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ROBUST BOUNDARY CONTROL OF CHEMOTAXIS REACTION DIFFUSION SYSTEM

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Abstract. This paper is concerned with the robust boundary control of the chemotaxis reaction diffusion system. That is, we show that the existence of the saddle point for the robust control problem when the control and the disturbance are given by the boundary condition.

1. Introduction

In this paper we consider the robust boundary control problem of the chemotaxis reaction diffusion system:

$$
\begin{aligned}
&\frac{\partial y}{\partial t} = a\frac{\partial^2 y}{\partial x^2} - b\frac{\partial}{\partial x}\Big(y\frac{\partial \rho}{\partial x}\Big) \qquad \text{in } (0,L)\times(0,T],\\
&\frac{\partial \rho}{\partial t} = d\frac{\partial^2 \rho}{\partial x^2} + fy - h\rho \qquad \text{in } (0,L)\times(0,T],\\
&\frac{\partial y}{\partial x}(0,t) = \frac{\partial y}{\partial x}(L,t) = \frac{\partial \rho}{\partial x}(0,t) = 0,\\
&\frac{\partial \rho}{\partial x}(L,t) = u(t) + \lambda(t) \qquad \text{on } (0,T],\\
&y(x,0) = y_0(x), \quad \rho(x,0) = \rho_0(x) \qquad \text{in } (0,L).
\end{aligned}
\tag{1.1}
$$

Here, $(0,L)$ is a bounded interval in $\mathbf{R}$. a, b, d, f, $h > 0$ are given positive numbers. $y = y(x,t)$ describes the cell concentration in $(0,L)$ at time t, and $\rho = \rho(x,t)$ the chemoattractant concentration in $(0,L)$ at time t. $u(t)$ and $\lambda(t)$ are the control function and the disturbance function. We refer to [6, 7, 8] and the references for (1.1).

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In this paper, we are to find the saddle point for the functional $J(u,\lambda)$ of the form

$$J(u,\lambda)=\int_0^T\|y(u,\lambda)-y_d\|^2_{H^1(0,L)}dt+\gamma\|u\|^2_{H^2(0,T)}-l\|\lambda\|^2_{H^2(0,T)}.$$

Optimal and robust control problems for nonlinear parabolic equations have already been published by many authors(see [2, 3, 4, 5, 7, 8, 9, 10, 11]). Ryu [11] has handled the optimal boundary control problem for the chemotaxis system. In [9, 10], Ryu and Yun studied the distributed robust control problems for the chemotaxis system with homogeneous boundary conditions. However, this paper is concerned with the robust boundary control problem for the chemotaxis system when the control and the disturbance are given by the boundary condition.

Notations and inequalities: **R** denotes the real line. Let $I=(0,l)$ be an interval in **R**. $L^p(I;\mathcal{H})$, $1\le p\le\infty$, denotes the L^p space of measurable functions in I with values in a Hilbert space $\mathcal{H}$. $\mathcal{C}(I;\mathcal{H})$ denotes the space of continuous functions in I with values in $\mathcal{H}$. For simplicity, we shall use a universal constant C to denote various constants which are determined in each occurrence in a specific way by a,b,d,f,h,L. In a case when C depends also on some parameter, say θ, it will be denoted by C_θ.

We shall state some inequalities on the Sobolev spaces ([1]). When $s>\frac{1}{2}$, $H^s(I)\subset\mathcal{C}(\bar{I})$ with the estimate

$$\|\cdot\|_{\mathcal{C}}\le C_s\|\cdot\|_{H^s}. \tag{1.2}$$

By (1.2), we observe that

$$\Big\|\frac{d}{dx}\Big(y\frac{d\rho}{dx}\Big)\Big\|_{(H^1)'}\le\begin{cases}C\|y\|_{L^\infty}\|\rho\|_{H^1}\\ C\|y\|_{L^2}\|\frac{\partial\rho}{\partial x}\|_{L^\infty}\end{cases}, \tag{1.3}$$

where $y\in H^1(I),\ \rho\in H^2_n(I)=\{\rho\in H^2(I):\frac{\partial\rho}{\partial x}(0)=\frac{\partial\rho}{\partial x}(l)=0\}$.

2. Existence of the solutions

In this section we recall the reformation of (1.1) as in [11]. First we construct a lifting function for the boundary conditions,

$$\phi(x,t)=m(t)\frac{x^2}{2L}.$$

Here, $m(t) = u(t)+\lambda(t)$ and $m \in H^2_\Gamma(0,T) = \{\alpha \in H^2(0,T) : \alpha(0) = 0\}$. Obviously

$$\left|\frac{\partial^i \phi(x,t)}{\partial x^i}\right| \le C|m(t)|, \quad \forall x \in (0,L), \ \forall t \in [0,T] \ (i = 0,1,2). \tag{2.1}$$

Let us set $w(x,t) = \rho(x,t) - \phi(x,t)$. Then the system (1.1) is equivalent to the one:

$$\begin{aligned}
&\frac{\partial y}{\partial t} = a\frac{\partial^2 y}{\partial x^2} - b\frac{\partial}{\partial x}\Big(y\frac{\partial (w+\phi)}{\partial x}\Big) && \text{in } (0,L)\times(0,T],\\
&\frac{\partial w}{\partial t} = d\frac{\partial^2 w}{\partial x^2} + fy - hw + g_m(x,t) && \text{in } (0,L)\times(0,T],\\
&\frac{\partial y}{\partial x}(0,t) = \frac{\partial y}{\partial x}(L,t) = \frac{\partial w}{\partial x}(0,t) = \frac{\partial w}{\partial x}(L,t) = 0 && \text{on } (0,T],\\
&y(x,0) = y_0(x), \quad w(x,0) = w_0 \quad \text{in } (0,L).
\end{aligned} \tag{2.2}$$

Here, $w_0 = \rho_0(x)$ and $g_m(x,t) = d\frac{\partial^2\phi}{\partial x^2} - h\phi - \frac{\partial \phi}{\partial t}$.

Let $A_1 = -a\frac{\partial^2}{\partial x^2} + a$ and $A_2 = -d\frac{\partial^2}{\partial x^2} + h$ with the same domain $\mathcal{D}(A_i) = H^2_n(0,L) = \{z \in H^2(0,L) : \frac{\partial z}{\partial x}(0) = \frac{\partial z}{\partial x}(L) = 0\}$ $(i = 1,2)$. Then, A_i are two positive definite self-adjoint operators in $L^2(0,L)$. We set two product Hilbert spaces $\mathcal{V} \subset \mathcal{H}$ as

$$\mathcal{V} = H^1(0,L)\times H^2_n(0,L), \quad \mathcal{H} = L^2(0,L)\times H^1(0,L).$$

By identifying $\mathcal{H}$ with its dual space, we consider $\mathcal{V} \subset \mathcal{H} = \mathcal{H}' \subset \mathcal{V}'$. It is then seen that

$$\mathcal{V}' = (H^1(0,L))' \times L^2(0,L).$$

We set also a symmetric bilinear form on $\mathcal{V}\times\mathcal{V}$:

$$a(Y,\widetilde{Y}) = a\int_0^L \frac{dy}{dx}\frac{d\tilde{y}}{dx}dx + a\int_0^L y\tilde{y}dx + \big(A_2^{1/2}w, A_2^{1/2}\tilde{w}\big)_{L^2},$$

where $Y = \binom{y}{w}, \widetilde{Y} = \binom{\tilde{y}}{\tilde{w}} \in \mathcal{V}$. Obviously, the form satisfies A is the positive definite self-adjoint operator on $\mathcal{H}$ defined by a symmetric bilinear form $a(Y,\widetilde{Y})$ on $\mathcal{V}$, $\langle AY,\widetilde{Y}\rangle_{\mathcal{V},\mathcal{V}'} = a(Y,\widetilde{Y})$, which satisfies

$$|a(Y,\widetilde{Y})| \le M\|Y\|_{\mathcal{V}}\|\widetilde{Y}\|_{\mathcal{V}}, \quad Y,\widetilde{Y} \in \mathcal{V}, \tag{a.i}$$

$$a(Y,Y) \ge \delta\|Y\|^2_{\mathcal{V}}, \quad Y \in \mathcal{V} \tag{a.ii}$$

with some δ and $M > 0$. This form then defines a linear isomorphism $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ from $\mathcal{V}$ to $\mathcal{V}'$, and the part of A in $\mathcal{H}$ is a positive definite self-adjoint operator in $\mathcal{H}$. Let U be a closed bounded convex subset in $H^2_\Gamma(0,T)$.

(2.2) is, then, formulated as an abstract equation

$$\frac{dY}{dt} + AY = F_m(Y) + G_m(t), \quad 0 < t \le T, \qquad Y(0) = Y_0 \tag{2.3}$$

in the space $\mathcal{V}'$. Here, $F_m(\cdot) : \mathcal{V} \to \mathcal{V}'$ is the mapping

$$F_m(Y) = \begin{pmatrix} ay - b\frac{\partial}{\partial x}\left(y\frac{\partial(w+\phi)}{\partial x}\right) \\ fy \end{pmatrix} \quad \text{and} \quad G_m(t) = \begin{pmatrix} 0 \\ g_m(x,t) \end{pmatrix}.$$

Y_0 is defined by $Y_0 = \binom{y_0}{w_0}$.

For all $m \in U$, $F_m(\cdot)$ satisfies the following conditions(see [11]):

For each $\eta > 0$, there exists an increasing continuous function $\mu_\eta : [0,\infty) \to [0,\infty)$ such that

(f.i) $\quad \|F_m(Y)\|_{\mathcal{V}'} \le \eta\|Y\|_{\mathcal{V}} + \mu_\eta(\|Y\|_{\mathcal{H}}), \ Y \in \mathcal{V}, \ \text{a.e. } (0,T);$

(f.ii) $\quad \|F_m(\widetilde{Y}) - F_m(Y)\|_{\mathcal{V}'} \le \eta\|\widetilde{Y} - Y\|_{\mathcal{V}}$
$+(\|\widetilde{Y}\|_{\mathcal{V}}+\|Y\|_{\mathcal{V}}+1)\mu_\eta(\|\widetilde{Y}\|_{\mathcal{H}}+\|Y\|_{\mathcal{H}})\|\widetilde{Y}-Y\|_{\mathcal{H}}, \ \widetilde{Y}, Y \in \mathcal{V}, \ \text{a.e. } (0,T).$

Since $m \in U$, we see that $G_m(\cdot) \in L^2(0,T;\mathcal{V}')$. We then obtain the following result (For the proof, see Ryu and Yagi [8]).

THEOREM 2.1. *If $Y_0 \in \mathcal{H}$, there exists a unique weak solution*

$$Y \in H^1(0,S;\mathcal{V}') \cap \mathcal{C}([0,S];\mathcal{H}) \cap L^2(0,S;\mathcal{V})$$

to (2.3), the number $S \in (0,T]$ is determined by the norm $\|G_m\|_{L^2(0,T;\mathcal{V}')}$ and $\|Y_0\|_{\mathcal{H}}$.

3. Existence of the robust control

Let U_{ad} and V_{ad} be closed, convex, bounded subsets of $H^2_\Gamma(0,T)$. Let $S > 0$ be such that for each $(u,\lambda) \in U_{ad} \times V_{ad}$, (2.3) has a unique weak solution $Y(u,\lambda) \in H^1(0,S;\mathcal{V}') \cap \mathcal{C}([0,S];\mathcal{H}) \cap L^2(0,S;\mathcal{V})$. Then, our problem is obviously formulated as follows:

$$(\mathbf{P}) \qquad J(\bar{u},\lambda) \le J(\bar{u},\bar{\lambda}) \le J(u,\bar{\lambda}) \quad \forall (u,\lambda) \in U_{ad} \times V_{ad},$$

where

$$J(u,\lambda) = \int_0^S \|DY(u,\lambda) - Y_d\|_{\mathcal{V}}^2 dt + \gamma\|u\|_{H^2(0,S)}^2 - l\|\lambda\|_{H^2(0,S)}^2.$$

Here, $D\binom{y}{w} = \binom{y}{0}$ and $Y_d = \binom{y_d}{0}$ is a fixed element of $L^2(0,S;\mathcal{V})$ with $y_d \in L^2(0,S;H^1(0,L))$. γ and l are positive constants.

The mapping $F_{u,\lambda}(\cdot) : \mathcal{V} \to \mathcal{V}'$ is defined by

$$F_{u,\lambda}(Y) = \begin{pmatrix} ay - b\frac{\partial}{\partial x}\left(y\frac{\partial(w+\phi)}{\partial x}\right) \\ fy \end{pmatrix}, \qquad Y = \begin{pmatrix} y \\ w \end{pmatrix} \in \mathcal{V},$$

where $\phi(x,t) = (u(t) + \lambda(t))\frac{x^2}{2L}$. Then, we have the following result.

LEMMA 3.1. *Let* (u_1,λ_1) *and* (u_2,λ_2) *in* $U_{ad} \times V_{ad}$. *Let* $Y_1 = Y(u_1,\lambda_1)$ *and* $Y_2 = Y(u_2,\lambda_2)$ *be solutions of* (2.3) *with respect to* (u_1,λ_1) *and* (u_2,λ_2), *respectively. Then, we have*

$$\|Y_1(t) - Y_2(t)\|_{\mathcal{H}}^2 + \delta\int_0^t \|Y_1(s) - Y_2(s)\|_{\mathcal{V}}^2 ds \le C(\|u_1(t) - u_2(t)\|_{H^2(0,S)}^2 + \|\lambda_1(t) - \lambda_2(t)\|_{H^2(0,S)}^2), \quad 0 \le t \le S. \tag{3.1}$$

Proof. The proof is similar to that of [11, Theorem 3.2]. □

Moreover, the mapping $F_{u,\lambda}(\cdot) : \mathcal{V} \to \mathcal{V}'$ must be a first-order Fréchet differentiable with the derivative

$$F'_{u,\lambda}(Y)Z = \begin{pmatrix} az_1 - b\frac{\partial}{\partial x}\left(z_1\frac{\partial(w+\phi)}{\partial x}\right) - b\frac{\partial}{\partial x}\left(y\frac{\partial z_2}{\partial x}\right) \\ fz_1 \end{pmatrix},$$

where $Y = \binom{y}{w}$, $Z = \binom{z_1}{z_2} \in \mathcal{V}$. Then, we have the following conditions.

LEMMA 3.2. *For each* $\eta > 0$, *there exists an increasing continuous functions* $\nu_\eta : [0,\infty) \to [0,\infty)$ *such that for* $Y, \widetilde{Y}, Z, P \in \mathcal{V}$,

$$\text{(f.iii)} \quad |\langle F'_{u,\lambda}(Y)Z, P\rangle_{\mathcal{V}'\times\mathcal{V}}| \le \begin{cases} \eta\|Z\|_{\mathcal{V}}\|P\|_{\mathcal{V}} + (\|Y\|_{\mathcal{V}} + 1)\nu_\eta(\|Y\|_{\mathcal{H}})\|Z\|_{\mathcal{H}}\|P\|_{\mathcal{V}}, \text{ a.e. } (0,S), \\ \eta\|Z\|_{\mathcal{V}}\|P\|_{\mathcal{V}} + (\|Y\|_{\mathcal{V}} + 1)\nu_\eta(\|Y\|_{\mathcal{H}})\|Z\|_{\mathcal{V}}\|P\|_{\mathcal{H}}, \text{ a.e. } (0,S), \end{cases}$$

$$\text{(f.iv)} \quad \|F'_{u,\lambda}(\widetilde{Y})Z - F'_{u,\lambda}(Y)Z\|_{\mathcal{V}'} \le C\|\widetilde{Y} - Y\|_{\mathcal{H}}\|Z\|_{\mathcal{V}}, \text{ a.e. } (0,S).$$

Proof. By (1.3) and (2.1), it is seen that

$$\begin{aligned}\Big\|\frac{\partial}{\partial x}\Big(z_1\frac{\partial(w+\phi)}{\partial x}\Big)+\frac{\partial}{\partial x}\Big(y\frac{\partial z_2}{\partial x}\Big)\Big\|_{(H^1)'}&\\ \le C(\|z_1\|_{L^\infty}\|w+\phi\|_{H^1}+\|y\|_{H^1}\|z_2\|_{H^1})&\\ \le C(\|z_1\|_{H^1}^{1/2}\|z_1\|_{L^2}^{1/2}(\|w\|_{H^1}+\|\phi\|_{H^1})+\|y\|_{H^1}\|z_2\|_{H^1})&\\ \le \varepsilon\|z_1\|_{H^1}+C_\varepsilon(\|y\|_{H^1}+1)(\|w\|_{H^1}^2+1)(\|z_1\|_{L^2}+\|z_2\|_{H^1})&\end{aligned}$$

with an arbitrary $\varepsilon>0$. Therefore, (f.iii) holds.

On the other hand, by (1.2) and (1.3),

$$\begin{aligned}\Big\|\frac{\partial}{\partial x}\Big((y-\tilde y)\frac{\partial z_2}{\partial x}\Big)-\frac{\partial}{\partial x}\Big(z_1\frac{\partial(w-\tilde w)}{\partial x}\Big)\Big\|_{(H^1)'}&\\ \le C(\|y-\tilde y\|_{L^2}\|z_2\|_{H^2}+\|w-\tilde w\|_{H^1}\|z_1\|_{H^1}).&\end{aligned}$$

Therefore, (f.iv) holds. □

Proposition 3.3. *For any fixed $\lambda\in V_{ad}$, the mapping $u\to Y(u,\lambda)$ from U_{ad} into $H^1(0,S;\mathcal{V}')\cap L^2(0,S;\mathcal{V})$ is differentiable in the sense*

$$\frac{Y(u+hv,\lambda)-Y(u,\lambda)}{h}\to Z \text{ in } H^1(0,S;\mathcal{V}')\cap L^2(0,S;\mathcal{V})$$

as $h\to 0$, where $u,v\in U_{ad}$ and $u+hv\in U_{ad}$. Moreover, $Z=Z(u,\lambda;v,0)$ satisfies the linear equation

$$(3.2)\quad \begin{aligned}&\frac{dZ}{dt}+AZ-F'_{u,\lambda}(Y(u,\lambda))Z=B_v(Y(u,\lambda))+G_v(t),\quad 0<t\le S,\\ &Z(0)=0,\end{aligned}$$

where $B_v(Y(u,\lambda))=\begin{pmatrix}-b\frac{\partial}{\partial x}\big(y\frac{\partial\phi_v}{\partial x}\big)\\ 0\end{pmatrix}$ *and* $\phi_v=v(t)\frac{x^2}{2L}$.

Proof. Let $u,v\in U_{ad}$ and $0\le h\le 1$. Let $Y_h=Y(u_h,\lambda)$ and $Y=Y(u,\lambda)$ be the solutions of (2.3) corresponding to $u_h=u+hv$ and u, respectively.

Step 1. $\Big\{\frac{Y_h-Y}{h}\Big\}_{h>0}$ *is bounded in $H^1(0,S;\mathcal{V}')\cap L^2(0,S;\mathcal{V})$.* Let $\widetilde Y=\frac{Y_h-Y}{h}$. We consider

$$(3.3)\quad \begin{aligned}&\frac{d\widetilde Y}{dt}+A\widetilde Y-\frac{F_{u_h,\lambda}(Y_h)-F_{u,\lambda}(Y)}{h}=G_v(t),\qquad 0<t\le S,\\ &\widetilde Y(0)=0.\end{aligned}$$

By (1.3) and (2.1), we have

$$(3.4)\quad \Big\|\frac{F_{u_h,\lambda}(Y_h)-F_{u,\lambda}(Y_h)}{h}\Big\|_{\mathcal{V}'}=\Big\|\begin{pmatrix}-b\frac{\partial}{\partial x}\Big(y_h\frac{\partial\phi_v}{\partial x}\Big)\\ 0\end{pmatrix}\Big\|_{\mathcal{V}'}$$

$$=\Big\|b\frac{\partial}{\partial x}\Big(y_h\frac{\partial\phi_v}{\partial x}\Big)\Big\|_{(H^1)'}\le C\|y_h\|_{L^2}\Big\|\frac{\partial\phi_v}{\partial x}\Big\|_{L^\infty}\le C\|Y_h\|_{\mathcal{H}},\ \text{a.e. } (0,S)$$

and

$$(3.5)\quad \Big\|\frac{F_{u,\lambda}(Y_h)-F_{u,\lambda}(Y)}{h}\Big\|_{\mathcal{V}'}$$

$$=\Big\|\begin{pmatrix}a\frac{y_h-y}{h}-b\frac{\partial}{\partial x}\Big(\frac{y_h-y}{h}\frac{\partial w_h}{\partial x}\Big)-b\frac{\partial}{\partial x}\Big(y\frac{\partial}{\partial x}\frac{w_h-w}{h}\Big)-b\frac{\partial}{\partial x}\Big(\frac{y_h-y}{h}\frac{\partial\phi}{\partial x}\Big)\\ f\frac{y_h-y}{h}\end{pmatrix}\Big\|_{\mathcal{V}'}$$

$$\le C\Big(\Big\|\frac{y_h-y}{h}\Big\|_{L^2}\|w_h\|_{H^2}+\|y\|_{H^1}\Big\|\frac{w_h-w}{h}\Big\|_{H^1}+\Big\|\frac{y_h-y}{h}\Big\|_{L^2}\Big)$$

$$\le C(\|Y_h\|_{\mathcal{V}}+\|Y\|_{\mathcal{V}}+1)\|\widetilde{Y}\|_{\mathcal{H}},\ \text{a.e. } (0,S).$$

Taking the scalar product with $\widetilde{Y}$ to (3.3) and using (3.4), (3.5), we obtain that

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{Y}(t)\|^2_{\mathcal{H}}+\frac{\delta}{2}\|\widetilde{Y}(t)\|^2_{\mathcal{V}}\le C(\|Y_h(t)\|^2_{\mathcal{V}}+\|Y(t)\|^2_{\mathcal{V}}+1)\|\widetilde{Y}(t)\|^2_{\mathcal{H}}$$
$$+C(\|Y_h(t)\|^2_{\mathcal{H}}+\|G_v(t)\|^2_{\mathcal{V}'}).$$

Using Gronwall's inequality, we obtain that

$$\|\widetilde{Y}(t)\|^2_{\mathcal{H}}+\delta\int_0^t\|\widetilde{Y}(t)\|^2_{\mathcal{V}}ds$$
$$\le C\Big(\|Y_h(t)\|^2_{L^\infty(0,S;\mathcal{H})}+\|G_v(t)\|^2_{L^2(0,S;\mathcal{V}')}\Big)e^{\int_0^S C(\|Y_h(t)\|^2_{\mathcal{V}}+\|Y(t)\|^2_{\mathcal{V}}+1)ds}$$

for all $t\in[0,S]$. Since $v\in U_{ad}$, $G_v(t)\in L^2(0,S;\mathcal{V}')$. Hence, $\frac{Y_h-Y}{h}$ is bounded in $H^1(0,S;\mathcal{V}')\cap L^2(0,S;\mathcal{V})$.

Step 2. *$\frac{Y_h-Y}{h}$ converges weakly to the unique solution $Z=\binom{z_1}{z_2}$ of (3.2) in $H^1(0,S;\mathcal{V}')\cap L^2(0,S;\mathcal{V})$ as $h\to 0$.* From Step 1, we see that

$$\frac{Y_h-Y}{h}\to\overline{Z}\ \text{weakly in } H^1(0,S;\mathcal{V}')\cap L^2(0,S;\mathcal{V})$$

and

$$\frac{Y_h - Y}{h} \to \overline{Z} \text{ strongly in } L^2(0,S;\mathcal{H}) \tag{3.6}$$

as $h \to 0$. Let us verify that $\overline{Z} = \binom{\bar{z}_1}{\bar{z}_2}$ is a solution of (3.2). First, we show that for $\Psi(t) = \binom{\psi_1(t)}{\psi_2(t)} \in \mathcal{C}([0,S];\mathcal{V})$,

(3.7)

$$\int_0^S \Big\langle \frac{F_{u,\lambda}(Y_h) - F_{u,\lambda}(Y)}{h}, \Psi(t) \Big\rangle_{\mathcal{V}',\mathcal{V}} dt \to \int_0^S \langle F'_{u,\lambda}(Y)\overline{Z}, \Psi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt$$

as $h \to 0$. Indeed, by direct calculation

$$\begin{aligned} &\frac{F_{u,\lambda}(Y_h) - F_{u,\lambda}(Y)}{h} - F'_{u,\lambda}(Y)\overline{Z} \\ &\quad = \begin{pmatrix} a\tilde{w}_1 - b\frac{\partial}{\partial x}\Big(\tilde{w}_1 \frac{\partial (w_h + \phi)}{\partial x}\Big) - b\frac{\partial}{\partial x}\Big(y\frac{\partial \tilde{w}_2}{\partial x}\Big) - b\frac{\partial}{\partial x}\Big(\bar{z}_1 \frac{\partial (w_h - w)}{\partial x}\Big) \\ f\tilde{w}_1 \end{pmatrix}, \end{aligned}$$

where $\tilde{w}_1 = \frac{y_h - y}{h} - \bar{z}_1$ and $\tilde{w}_2 = \frac{w_h - w}{h} - \bar{z}_2$. For $\psi_1 \in \mathcal{C}([0,S];H^1(0,L))$,

$$\begin{aligned} \int_0^S &\Big\langle \frac{\partial}{\partial x}\Big(\tilde{w}_1 \frac{\partial (w_h + \phi)}{\partial x}\Big) + \frac{\partial}{\partial x}\Big(y \frac{\partial \tilde{w}_2}{\partial x}\Big), \psi_1 \Big\rangle_{(H^1)',H^1} dt \\ &\le C(\|\tilde{w}_1\|_{L^2(0,S;L^2)}(\|w_h\|_{L^2(0,S;H^2)} + 1)\|\psi_1\|_{\mathcal{C}([0,S];H^1)} \\ &\quad + \|y\|_{L^2(0,S;H^1)}\|\tilde{w}_2\|_{L^2(0,S;H^1)}\|\psi_1\|_{\mathcal{C}([0,S];H^1)}) \end{aligned}$$

and

$$\begin{aligned} \int_0^S &\Big\langle \frac{\partial}{\partial x}\Big(\bar{z}_1 \frac{\partial (w_h - w)}{\partial x}\Big), \psi_1 \Big\rangle_{(H^1)',H^1} dt \\ &\le C\|\bar{z}_1\|_{L^2(0,S;H^1)}\|w_h - w\|_{L^2(0,S;H^1)}\|\psi_1\|_{\mathcal{C}([0,S];H^1)}. \end{aligned}$$

From (3.1) and (3.6), it is seen that (3.7) holds.

Moreover, since

$$\begin{aligned} \int_0^S &\Big\langle \frac{\partial}{\partial x}\Big((y_h - y)\frac{\partial \phi_v}{\partial x}\Big), \psi_1 \Big\rangle_{(H^1)',H^1} dt \\ &\le C\|y_h - y\|_{L^2(0,S;L^2)}\|\psi_1\|_{L^2(0,S;H^1)}, \end{aligned}$$

it is seen from (3.1) that

$$\int_0^S \Big\langle \frac{F_{u_h,\lambda}(Y_h) - F_{u,\lambda}(Y_h)}{h}, \Psi(t) \Big\rangle_{\mathcal{V}',\mathcal{V}} dt \to \int_0^S \langle B_v(Y), \Psi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt.$$

By the uniqueness, we see that $\overline{Z} = Z$. Hence, $\frac{Y_h - Y}{h}$ converges weakly to the unique solution Z of (3.2) in $H^1(0, S; \mathcal{V}') \cap L^2(0, S; \mathcal{V})$ as $h \to 0$.

Step 3. $\frac{Y_h - Y}{h} \to Z$ *strongly in* $H^1(0, S; \mathcal{V}') \cap L^2(0, S; \mathcal{V})$ *as* $h \to 0$. $\widetilde{W} = \frac{Y_h - Y}{h} - Z$ satisfies

$$\tag{3.8} \begin{aligned} &\frac{dW}{dt} + AW - \Big(\frac{F_{u,\lambda}(Y_h) - F_{u,\lambda}(Y)}{h} - F'_{u,\lambda}(Y)Z\Big) \\ &\qquad = \Big(\frac{F_{u_h,\lambda}(Y_h) - F_{u,\lambda}(Y_h)}{h} - B_v(Y)\Big), \qquad 0 < t \le S, \\ &W(0) = 0. \end{aligned}$$

Applying (1.2), (1.3) and (2.1), we obtain that

$$\tag{3.9} \begin{aligned} &\Big\| \frac{F_{u,\lambda}(Y_h) - F_{u,\lambda}(Y)}{h} - F'_{u,\lambda}(Y)Z \Big\|_{\mathcal{V}'} \\ &\le C(\|Y_h\|_{\mathcal{V}} + \|Y\|_{\mathcal{V}} + 1)\|\widetilde{W}\|_{\mathcal{H}} + C\|Y_h - Y\|_{\mathcal{H}}\|Z\|_{\mathcal{V}}, \text{ a.e. } (0, S) \end{aligned}$$

and

$$\tag{3.10} \Big\| \frac{F_{u_h,\lambda}(Y_h) - F_{u,\lambda}(Y_h)}{h} - B_v(Y) \Big\|_{\mathcal{V}'} \le C\|Y_h - Y\|_{\mathcal{H}}, \text{ a.e. } (0, S).$$

Taking the scalar product of the equation of (3.8) with $\widetilde{W}$ and using (3.9), (3.10), we obtain that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\widetilde{W}(t)\|^2_{\mathcal{H}} + \frac{\delta}{2}\|\widetilde{W}(t)\|^2_{\mathcal{V}} &\le C(\|Y_h(t)\|^2_{\mathcal{V}} + \|Y(t)\|^2_{\mathcal{V}} + 1)\|\widetilde{W}(t)\|^2_{\mathcal{H}} \\ &\quad + C(\|Z(t)\|^2_{\mathcal{V}} + 1)\|Y_h(t) - Y(t)\|^2_{\mathcal{H}}. \end{aligned}$$

From Gronwall's inequality,

$$\|\widetilde{W}(t)\|^2_{\mathcal{H}} + \delta \int_0^t \|\widetilde{W}(s)\|^2_{\mathcal{V}} ds \le C\|Y_h(t) - Y(t)\|^2_{L^\infty(0,S;\mathcal{H})}(\|Z\|^2_{L^2(0,S;\mathcal{V})} + 1).$$

Since $Y_h \to Y$ strongly in $L^\infty(0, S; \mathcal{H})$, it follows that $\frac{Y_h - Y}{h}$ is strongly convergent to Z in $H^1(0, S; \mathcal{V}') \cap L^2(0, S; \mathcal{V})$. □

Proposition 3.4. *The solution* $Z(u, \lambda; v, 0)$ *of* (3.2) *satisfies the estimates* ($\forall u, v, u_1, u_2 \in U_{ad}$):

$$\tag{3.11} \|Z(u, \lambda; v, 0)\|_{L^\infty(0,S;\mathcal{H}) \cap L^2(0,S;\mathcal{V})} \le C\|v(t)\|_{H^2(0,S)},$$

$$\tag{3.12} \begin{aligned} &\|Z(u_1, \lambda; v, 0) - Z(u_2, \lambda; v, 0)\|_{L^2(0,S;\mathcal{V})} \\ &\qquad \le C\|u_1(t) - u_2(t)\|_{H^2(0,S)}\|v(t)\|_{H^2(0,S)}. \end{aligned}$$

Proof. Let $Z = Z(u, \lambda; v, 0)$ be the solution of (3.2). From (1.3) and (2.1), we have

$$\|B_v(Y)\|_{\mathcal{V}'} = \Big\| b\frac{\partial}{\partial x}\Big(y\frac{\partial \phi_v}{\partial x}\Big)\Big\|_{(H^1)'} \le \|y\|_{L^2}\Big\|\frac{\partial \phi_v}{\partial x}\Big\|_{L^\infty} \le C|v(t)|\|Y\|_{\mathcal{H}} \tag{3.13}$$

and

$$\|G_v(t)\|_{\mathcal{V}'} = \|g_v(x,t)\|_{L^2} \le C\Big(|v(t)| + \Big|\frac{dv(t)}{dt}\Big|\Big). \tag{3.14}$$

Taking the scalar product with Z to (3.2) and using (f.iii), (3.13), (3.14) we have

$$\frac{d}{dt}\|Z(t)\|^2_{\mathcal{H}} + \delta\|Z(t)\|^2_{\mathcal{V}} \le (\|Y(u,\lambda)\|^2_{\mathcal{V}} + 1)\tilde{\nu}(\|Y(u,\lambda)\|^2_{\mathcal{H}})\|Z(t)\|^2_{\mathcal{H}} + C(1 + \|Y(t)\|^2_{\mathcal{H}})\Big(|v(t)|^2 + \Big|\frac{dv(t)}{dt}\Big|^2\Big),$$

where $\tilde{\nu} : [0,\infty) \to [0,\infty)$ is some increasing continuous function. Using Gronwall's inequality, we obtain

$$\|Z(t)\|^2_{\mathcal{H}} + \delta\int_0^S \|Z(t)\|^2_{\mathcal{V}} dt \le C\|v(t)\|^2_{H^2(0,S)} e^{\int_0^S (\|Y(u,\lambda)\|^2_{\mathcal{V}}+1)\tilde{\nu}(\|Y(u,\lambda)\|^2_{\mathcal{H}})ds} \le C\|v(t)\|^2_{H^2(0,S)}.$$

Hence, (3.11) is verified.

On the other hand, let $Z_i = Z(u_i, \lambda; v, 0)$ $(i = 1, 2)$ be the solutions of

$$\frac{dZ_i}{dt} + AZ_i - F'_{u_i,\lambda}(Y_i)Z_i = B_v(Y_i) + G_v(t), \qquad 0 < t \le S,$$
$$Z_i(0) = 0,$$

where $Y_1 = Y(u_1, \lambda)$ and $Y_2 = Y(u_2, \lambda)$ are the solution of (2.3) with respect to (u_1, λ) and (u_2, λ), respectively. Then $Z_3 = Z_1 - Z_2$ satisfies the equation

$$\frac{dZ_3}{dt} + AZ_3 - F'_{u_1,\lambda}(Y_1)Z_3 = (F'_{u_1,\lambda}(Y_1) - F'_{u_2,\lambda}(Y_1))Z_2 + (F'_{u_2,\lambda}(Y_1) - F'_{u_2,\lambda}(Y_2))Z_2 + B_v(Y_1) - B_v(Y_2), \tag{3.15}$$
$$Z_3(0) = 0.$$

From (1.3) and (2.1), we have

$$\|(F'_{u_1,\lambda}(Y_1) - F'_{u_2,\lambda}(Y_1))Z_2\|_{\mathcal{V}'} \le C|u_1(t) - u_2(t)|\|Z_2\|_{\mathcal{H}}, \tag{3.16}$$

$$\|B_v(Y_1) - B_v(Y_2)\|_{\mathcal{V}'} \le C|v(t)|\|Y_1 - Y_2\|_{\mathcal{H}}. \tag{3.17}$$

Taking the scalar product with Z_3 to (3.15) and using (f.iii), (f.iv), (3.16), (3.17), we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|Z_3(t)\|^2_{\mathcal{H}} + \frac{\delta}{2}\|Z_3(t)\|^2_{\mathcal{V}} \\
&\le (\|Y_1(t)\|^2_{\mathcal{V}} + 1)\tilde{\nu}(\|Y_1(t)\|^2_{\mathcal{H}})\|Z_3(t)\|^2_{\mathcal{H}} + C(\|Z_2(t)\|^2_{\mathcal{V}}\|Y_1(t) - Y_2(t)\|^2_{\mathcal{H}} \\
&\qquad + |u_1(t) - u_2(t)|^2\|Z_2\|^2_{\mathcal{H}} + |v(t)|^2\|Y_1(t) - Y_2(t)\|^2_{\mathcal{H}}),
\end{aligned}$$

where $\tilde{\nu} : [0,\infty) \to [0,\infty)$ is some increasing continuous function. Using Gronwall's inequality and applying (3.1) and (3.11),

$$\begin{aligned}
\|Z_3(t)\|^2_{\mathcal{H}} + \delta\int_0^t \|Z_3(s)\|^2_{\mathcal{V}}ds \le & C\Big(\|Y_1(t) - Y_2(t)\|^2_{L^\infty(0,S;\mathcal{H})}\|Z_2\|^2_{L^2(0,S;\mathcal{V})} \\
& + \|u_1(t) - u_2(t)\|^2_{L^2(0,S)}\|Z_2\|^2_{L^\infty(0,S;\mathcal{H})} \\
& + \|v(t)\|^2_{L^2(0,S)}\|Y_1(t) - Y_2(t)\|^2_{L^\infty(0,S;\mathcal{H})}\Big) \\
\le & C\|u_1(t) - u_2(t)\|^2_{H^2(0,S)}\|v(t)\|^2_{H^2(0,S)}
\end{aligned}$$

for all $t \in [0,S]$. Hence, (3.12) is verified. □

PROPOSITION 3.5. *For any fixed $u \in U_{ad}$, the mapping $\lambda \to Y(u,\lambda)$ from V_{ad} into $H^1(0,S;\mathcal{V}') \cap L^2(0,S;\mathcal{V})$ is differentiable in the sense*

$$\frac{Y(u,\lambda + h\tilde{\lambda}) - Y(u,\lambda)}{h} \to \widetilde{Z} \text{ in } H^1(0,S;\mathcal{V}') \cap L^2(0,S;\mathcal{V})$$

as $h \to 0$, for $\lambda, \tilde{\lambda} \in V_{ad}$ and $\lambda + h\tilde{\lambda} \in V_{ad}$. Moreover, $\widetilde{Z} = \widetilde{Z}(u,\lambda;0,\tilde{\lambda})$ satisfies the linear equation

$$\begin{aligned}
&\frac{d\widetilde{Z}}{dt} + A\widetilde{Z} - F'_{u,\lambda}(Y(u,\lambda))\widetilde{Z} = B_{\tilde{\lambda}}(Y(u,\lambda)) + G_{\tilde{\lambda}}(t), \qquad 0 < t \le S, \\
&\widetilde{Z}(0) = 0.
\end{aligned}$$

Moreover, the solution $\widetilde{Z}(u,\lambda;0,\tilde{\lambda})$ satisfies the estimates ($\forall\tilde{\lambda}, \lambda, \lambda_1, \lambda_2 \in V_{ad}$):

$$\begin{aligned}
\|\widetilde{Z}(u,\lambda;0,\tilde{\lambda})\|_{L^\infty(0,S;\mathcal{H})\cap L^2(0,S;\mathcal{V})} &\le C\|\tilde{\lambda}\|_{H^2(0,S)}, \\
\|\widetilde{Z}(u,\lambda_1;0,\tilde{\lambda}) - \widetilde{Z}(u,\lambda_2;0,\tilde{\lambda})\|_{L^2(0,S;\mathcal{V})} &\le C\|\lambda_1 - \lambda_2\|_{H^2(0,S)}\|\tilde{\lambda}\|_{H^2(0,S)}.
\end{aligned}$$

Proof. The proof is similar to that of Proposition 3.3 and Proposition 3.4. □

PROPOSITION 3.6. *There exist $\bar{\gamma}$ and $\bar{l}$ such that, for $\gamma \geq \bar{\gamma}$ and $l \geq \bar{l}$, we have*
(1) $\forall \lambda \in V_{ad}$, $u \to J(u, \lambda)$ is convex lower semicontinuous,
(2) $\forall u \in U_{ad}$, $\lambda \to J(u, \lambda)$ is concave upper semicontinuous.

Proof. Let J_1 be the map $u \to J(u, \lambda)$ and let J_2 be the map $\lambda \to J(u, \lambda)$. To obtain the existence of the robust control problem, we prove that J_1 is convex and lower semicontinuous for all $\lambda \in V_{ad}$, and J_2 is concave and upper semicontinuous for all $u \in U_{ad}$. Firstly we prove that the convexity of J_1 and the concavity of J_2. In order to prove the convexity, it is enough to prove that we have

$$J_1'(u)(u-v) - J_1'(v)(u-v) \geq 0$$

for all $u, v \in U_{ad}$([2], [12]). According to the definition of J_1,

(3.18)
$$J_1'(u)(u-v) - J_1'(v)(u-v) = \int_0^S \langle Z_1 - Z_2, D^*\mathcal{J}(DY_2 - Y_d)\rangle_{\mathcal{V},\mathcal{V}'} dt + \int_0^S \langle D(Y_1 - Y_2), DZ_2\rangle_{\mathcal{V}} dt + \gamma\langle u-v, u-v\rangle_{H^2(0,S)}.$$

Here, $\mathcal{J} : \mathcal{V} \to \mathcal{V}'$ is a canonical isomorphism and $Z_1 = Z(u, \lambda; u-v, 0)$ and $Z_2 = Z(v, \lambda; u-v, 0)$ satisfies

$$\begin{aligned} &\frac{dZ_i}{dt} + AZ_i - F'_{u,\lambda}(Y_i)Z_i = B_{u-v}(Y_i) + G_{u-v}(t), \qquad 0 < t \leq S, \\ &Z_i(0) = 0, \end{aligned}$$

where $Y_1 = Y(u, \lambda)$ and $Y_2 = Y(v, \lambda)$. According to (3.1), (3.11) and (3.12), we have

$$\text{(3.19)} \quad \int_0^S \langle Z_1 - Z_2, D^*\mathcal{J}(DY_2 - Y_d)\rangle_{\mathcal{V},\mathcal{V}'} dt \leq \Big(\int_0^S \|Z_1 - Z_2\|_{\mathcal{V}}^2 dt\Big)^{1/2} \times \|D^*\| \Big(\int_0^S \|(DY_2 - Y_d)\|_{\mathcal{V}}^2 dt\Big)^{1/2} \leq C_1 \|u(t) - v(t)\|_{H^2(0,S)}^2$$

and

$$(3.20)\quad \int_0^S \langle D(Y_1-Y_2), DZ_2\rangle_{\mathcal{V}} dt \le \|D\|^2\Big(\int_0^S \|(Y_1-Y_2)\|_{\mathcal{V}}^2 dt\Big)^{1/2} \times \Big(\int_0^S \|Z_2\|_{\mathcal{V}}^2 dt\Big)^{1/2} \le C_2\|u-v\|^2_{H^2(0,S)}.$$

From (3.18), (3.19) and (3.20), we have under assumption $\gamma \ge \bar\gamma = C_1 + C_2$,

$$J_1'(u)(u-v) - J_1'(v)(u-v) \ge 0$$

and obtain the convexity of J_1. By using Proposition 3.5, we can find $\bar l$ such that for $l \ge \bar l$ we have the concavity of J_2.

As proved in [9, Proposition 3.6], we can obtain the lower semicontinuous of J_1 for all $\lambda \in V_{ad}$ and the upper semicontinuous of J_2 for all $u \in U_{ad}$. □

Then, we have the following result.

Theorem 3.7. *Assume that U_{ad} and V_{ad} are non-empty, closed, convex, bounded subsets of $H^2_\Gamma(0,S)$ and $\gamma \ge \bar\gamma$ and $l \ge \bar l$. Then, there exists a saddle point $(\bar u, \bar\lambda)$ such that*

$$J(\bar u, \lambda) \le J(\bar u, \bar\lambda) \le J(u, \bar\lambda) \quad \forall (u,\lambda) \in U_{ad} \times V_{ad}.$$

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