and $BX \subset SX$,*
(2) *the pair (B, T) satisfies property (E-A)(resp., (A, S) satisfies property (E-A)),*
(3) *for any $x, y \in X$, $\alpha(d(Ax, By)) \leq \phi(\alpha(M(x, y)))$, where $M(x, y) = max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx)\}$,*
(4) *the pairs (A, S) and (B, T) are weakly compatible,*
(5) *SX is a d-closed($\tau(d)$-closed) subset of X(resp., TX is a d-closed ($\tau(d)$-closed) subset of X).*
Then A, B, S and T have a unique common fixed point in X.

Proof. From (2), there exist a sequence $\{x_n\}$ in X and a point $t \in X$ such that $lim_{n\to\infty}d(Tx_n, t) = lim_{n\to\infty}d(Bx_n, t) = 0$.

From (1), there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$ and hence $lim_{n\to\infty}d(Sy_n, t) = 0$. By (H.E), $lim_{n\to\infty}d(Bx_n, Tx_n) = lim_{n\to\infty}d(Sy_n, Tx_n) = 0$. From (5), there exists a point $u \in X$ such that $Su = t$.

We show $Au = Su$. From (3) we have

$$\begin{aligned}&\alpha(d(Au, Bx_n))\\ \leq&\phi(\alpha(max\{d(Su, Tx_n), d(Au, Su), d(Bx_n, Tx_n),\\ &d(Au, Tx_n), d(Bx_n, Su)\})).\end{aligned}$$

In the above inequality, we take $n \to \infty$, by (C.C) and (H.E), we have

$$
\begin{aligned}
&\alpha(d(Au, Su))\\
\leq&\phi(\alpha(max\{0, d(Au, Su), 0, d(Au, Su), 0\}))\\
=&\phi(\alpha(d(Au, Su)))
\end{aligned}
$$

which implies $\alpha(d(Au, Su)) = 0$. By ($\alpha 1$), we have $d(Au, Su) = 0$. Hence $Au = Su$.

Since $AX \subset TX$, there exists a point $w \in X$ such that $Au = Tw$. Thus we get $Au = Su = Tw$.

We show that $Tw = Bw$. From (3) we have

$$
\begin{aligned}
&\alpha(d(Tw, Bw))\\
=&\alpha(d(Au, Bw))\\
\leq&\phi(\alpha(max\{d(Su, Tw), d(Au, Su), d(Bw, Tw),\\
&d(Au, Tw), d(Bw, Su)\}))\\
=&\phi(\alpha(max\{d(Tw, Tw), d(Au, Au), d(Bw, Tw),\\
&d(Au, Au), d(Bw, Tw)\}))\\
=&\phi(\alpha(d(Bw, Tw))).
\end{aligned}
$$

Thus we get $\alpha(d(Tw, Bw)) = 0$. Hence $d(Tw, Bw) = 0$ or $Tw = Bw$. Therefore we have

$$z = Au = Su = Bw = Tw. \tag{3.1.1}$$

From (4), we have

$$AAu = ASu = SAu = SSu \tag{3.1.2}$$

and

$$BTw = TBw = TTw = BBw. \tag{3.1.3}$$

We show $z = Az$. From (3), (3.1.1) and (3.1.2) we have

$$\begin{aligned}
&\alpha(d(z, Az))\\
=&\alpha(d(Au, AAu))\\
=&\alpha(d(AAu, Bw))\\
\leq&\phi(\alpha(max\{d(SAu, Tw), d(AAu, SAu), d(Bw, Tw),\\
&d(AAu, Tw), d(Bw, SAu)\}))\\
=&\phi(\alpha(max\{d(AAu, Au), 0, 0, d(AAu, Au), d(AAu, Au)\}))\\
=&\phi(\alpha(d(AAu, Au)))\\
=&\phi(\alpha(d(z, Az)))
\end{aligned}$$

which implies $\alpha(d(z, Az)) = 0$. Thus we have $d(z, Az) = 0$ or $z = Az$. From (3.1.1) and (3.1.2) we get

$$z = Az = Sz. \tag{3.1.4}$$

Next, we show $z = Bz$. Again, from (3), (3.1.1) and (3.1.3) we have

$$\begin{aligned}
&\alpha(d(z, Bz))\\
=&\alpha(d(Bw, BBw))\\
=&\alpha(d(Au, BBw))\\
\leq&\phi(\alpha(max\{d(Su, TBw), d(Au, Su), d(BBw, TBw),\\
&d(Au, TBw), d(BBw, Su)\}))\\
=&\phi(\alpha(max\{d(Bw, BBw), d(Bw, Bw), d(BBw, BBw),\\
&d(Bw, BBw), d(BBw, Bw)\}))\\
=&\phi(\alpha(max\{d(Bw, BBw), 0, 0, d(Bw, BBw), d(Bw, BBw)\}))\\
=&\phi(\alpha(d(Bw, BBw)))\\
=&\phi(\alpha(d(z, Bz)))
\end{aligned}$$

which implies $\alpha(d(z, Bz)) = 0$. Thus we have $d(z, Bz) = 0$ or $z = Bz$. Thus from (3.1.1) and (3.1.3) we get $z = Bz = Tz$.

Therefore, by (3.1.4), we have $z = Az = Sz = Tz = Bz$.

For the uniqueness, let w be an another common fixed point of A, B, S and T. If $w \neq z$, then from (3) we get

$$\begin{aligned}
&\alpha(d(z,w))\\
=&\alpha(d(Az,Bw))\\
\leq&\phi(\alpha(max\{d(Sz,Tw),d(Az,Sz),d(Bw,Tw),\\
&d(Az,Tw),d(Bw,Sz)\}))\\
=&\phi(\alpha(max\{d(z,w),d(z,z),d(w,w),d(z,w),d(w,z)\}))\\
=&\phi(\alpha(max\{d(z,w),0,0,d(w,z),d(w,z)\}))\\
=&\phi(\alpha(d(z,w)))\\
<&\alpha(d(z,w)
\end{aligned}$$

which is a contradiction. Thus $\alpha(d(z,w)) = 0$ and so $d(z,w) = 0$. Hence $w = z$. □

Example 3.2. Let $X = [0,1]$ and $d(x,y) = (x-y)^2$. Define self mappings A, B, S and T by $Ax = Bx = \frac{1}{2}x$ and $Sx = Tx = x$ for all $x \in X$. Let $\alpha(s) = s$ for all $s \in [0,\infty)$ and $\phi(t) = \frac{1}{2}t$ for all $t \in [0,\infty)$. Then we have
(0) (X,d) is a symmetric space stisfying the properties (H.E) and (C.C),
(1) $AX \subset TX$ and $BX \subset SX$,
(2) the pair (B,T) satisfies property (E-A) for the sequence $x_n = \frac{1}{n}, n = 1,2,3\cdots$,
(3) the pairs (A,S) and (B,T) are weakly compatible,
(4) for any $x, y \in X$,

$$\alpha(d(Ax,By)) \leq \phi(\alpha(d(Sx,Ty))) \leq \phi(\alpha(M(x,y))),$$

(5) SX is a d-closed($\tau(d)$-closed) subset of X,
(6) $A0 = B0 = S0 = T0 = 0$.

Corollary 3.3. *Let (X,d) be a symmetric(semi-metric) space that satisfies (H.E) and (C.C) and let A, B, S and T be self mappings of X and $\alpha \in \Lambda$ and $\phi \in \Phi$ satisfying*
(1) *$AX \subset TX$ and $BX \subset SX$,*
(2) *the pair (B,T) satisfies property (E-A)(resp., (A,S) satisfies property (E-A)),*
(3) *for any $x, y \in X$, $\alpha(d(Ax,By)) \leq \phi(\alpha(m(x,y)))$, where $m(x,y) =$*

$max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \frac{1}{2}\{d(Ax,Ty)+d(By,Sx)\}\}$,
(4) *the pairs* (A,S) *and* (B,T) *are weakly compatible,*
(5) SX *is a d-closed(*$\tau(d)$*-closed) subset of* X *(resp.,* TX *is a d-closed (*$\tau(d)$*-closed) subset of* X*).*
Then A, B, S *and* T *have a unique common fixed point in* X.

COROLLARY 3.4. *Let* (X,d) *be a symmetric(semi-metric) space that satisfies (H.E) and (C.C) and let* A, B, S *and* T *be self mappings of* X *and* $\alpha \in \Lambda$ *and* $\phi \in \Phi$ *satisfying*
(1) $AX \subset TX$ *and* $BX \subset SX$,
(2) *the pair* (B,T) *satisfies property (E-A)(resp.,* (A,S) *satisfies property (E-A)),*
(3) *for any* $x, y \in X$,
$\alpha(d(Ax,By)) \leq \phi(\alpha(max\{d(Sx,Ty), d(By,Ty), d(By,Sx)\}))$,
(4) *the pairs* (A,S) *and* (B,T) *are weakly compatible,*
(5) SX *is a d-closed(*$\tau(d)$*-closed) subset of* X *(resp.,* TX *is a d-closed (*$\tau(d)$*-closed) subset of* X*).*
Then A, B, S *and* T *have a unique common fixed point in* X.

REMARK 3.5. If we have $\alpha(s) = \int_0^s \varphi(t)dt$ in Theorem 3.1(Corollary 3.3, Corollary 3.4), then the conclusion is still true where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable, nonnegative and $\int_0^u \varphi(t)dt > 0$ for each $u > 0$.

THEOREM 3.6. *Let* (X,d) *be a symmetric(semi-metric) space that satisfies (H.E) and (W4) and let* A, B, S *and* T *be self mappings of* X *and* $\alpha \in \Lambda$ *and* $\phi \in \Phi$ *satisfying*
(1) $AX \subset TX$ *and* $BX \subset SX$,
(2) *the pair* (B,T) *satisfies property (E-A)(resp.,* (A,S) *satisfies property (E-A)),*
(3) *for any* $x, y \in X$,
$\alpha(d(Ax,By)) \leq \phi(\alpha(max\{d(Sx,Ty), d(By,Ty), d(By,Sx)\}))$,
(4) *the pairs* (A,S) *and* (B,T) *are weakly compatible,*
(5) *one of* AX, BX, SX *and* TX *is complete subspace of* X.
Then A, B, S *and* T *have a unique common fixed point in* X.

Proof. As in proof of Theorem 3.1, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X and a point $t \in X$ such that
$lim_{n\to\infty} d(Tx_n, t) = lim_{n\to\infty} d(Bx_n, t) = lim_{n\to\infty} d(Bx_n, Tx_n)$

$= lim_{n\to\infty} d(Sy_n, Tx_n) = 0$ and $Bx_n = Sy_n$.

We now show that $lim_{n\to\infty} d(Ay_n, t) = 0$. From (3) we have
$\alpha(d(Ay_n, Bx_n))$
$\leq \phi(\alpha(max\{d(Sy_n, Tx_n), d(Bx_n, Tx_n), d(Bx_n, Sy_n)\}))$.

Letting $n \to \infty$, we have $lim_{n\to\infty}\alpha(d(Ay_n, Bx_n)) \leq \phi(\alpha(0)) = 0$. Thus $lim_{n\to\infty} d(Ay_n, Bx_n) = 0$. By $(W4)$, we get $lim_{n\to\infty} d(Ay_n, t) = 0$. If SX is complete subspace of X, then there exists $u \in X$ such that $t = Su$. Thus we have

$$lim_{n\to\infty} d(Ay_n, Su) = lim_{n\to\infty} d(Bx_n, Su)$$

$$= lim_{n\to\infty} d(Tx_n, Su) = lim_{n\to\infty} d(Sy_n, Su) = 0.$$

We now show that $Au = Su$. From (3) we have

$$\alpha(d(Au, Bx_n)) \leq \phi(\alpha(max\{d(Su, Tx_n), d(Bx_n, Tx_n), d(Bx_n, Su)\})).$$

Taking $n \to \infty$, we get $lim_{n\to\infty}\alpha(d(Au, Bx_n)) \leq \phi(\alpha(0)) = 0$. So $lim_{n\to\infty} d(Au, Bx_n) = 0$. By Lemma 2.1, (X, d) satisfies $(W3)$ and so we have $Su = Au = z$. By (4) we have

$$Az = Sz. \tag{3.6.1}$$

From (1) there exists $v \in X$ such that $Au = Tv$. Thus we get $Au = Tv = Su = z$. We claim that $Bv = Tv$. If not, then we have

$$\begin{aligned}
&\alpha(d(Tv, Bv)) \\
=&\alpha(d(Au, Bv)) \\
\leq&\phi(\alpha(max\{d(Su, Tv), d(Bv, Tv), d(Bv, Su)\})) \\
=&\phi(\alpha(max\{d(Tv, Tv), d(Bv, Tv), d(Bv, Tv)\})) \\
=&\phi(\alpha(d(Bv, Tv))) \\
<&\alpha(d(Bv, Tv))
\end{aligned}$$

which is a contradiction. Thus we have $Bv = Tv$.

Therefore, we get

$$Bv = Tv = Su = Au = z. \tag{3.6.2}$$

From (4) we have

$$Bz = Tz. \tag{3.6.3}$$

We show that $z = Az$.

From (3), (3.6.1) and (3.6.2) we have

$$\begin{aligned}
&\alpha(d(z, Az)) \\
=&\alpha(d(Az, Bv)) \\
\le&\phi(\alpha(max\{d(Sz, Tv), d(Bv, Tv), d(Bv, Sz)\})) \\
=&\phi(\alpha(max\{d(Az, z), d(z, z), d(z, Az)\})) \\
=&\phi(\alpha(d(z, Az))),
\end{aligned}$$

which implies $\alpha(d(z, Az)) = 0$ and so $d(z, Az) = 0$. Hence $z = Az$. From (3.6.1) we have

$$z = Az = Sz. \tag{3.6.4}$$

We show that $z = Bz$.

From (3), (3.6.3) and (3.6.4) we have

$$\begin{aligned}
&\alpha(d(z, Bz)) \\
=&\alpha(d(Az, Bz)) \\
\le&\phi(\alpha(max\{d(Sz, Tz), d(Bz, Tz), d(Bz, Sz)\})) \\
=&\phi(\alpha(max\{d(Bz, z), 0, d(z, Bz)\})) \\
=&\phi(\alpha(d(z, Bz)))
\end{aligned}$$

which implies $\alpha(d(z, Bz)) = 0$ and so $d(z, Bz) = 0$. Hence $z = Bz$. By (3.6.3), we have

$$z = Bz = Tz. \tag{3.6.5}$$

Therefore, by (3.6.4) and (3.6.4), we have $z = Az = Bz = Tz = Sz$.

For the uniqueness, let w be an another common fixed point of A, B, S and T. If $w \neq z$, then from (3) we get

$$\begin{aligned}
&\alpha(d(z,w))\\
=&\alpha(d(Az,Bw))\\
\leq&\phi(\alpha(max\{d(Sz,Tw),d(Bw,Tw),d(Bw,Sz)\}))\\
=&\phi(\alpha(max\{d(z,w),d(w,w),d(w,z)\}))\\
=&\phi(\alpha(max\{d(z,w),0,d(w,z)\}))\\
=&\phi(\alpha(d(z,w))\\
<&\alpha(d(z,w))
\end{aligned}$$

which is a contradiction. Thus we have $\alpha(d(z,w)) = 0$ and so $d(z,w) = 0$. Hence $w = z$. □

REMARK 3.7. In Theorem 3.6, if $\alpha(s) = \int_0^s \varphi(t)dt$ then we have the theorem 1 of [2] without condition $(W3)$ where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable, nonnegative and $\int_0^u \varphi(t)dt > 0$ for each $u > 0$.

REMARK 3.8. In the case of $A = B = g$ and $S = T = f$ in Theorem 3.1 and Theorem 3.6, we can show that f and g have a unique common fixed point without the condition (1), that is, $gX \subset fX$.

The following example shows that the axioms (H.E) and (C.C) can not be dropped in Theorem 3.1.

EXAMPLE 3.9. Let $X = [0, \infty)$ and let

$$d(x,y) = \begin{cases} |x-y| & (x \neq 0, y \neq 0), \\ \frac{1}{x} & (x \neq 0). \end{cases}$$

Then (X, d) is a symmetric space which satisfies (W4) but does not satisfy (H.E) for $x_n = n, y_n = n+1$. Also (X, d) does not satisfy (C.C).

Let $S = T = f$ and $A = B = g$ be self mappings of X defined as follows:

$$fx = x (x \geq 0) \text{ and } gx = \begin{cases} \frac{1}{3}x & (x > 0), \\ \frac{1}{3} & (x = 0). \end{cases}$$

Let $\alpha(s) = 2s$ for all $s \in [0, \infty)$ and $\phi(t) = \frac{1}{2}t$ for all $t \in [0, \infty)$.

Then the condition (3) of Theorem 3.1 is satisfied.

To show this, let $n(x,y) = max\{d(fx,fy), d(fx,gx), d(fy,gy), d(fy,gx), d(fx,gy)\}$. We consider four cases.

Case 1. $x = 0, y = 0$.

Obviousely, we have $\alpha(d(gx, gy)) \le \phi(\alpha(n(x, y)))$.

Case 2. $x = 0, 0 < y < 1$.

$\alpha(d(gx, gy)) = 2\frac{1}{3}|y-1| \le \frac{1}{2}\cdot 2\cdot 3 = \phi(\alpha(d(fx, gx))) \le \phi(\alpha(n(x, y)))$.

Case 3. $x = 0, y \ge 1$.

$\alpha(d(gx, gy)) = 2\frac{1}{3}|y - 1| \le \frac{1}{2} \cdot 2|y - \frac{1}{3}| = \phi(\alpha(d(fx, gy)))$
$\le \phi(\alpha(n(x, y)))$.

Case 4. $x > 0, y > 0 (x \ne y)$.

$\alpha(d(gx, gy)) = \frac{2}{3}|x-y| \le |x-y| = \phi(\alpha(d(fx, fy))) \le \phi(\alpha(n(x, y)))$.

Thus the condition (3) of Theorem 3.1 is satisfied. Note that fX is a d-closed($\tau(d)$-closed) subuset of X. Also, the pair (f, g) satisfies property (E-A) for $x_n = n$. Also, the pair (f, g) has no coincidence points and so (f, g) is weakly compatible but the pair (f, g) has no common fixed points.

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