

MULTIPLICITY RESULTS FOR THE ELLIPTIC SYSTEM USING THE MINIMAX THEOREM

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ABSTRACT. In this paper, we consider an elliptic system of three equations using the minimax theorem. We prove the existence of two solutions for suitable forcing terms, under a condition on the linear part which prevents resonance with eigenvalues of the operator.

1. Introduction

In this work we consider the problem

$$(1) \quad \begin{cases} -\Delta u = au + bv + bw + (u^+)^{p_1} + f_1 + t\phi_1 & \text{in } \Omega \\ -\Delta v = bu + av + bw + (v^+)^{p_2} + f_2 + r\phi_1 & \text{in } \Omega \\ -\Delta w = bu + bv + aw + (w^+)^{p_3} + f_3 + s\phi_1 & \text{in } \Omega \\ u = v = w = 0 & \text{on } \partial\Omega \end{cases}$$

where $u^+ = \max\{0, u(x)\}$, $\phi_1 > 0$ is the first eigenfunction of the Laplacian with Dirichlet boundary conditions and $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain with $N \geq 2$.

The nonlinearities will be assumed both superlinear and subcritical, that is, $1 < p_1, p_2, p_3 < 2^* - 1$, where $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = \infty$ if $N = 2$.

We may write (1) in vectorial form as

$$\begin{cases} -\Delta \begin{bmatrix} u \\ v \\ w \end{bmatrix} = A \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} (u^+)^{p_1} \\ (v^+)^{p_2} \\ (w^+)^{p_3} \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} + \begin{bmatrix} t \\ r \\ s \end{bmatrix} \phi_1 & \text{in } \Omega \\ u = v = w = 0 & \text{in } \partial\Omega \end{cases}$$

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where $A = \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix}$; we will assume that A has real eigenvalues

$$\nu_{i,1} = \nu_{i,2} = a - b \text{ and } \nu_{i,3} = a + 2b.$$

Throughout the paper, we will denote by $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_i \leq \dots$ the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$ and by $\{\phi_i\}_{i \in \mathbb{N}}$ the corresponding eigenfunctions, taken orthogonal and normalized with $\|\phi_i\|_{L^2} = 1$ and $\phi_1 > 0$; by $\sigma(-\Delta)$ we will denote the spectrum of the Laplacian, that is, the set $\{\lambda_i : i \in \mathbb{N}\}$.

Our results are

THEOREM 1.1. *If A has real eigenvalues that are not $\sigma(-\Delta)$ and $f_1, f_2, f_3 \in L^n(\Omega)$ with $n > N \geq 2$ then there exists $(t_0, r_0, s_0) \in \mathbb{R}^3$ such that if*

$$(t, r, s)^T = (t_0, r_0, s_0)^T + (\lambda_1 I - A)(\tau, \rho, \sigma)^T$$

with $\tau, \rho, \sigma < 0$ then a negative solution $(u_{neg}, v_{neg}, w_{neg})$ of (1) exists.

THEOREM 1.2. *Let $-a + b \notin \sigma(-\Delta)$ and $\frac{a+2b}{2} \notin \sigma(-\Delta)$, $f_1, f_2, f_3 \in L^n(\Omega)$ with $n > N \geq 2$ and (t, r, s) as in Theorem 1.1; then there exists a second solution for system (1).*

2. The negative solution

In this section, we will look for negative solutions, in the sense that both components are negative: this is relatively simple since in this case the nonlinear term disappears in (1).

We will need the following.

LEMMA 2.1. *If A has real eigenvalues that are not in $\sigma(-\Delta)$ and $f_1, f_2, f_3 \in L^n(\Omega)$ with $n > N$ then there exists a unique solution (u_0, v_0, w_0) of the problem*

$$(2) \quad \begin{cases} -\Delta \begin{bmatrix} u \\ v \\ w \end{bmatrix} = A \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} & \text{in } \Omega \\ u = v = w = 0 & \text{in } \partial\Omega \end{cases}$$

Proof. For the matrix A eigenvalue-eigenvector pairs are

$$\nu_{i,1} = a - b, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}; \quad \nu_{i,2} = a - b, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}; \quad \nu_{i,3} = a + 2b, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Hance A is diagonalizable, that is, $X^{-1}AX = D$ where

$$X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} a-b & 0 & 0 \\ 0 & a-b & 0 \\ 0 & 0 & a+2b \end{bmatrix}.$$

Let $\begin{bmatrix} u \\ v \\ w \end{bmatrix} = X \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix}$ then we written the equation (2) as

$$(3) \quad \begin{cases} -\Delta \tilde{u} = (a-b)\tilde{u} + f_1 & \text{in } \Omega \\ -\Delta \tilde{v} = (a-b)\tilde{v} + f_2 & \text{in } \Omega \\ -\Delta \tilde{w} = (a+2b)\tilde{w} + f_3 & \text{in } \Omega \\ \tilde{u} = \tilde{v} = \tilde{w} = 0 & \text{on } \partial\Omega \end{cases}$$

Since each real eigenvalue of A is not in $\sigma(-\Delta)$, equation (3) are uniquely solvable.

The hypothesis $f_1, f_2, f_3 \in L^n(\Omega)$ implies, by regularity theory and General Sobolev inequalities, that $u_0, v_0, w_0 \in W^{2,n}(\Omega) \subseteq C^{1,\alpha}(\bar{\Omega})$ for $\alpha = 1 - \frac{N}{n}$. \square

With this result we may obtain the negative solution:

Proof of Theorem 1.1. Let (u_0, v_0, w_0) be the corresponding solution for (2). Consider the problem

$$\begin{cases} -\Delta \begin{bmatrix} u \\ v \\ w \end{bmatrix} = A \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} t \\ r \\ s \end{bmatrix} \phi_1 & \text{in } \Omega \\ u = v = w = 0 & \text{in } \partial\Omega. \end{cases}$$

be looking for a solution of the form $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \phi_1$, one obtains $(\lambda_1 I -$

$$A) \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} t \\ r \\ s \end{bmatrix} \text{ and then by the superposition principle}$$

$$(4) \quad (\lambda_1 I - A)^{-1} \begin{bmatrix} t \\ r \\ s \end{bmatrix} \phi_1 + \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix}$$

is a solution of (1), provided it is nonpositive. Since $u_0, v_0, w_0 \in C^{1,\alpha}$, there exist finite

$$\begin{aligned}\alpha_0 &= \sup\{\alpha \mid \alpha\phi_1 + u_0 < 0\}, \\ \beta_0 &= \sup\{\beta \mid \beta\phi_1 + v_0 < 0\}, \\ \gamma_0 &= \sup\{\gamma \mid \gamma\phi_1 + w_0 < 0\}\end{aligned}$$

so (4) is negative provided $\alpha < \alpha_0$, $\beta < \beta_0$, and $\gamma < \gamma_0$ that is

$$\begin{bmatrix} t \\ r \\ s \end{bmatrix} = (\lambda_1 I - A) \left(\begin{bmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{bmatrix} + \begin{bmatrix} \alpha - \alpha_0 \\ \beta - \beta_0 \\ \gamma - \gamma_0 \end{bmatrix} \right)$$

by setting $(t_0, r_0, s_0)^T = (\lambda_1 I - A)^{-1}(\alpha_0, \beta_0, \gamma_0)^T$, $\tau = \alpha - \alpha_0 < 0$, $\rho = \beta - \beta_0 < 0$, $\sigma = \gamma - \gamma_0 < 0$. We get the condition in the claim. \square

3. The second solution

We will find the second solution by using a minimax theorem due to Felmer [3].

3.1. The variational structure.

We consider the Hilbert space $E = H_0^1 \times H_0^1 \times H_0^1$ equipped with the scalar product

$$\langle (u_1, v_1, w_1), (u_2, v_2, w_2) \rangle_E = \int_{\Omega} \nabla u_1 \nabla u_2 + \nabla v_1 \nabla v_2 + \nabla w_1 \nabla w_2$$

the related norm $\|(u_1, v_1, w_1)\|_E$ and the bounded symmetric quadratic form

$$B((u_1, v_1, w_1), (u_2, v_2, w_2)) = \int_{\Omega} Q - \int_{\Omega} aS + bR,$$

where

$$\begin{aligned}Q &= \nabla u_1 \nabla v_2 + \nabla u_1 \nabla w_2 + \nabla v_1 \nabla u_2 \\ &\quad + \nabla v_1 \nabla w_2 + \nabla w_1 \nabla u_2 + \nabla w_1 \nabla v_2, \\ S &= u_1 u_2 + v_1 v_2 + w_1 w_2, \\ R &= u_1 v_2 + u_1 w_2 + v_1 u_2 + v_1 w_2 + w_1 u_2 + w_1 v_2.\end{aligned}$$

Let (t, r, s) be as in Theorem 1.1 and $(u_{neg}, v_{neg}, w_{neg})$ be the corresponding negative solution for (1), then we define the functional $F : E \rightarrow \mathbb{R}$ for $\mathbf{u} = (u, v, w) \in E$ by

$$F(\mathbf{u}) = \frac{1}{2}B(\mathbf{u}, \mathbf{u}) - H(\mathbf{u})$$

where

$$H(\mathbf{u}) = \int_{\Omega} \frac{[(u + u_{neg})^+]^{p_1+1}}{p_1 + 1} + \int_{\Omega} \frac{[(v + v_{neg})^+]^{p_2+1}}{p_2 + 1} + \int_{\Omega} \frac{[(w + w_{neg})^+]^{p_3+1}}{p_3 + 1}.$$

Then it is simple to see that the functional F is $C^1(E; \mathbb{R})$ and its critical point (u, v, w) are such that $(u + u_{neg}, v + v_{neg}, w + w_{neg})$ are solutions of (1); in particular, the origin is a critical point at level zero and corresponds to the already found negative solution.

In order to find an orthogonal base for E which diagonalizes B , we consider, in a way similar to what was done in [1], the eigenvalue problem

$$B((u, v, w), (\phi, \varphi, \psi)) = \mu \langle (u, v, w), (\phi, \varphi, \psi) \rangle_E, \quad \forall (\phi, \varphi, \psi) \in E :$$

this gives (use $(\phi_i, 0, 0)$, $(0, \phi_i, 0)$, $(0, 0, \phi_i)$ as test function and let u_i, v_i , and w_i be the Fourier's coefficients for u, v , and w)

$$(5) \quad \begin{bmatrix} \mu\lambda_i + a & b - \lambda_i & b - \lambda_i \\ b - \lambda_i & \mu\lambda_i + a & b - \lambda_i \\ b - \lambda_i & b - \lambda_i & \mu\lambda_i + a \end{bmatrix} \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (i \in \mathbb{N}),$$

so we get nontrivial solutions when μ is such that the determinant of the above matrix is zero for some $i \in \mathbb{N}$. This gives $(\mu\lambda_i + a)^3 - 3(b - \lambda_i)^2(\mu\lambda_i + a) + 2(b - \lambda_i)^3 = 0$ and so

$$\mu_{i,1} = \mu_{i,2} = \frac{-a + (b - \lambda_i)}{\lambda_i}, \quad \mu_{i,3} = \frac{-a - 2(b - \lambda_i)}{\lambda_i} \quad (i \in \mathbb{N});$$

from (5) we also get the related eigenvectors $\Phi_{i,j}$, $j = 1, 2, 3$:

$$\Phi_{i,1} = (\phi_i, 0, -\phi_i), \quad \Phi_{i,2} = (0, \phi_i, -\phi_i), \quad \Phi_{i,3} = (\phi_i, \phi_i, \phi_i) \quad (i \in \mathbb{N}).$$

Because that $\Phi_{i,j}$, $j = 1, 2, 3$ are not orthogonal, we apply the Gram-Schmidt Process to $\Phi_{i,j}$, $j = 1, 2, 3$ and normalize to obtain $\Psi_{i,j}$, $j = 1, 2, 3$, that is , $\|\Psi_{i,j}\|_E = 1$:

$$\Psi_{i,1} = \frac{(\phi_i, 0, -\phi_i)}{\sqrt{2\lambda_i}}, \quad \Psi_{i,2} = \frac{(-\frac{1}{2}\phi_i, \phi_i, -\frac{1}{2}\phi_i)}{\sqrt{\frac{3}{2}\lambda_i}}, \quad \Psi_{i,3} = \frac{(\phi_i, \phi_i, \phi_i)}{\sqrt{3\lambda_i}} \quad (i \in \mathbb{N}).$$

With this structure we have

$$\begin{aligned}\langle \Psi_{i,j}, \Psi_{k,l} \rangle_E &= \begin{cases} 1 & i = k \text{ and } j = l \\ 0 & i \neq k \text{ or } j \neq l \end{cases}, \\ B(\Psi_{i,j}, \Psi_{k,l}) &= \begin{cases} \mu_{i,j} & i = k \text{ and } j = l \\ 0 & i \neq k \text{ or } j \neq l \end{cases}, \\ \langle \Psi_{i,j}, \Psi_{k,l} \rangle_{[L^2]^3} &= \begin{cases} \mu_{i,j}^{-1} & i = k \text{ and } j = l \\ 0 & i \neq k \text{ or } j \neq l \end{cases},\end{aligned}$$

so if we write $(u, v, w) = \sum_{i \in \mathbb{N}, j=1,2,3} C_{i,j} \Psi_{i,j}$, we get

$$\begin{aligned}\|(u, v, w)\|_E^2 &= \sum_{i \in \mathbb{N}, j=1,2,3} C_{i,j}^2, \\ B((u, v, w), (u, v, w)) &= \sum_{i \in \mathbb{N}, j=1,2,3} \mu_{i,j} C_{i,j}^2, \\ \|(u, v, w)\|_{[L^2]^3} &= \sum_{i \in \mathbb{N}, j=1,2,3} \mu_{i,j}^{-1} C_{i,j}^2,\end{aligned}$$

In view of this structure we may define

$$\begin{aligned}E^+ &= \overline{\text{span}\{\Psi_{i,j} : \mu_{i,j} > 0, i \in \mathbb{N}, j = 1, 2, 3\}}, \\ E^- &= \overline{\text{span}\{\Psi_{i,j} : \mu_{i,j} < 0, i \in \mathbb{N}, j = 1, 2, 3\}}, \\ E^0 &= \text{span}\{\Psi_{i,j} : \mu_{i,j} = 0, i \in \mathbb{N}, j = 1, 2, 3\},\end{aligned}$$

and we have

LEMMA 3.1. *There exists $\xi^* > 0$ such that*

$$(6) \quad B(\mathbf{u}, \mathbf{u}) \geq 2\xi^* \|\mathbf{u}\|_E^2 \quad \text{for } \mathbf{u} \in E^+$$

$$(7) \quad B(\mathbf{u}, \mathbf{u}) \leq -2\xi^* \|\mathbf{u}\|_E^2 \quad \text{for } \mathbf{u} \in E^-.$$

Moreover, $-a + b \notin \sigma(-\Delta)$ and $\frac{a+2b}{2} \notin \sigma(-\Delta)$, then $E^0 = \{0\}$.

Proof. The claim is satisfied by setting

$$2\xi^* := \inf\{|\mu_{i,j}| : |\mu_{i,j}| > 0, i \in \mathbb{N}, j = 1, 2, 3\}$$

Since

$$\lim_{i \rightarrow \infty} \mu_{i,1} = \lim_{i \rightarrow \infty} \mu_{i,2} = -1, \quad \lim_{i \rightarrow \infty} \mu_{i,3} = 2,$$

$2\xi^*$ is strictly positive.

The condition $-a + b \notin \sigma(-\Delta)$ and $\frac{a+2b}{2} \notin \sigma(-\Delta)$ implies $\mu_{i,j} \neq 0$ for any $i \in \mathbb{N}, j = 1, 2, 3$. \square

For later use, we also define \tilde{n} such that for $i \geq \tilde{n}$ we have $-(\lambda_i - b) < a < 2(\lambda_i - b)$ and

$$E_h = \overline{\text{span}\{\Psi_{i,j} : i \geq \tilde{n}, i \in \mathbb{N}, j = 1, 2, 3\}},$$

$$E_l = \text{span}\{\Psi_{i,j} : i \leq \tilde{n}, i \in \mathbb{N}, j = 1, 2, 3\} :$$

we have the following

LEMMA 3.2. $(u, v, w) \in E^+ \cap E_h$ implies $u = v = w$ and $(u, v, w) \in E^- \cap E_h$ implies $u + v + w = 0$.

Proof. It follows readily from the fact that for $i \geq \tilde{n}$ we have $\mu_{i,3} > 0$ and $\mu_{i,1} = \mu_{i,2} < 0$ and that $\Psi_{i,1} = \frac{(\phi_i, 0, -\phi_i)}{\sqrt{2\lambda_i}}$, $\Psi_{i,2} = \frac{(-\frac{1}{2}\phi_i, \phi_i, -\frac{1}{2}\phi_i)}{\sqrt{\frac{3}{2}\lambda_i}}$, and $\Psi_{i,3} = \frac{(\phi_i, \phi_i, \phi_i)}{\sqrt{3\lambda_i}}$. □

3.2. Estimates for the linking structure.

In this section we will prove the estimates we need in order to apply the minimax Theorem.

LEMMA 3.3. *There exist $\rho, \xi > 0$ such that*

$$F(\mathbf{u}) \geq \xi \quad \text{for } \mathbf{u} \in E^+ \quad \text{and} \quad \|\mathbf{u}\|_E = \rho.$$

Proof. Let \mathbf{u} be as above. By the continuous embedding of H_0^1 in L^{p_1+1} , L^{p_2+1} , and L^{p_3+1} we get

$$\int_{\Omega} \frac{[(u + u_{neg})^+]^{p_1+1}}{p_1 + 1} \leq \int_{\Omega} \frac{|u|^{p_1+1}}{p_1 + 1} \leq C_1 \|u\|_{H_0^1}^{p_1+1},$$

$$\int_{\Omega} \frac{[(v + v_{neg})^+]^{p_2+1}}{p_2 + 1} \leq \int_{\Omega} \frac{|v|^{p_2+1}}{p_2 + 1} \leq C_2 \|v\|_{H_0^1}^{p_2+1},$$

$$\int_{\Omega} \frac{[(w + w_{neg})^+]^{p_3+1}}{p_3 + 1} \leq \int_{\Omega} \frac{|w|^{p_3+1}}{p_3 + 1} \leq C_3 \|w\|_{H_0^1}^{p_3+1},$$

where C_1, C_2 and C_3 are positive constants. By (6) in Lemma 3.1,

$$\frac{1}{2}B(\mathbf{u}, \mathbf{u}) \geq \xi^* \|\mathbf{u}\|_E^2 = \xi^* \left(\|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2 + \|w\|_{H_0^1}^2 \right).$$

We get

$$\begin{aligned}
 F(\mathbf{u}) &\geq \xi^* \left(\|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2 + \|w\|_{H_0^1}^2 \right) \\
 &\quad - C \left(\|u\|_{H_0^1}^{p_1+1} + \|v\|_{H_0^1}^{p_2+1} + \|w\|_{H_0^1}^{p_3+1} \right) \\
 &\geq \|u\|_{H_0^1}^2 (\xi^* + C\rho^{p_1-1}) + \|v\|_{H_0^1}^2 (\xi^* + C\rho^{p_2-1}) \\
 &\quad + \|w\|_{H_0^1}^2 (\xi^* + C\rho^{p_3-1})
 \end{aligned}$$

where $C = \max\{C_1, C_2, C_3\}$ is a positive number. Since $p_1, p_2, p_3 > 1$, for $\rho > 0$ small enough we obtain $\xi^* + C\rho^{p_j-1} > 0$ $j = 1, 2, 3$. Let $C^* = \min\{\xi^* + C\rho^{p_j-1} : j = 1, 2, 3\} > 0$, then

$$F(\mathbf{u}) \geq C^* \|\mathbf{u}\|_E^2 = C^* \rho^2.$$

Let $\xi = C^* \rho^2$, then $F(\mathbf{u}) \geq \xi > 0$. □

LEMMA 3.4. *There exists $\mathbf{g} = (g, g, g) \in E^+ \cap E_h$ with $\|\mathbf{g}\|_E = 1$ and $\|g^+\|_{L^\infty} = +\infty$.*

Proof. Since H_0^1 is not embedded in L^∞ (here is where we need the condition $N \geq 2$), there exists $u \in H_0^1$ such that $\|u^+\|_{L^\infty} = +\infty$; by removing the components of u in the directions of the eigenvectors ϕ_i with $i < \tilde{n}$ we maintain this property since we simply subtract a finite linear combination of regular functions, so we may assume that such components are zero.

Since $\mu_{i,3} > 0$ and $\Psi_{i,3} = \frac{(\phi_i, \phi_i, \phi_i)}{\sqrt{3\lambda_i}}$, for $i \geq \tilde{n}$, we have that $(u, u, u) \in E^+ \cap E_h$.

Finally, we obtain $\|(g, g, g)\|_E = 1$ by a suitable rescaling of (u, u, u) . □

LEMMA 3.5. *Let $\mathbf{g} = (g, g, g)$ as in the lemma above. Then there exist $R, \theta > 0$ with $R\theta > \rho$ such that $F(\mathbf{u}) \leq 0$ for*

- (a) $\mathbf{u} \in E^-$,
- (b) $\mathbf{u} = \mathbf{w} + \tau\mathbf{g}$; $\mathbf{w} \in E^-$, $\|\mathbf{w}\|_E = R$, $0 \leq \tau \leq \theta R$,
- (c) $\mathbf{u} = \mathbf{w} + \tau\mathbf{g}$; $\mathbf{w} \in E^-$, $\|\mathbf{w}\|_E \leq R$, $\tau = \theta R$.

Proof. (a) Let $\mathbf{u} \in E^-$. By (7) in Lemma 3.1,

$$F(\mathbf{u}) \leq \frac{1}{2}B(\mathbf{u}, \mathbf{u}) \leq -\xi^* \|\mathbf{u}\|_E^2 \leq 0.$$

(b) Let $\mathbf{w} \in E^-$ with $\|\mathbf{w}\|_E = R$ and $0 \leq \tau \leq \theta R$. Observe that \mathbf{g} is orthogonal to \mathbf{w} , that is, $\langle \mathbf{w}, \mathbf{g} \rangle_E = 0 = B(\mathbf{w}, \mathbf{g})$; then we estimate, by

using (7) in Lemma 3.1,

$$\begin{aligned} F(\mathbf{u}) &\leq \frac{1}{2}B(\mathbf{u}, \mathbf{u}) = \frac{1}{2}B(\mathbf{w} + \tau\mathbf{g}, \mathbf{w} + \tau\mathbf{g}) = \frac{1}{2}B(\mathbf{w}, \mathbf{w}) + \frac{1}{2}\tau^2B(\mathbf{g}, \mathbf{g}) \\ &\leq -\xi^*\|\mathbf{w}\|_E^2 + \frac{1}{2}\tau^2B(\mathbf{g}, \mathbf{g}) = R^2 \left(-\xi^* + \frac{1}{2} \left(\frac{\tau}{R} \right)^2 B(\mathbf{g}, \mathbf{g}) \right) \\ &\leq R^2 \left(-\xi^* + \frac{1}{2}\theta^2B(\mathbf{g}, \mathbf{g}) \right) \end{aligned}$$

Since $\|\mathbf{g}\|_E = 1$, $B(\mathbf{g}, \mathbf{g}) \geq 2\xi^* > 0$ (by (6) in Lemma 3.1) and then $0 < \frac{2\xi^*}{B(\mathbf{g}, \mathbf{g})}$. By fixing $0 < \theta < \sqrt{\frac{2\xi^*}{B(\mathbf{g}, \mathbf{g})}}$, such that last term is negative, the claim (b) is proved.

(c) Consider now $\|\mathbf{w}\|_E \leq R$, $\tau = \theta R$, and let

$$P_l\mathbf{w} = (\sigma_1, \sigma_2, \sigma_3), \quad P_h\mathbf{w} = (\delta_1, \delta_2, \delta_3)$$

where P_l and P_h are the orthogonal projections onto E_l and E_h , respectively. In this way, $P_h\mathbf{w} \in E^- \cap E_h$ and then it is of the form $P_h\mathbf{w} = (\delta_1, \delta_2, -\delta_1 - \delta_2)$, by Lemma 3.2.

Write now

$$\begin{aligned} (8) \int_{\Omega} [(u + u_{neg})^+]^{p_1+1} &= \int_{\Omega} [(\sigma_1 + \delta_1 + \theta Rg + u_{neg})^+]^{p_1+1} \\ &= R^{p_1+1} \int_{\Omega} \left[\left(\frac{\sigma_1 + \delta_1 + u_{neg}}{R} + \theta g \right)^+ \right]^{p_1+1} \end{aligned}$$

$$\begin{aligned} (9) \int_{\Omega} [(v + v_{neg})^+]^{p_2+1} &= \int_{\Omega} [(\sigma_2 + \delta_2 + \theta Rg + v_{neg})^+]^{p_2+1} \\ &= R^{p_2+1} \int_{\Omega} \left[\left(\frac{\sigma_2 + \delta_2 + v_{neg}}{R} + \theta g \right)^+ \right]^{p_2+1} \end{aligned}$$

$$\begin{aligned} \int_{\Omega} [(w + w_{neg})^+]^{p_3+1} &= \int_{\Omega} [(\sigma_3 - \delta_1 - \delta_2 + \theta Rg + w_{neg})^+]^{p_3+1} \\ (10) \quad &= R^{p_3+1} \int_{\Omega} \left[\left(\frac{\sigma_3 - \delta_1 - \delta_2 + w_{neg}}{R} + \theta g \right)^+ \right]^{p_3+1} \end{aligned}$$

Since u_{neq} , v_{neq} , and w_{neq} are fixed and bounded, and $\sigma_1, \sigma_2, \sigma_3$ are linear combinations of a finite number of eigenvectors of the Laplacian (because of $P_l \mathbf{w} \in E_l$), there exists a constant C such that

$$|u_{neq}|, |v_{neq}|, |w_{neq}| < \frac{C}{2} \quad \text{and} \quad |\sigma_1|, |\sigma_2|, |\sigma_3| < \frac{C}{2},$$

so, for $R > 1$,

$$\frac{|\sigma_1 + u_{neq}|}{R}, \frac{|\sigma_2 + v_{neq}|}{R}, \frac{|\sigma_3 + w_{neq}|}{R} < C.$$

Moreover, since \mathbf{g} and θ have already been fixed and $\|g^+\|_{L^\infty} = \infty$, we know that

$$\Omega^* = \{x \in \Omega : \theta g > C + 1\}$$

has positive measure; we observe that $\|g^+\|_{L^\infty} = \infty$ implies

$$\max\{\theta g + \delta_1/R\} > C + 1, \quad \max\{\theta g + \delta_2/R\} > C + 1$$

and

$$\max\{\theta g - \delta_1/R - \delta_2/R\} > C + 1$$

for any bounded function δ_1, δ_2 and any $R > 1$: then $\Omega^* \subset \Omega_1^* \cup \Omega_2^* \cup \Omega_3^*$, where

$$\begin{aligned} \Omega_1^* &= \{x \in \Omega : \theta g + \delta_1/R > C + 1\}, \\ \Omega_2^* &= \{x \in \Omega : \theta g + \delta_2/R > C + 1\}, \\ \Omega_3^* &= \{x \in \Omega : \theta g - \delta_1/R - \delta_2/R > C + 1\} \end{aligned}$$

(observe that both Ω_i^* ($i = 1, 2, 3$) depend on \mathbf{w} and R , but Ω^* does not).

Then $|\Omega_1^*| \geq |\Omega^*|/3$ or $|\Omega_2^*| \geq |\Omega^*|/3$ or $|\Omega_3^*| \geq |\Omega^*|/3$ and, as a consequence, for any \mathbf{w} as assumed and $R > 1$, one of the following three cases hold:

- (i) Let $|\Omega_1^*| \geq |\Omega^*|/3$.
For any $x \in \Omega_1^*$,

$$\frac{\sigma_1 + \delta_1 + u_{neq}}{R} + \theta g > 1,$$

since $\theta g + \delta_1/R > C + 1$ and $-C < \sigma_1/R + u_{neq}/R < C$.

We conclude from (8) that

$$\begin{aligned}
 H(\mathbf{u}) &\geq \int_{\Omega} \frac{[(u + u_{neg})^+]^{p_1+1}}{p_1 + 1} \\
 &= \frac{R^{p_1+1}}{p_1 + 1} \int_{\Omega} \left[\left(\frac{\sigma_1 + \delta_1 + u_{neg}}{R} + \theta g \right)^+ \right]^{p_1+1} \\
 &\geq \frac{R^{p_1+1}}{p_1 + 1} \int_{\Omega_1^*} \left[\left(\frac{\sigma_1 + \delta_1 + u_{neg}}{R} + \theta g \right)^+ \right]^{p_1+1} \\
 &\geq \frac{R^{p_1+1}}{p_1 + 1} |\Omega_1^*| \geq \frac{|\Omega^*| R^{p_1+1}}{3(p_1 + 1)}
 \end{aligned}$$

- (ii) Let $|\Omega_2^*| \geq |\Omega^*|/3$.
 For any $x \in \Omega_2^*$,

$$\frac{\sigma_2 + \delta_2 + v_{neg}}{R} + \theta g > 1,$$

since $\theta g + \delta_2/R > C + 1$ and $-C < \sigma_2/R + v_{neg}/R < C$.
 We conclude from (9) that

$$\begin{aligned}
 H(\mathbf{u}) &\geq \int_{\Omega} \frac{[(v + v_{neg})^+]^{p_2+1}}{p_2 + 1} \\
 &= \frac{R^{p_2+1}}{p_2 + 1} \int_{\Omega} \left[\left(\frac{\sigma_2 + \delta_2 + v_{neg}}{R} + \theta g \right)^+ \right]^{p_2+1} \\
 &\geq \frac{R^{p_2+1}}{p_2 + 1} \int_{\Omega_2^*} \left[\left(\frac{\sigma_2 + \delta_2 + v_{neg}}{R} + \theta g \right)^+ \right]^{p_2+1} \\
 &\geq \frac{R^{p_2+1}}{p_2 + 1} |\Omega_2^*| \geq \frac{|\Omega^*| R^{p_2+1}}{3(p_2 + 1)}
 \end{aligned}$$

- (iii) Let $|\Omega_3^*| \geq |\Omega^*|/3$.
 For any $x \in \Omega_3^*$,

$$\frac{\sigma_3 - \delta_1 - \delta_2 + w_{neg}}{R} + \theta g > 1,$$

since $\theta g - \delta_1/R - \delta_2/R > C + 1$ and $-C < \sigma_3/R + w_{neg}/R < C$.

We conclude from (10) that

$$\begin{aligned}
 H(\mathbf{u}) &\geq \int_{\Omega} \frac{[(w + w_{neg})^+]^{p_3+1}}{p_3 + 1} \\
 &= \frac{R^{p_3+1}}{p_3 + 1} \int_{\Omega} \left[\left(\frac{\sigma_3 - \delta_1 - \delta_2 + w_{neg}}{R} + \theta g \right)^+ \right]^{p_3+1} \\
 &\geq \frac{R^{p_3+1}}{p_3 + 1} \int_{\Omega_3^*} \left[\left(\frac{\sigma_3 - \delta_1 - \delta_2 + w_{neg}}{R} + \theta g \right)^+ \right]^{p_3+1} \\
 &\geq \frac{R^{p_3+1}}{p_3 + 1} |\Omega_3^*| \geq \frac{|\Omega^*| R^{p_3+1}}{3(p_3 + 1)}
 \end{aligned}$$

Let $\tilde{C} = \min\{\frac{|\Omega^*|}{3(p_i+1)} : i = 1, 2, 3\}$ then $\tilde{C} > 0$ does not depend on R and \mathbf{w} . And we conclude that $H(\mathbf{u}) \geq \tilde{C}R^{\min\{p_1, p_2, p_3\}+1}$.

Finally, by estimating the first terms as in point (b), we get

$$\begin{aligned}
 F(\mathbf{u}) &= \frac{1}{2}B(\mathbf{w} + \theta R\mathbf{g}, \mathbf{w} + \theta R\mathbf{g}) - H(\mathbf{u}) \\
 &\leq \frac{1}{2}B(\mathbf{w} + \theta R\mathbf{g}, \mathbf{w} + \theta R\mathbf{g}) - \tilde{C}R^{\min\{p_1, p_2, p_3\}+1} \\
 &\leq -\xi^* \|\mathbf{w}\|_E^2 + \frac{1}{2}\theta^2 R^2 B(\mathbf{g}, \mathbf{g}) - \tilde{C}R^{\min\{p_1, p_2, p_3\}+1} \\
 &\leq R^2 \left(\frac{1}{2}\theta^2 B(\mathbf{g}, \mathbf{g}) - \tilde{C}R^{\min\{p_1, p_2, p_3\}-1} \right) :
 \end{aligned}$$

since $p_1, p_2, p_3 > 1$, we may choose $R > 1$ (and also $R > \rho/\theta$) large enough to make the last expression negative; this concludes the proof of the claim (c). □

3.3. The PS conditions.

In this section we will prove the PS condition, which was required for the application of the minimax theorem.

LEMMA 3.6. (PS condition). *Under the considered hypotheses, the functional F satisfies the PS condition, that is, let ϵ_n be a sequence of positive reals converging to zero and $\{\mathbf{u}_n\}_{n \in \mathbb{N}} \subseteq E$ be such that*

$$(11) \quad |F(\mathbf{u}_n)| \leq T,$$

$$(12) \quad |F'(\mathbf{u}_n)[\phi, \varphi, \psi]| \leq \epsilon_n \|(\phi, \varphi, \psi)\|_E \quad \forall (\phi, \varphi, \psi) \in E :$$

then $\{\mathbf{u}_n\}$ admits a convergent subsequence.

Proof. First, we want to prove that $\|\mathbf{u}_n\|_E$ is bounded: so we consider for the sake of contradiction a subsequence such that $\|\mathbf{u}_n\|_E \rightarrow \infty$ and we define

$$(U_n, V_n, W_n) = \frac{1}{\|\mathbf{u}_n\|_E} (u_n, v_n, w_n),$$

so that (up to a further subsequence) $(U_n, V_n, W_n) \rightarrow (U, V, W)$ weakly in E .

Applying the definition of the functional F ,

$$\begin{aligned} F(\mathbf{u}_n) &= \frac{1}{2} B(\mathbf{u}_n, \mathbf{u}_n) - \int_{\Omega} \frac{[(u_n + u_{neq})^+]^{p_1+1}}{p_1 + 1} \\ &\quad - \int_{\Omega} \frac{[(v_n + v_{neq})^+]^{p_2+1}}{p_2 + 1} - \int_{\Omega} \frac{[(w_n + w_{neq})^+]^{p_3+1}}{p_3 + 1} \end{aligned}$$

and

$$\begin{aligned} F'(\mathbf{u}_n)\mathbf{u}_n &= B(\mathbf{u}_n, \mathbf{u}_n) - \int_{\Omega} [(u_n + u_{neq})^+]^{p_1} u_n \\ &\quad - \int_{\Omega} [(v_n + v_{neq})^+]^{p_2} v_n - \int_{\Omega} [(w_n + w_{neq})^+]^{p_3} w_n. \end{aligned}$$

Now observe that

$$\int_{\Omega} [(u_n + u_{neq})^+]^p u_n = \int_{\Omega} [(u_n + u_{neq})^+]^{p+1} + \int_{\Omega} [(u_n + u_{neq})^+]^p (-u_{neq})$$

(and an analogous relation holds for the term in v_n and w_n); then,

$$\begin{aligned} F'(\mathbf{u}_n)\mathbf{u}_n &= B(\mathbf{u}_n, \mathbf{u}_n) \\ &\quad - \int_{\Omega} [(u_n + u_{neq})^+]^{p_1+1} - \int_{\Omega} [(u_n + u_{neq})^+]^{p_1} (-u_{neq}) \\ &\quad - \int_{\Omega} [(v_n + v_{neq})^+]^{p_2+1} - \int_{\Omega} [(v_n + v_{neq})^+]^{p_2} (-v_{neq}) \\ &\quad - \int_{\Omega} [(w_n + w_{neq})^+]^{p_3+1} - \int_{\Omega} [(w_n + w_{neq})^+]^{p_3} (-w_{neq}) \end{aligned}$$

and by considering $F(\mathbf{u}_n) - \frac{1}{2}F(\mathbf{u}_n)\mathbf{u}_n$, we get

$$\begin{aligned} & \kappa_1 \int_{\Omega} [(u_n + u_{neq})^+]^{p_1+1} + \kappa_2 \int_{\Omega} [(v_n + v_{neq})^+]^{p_2+1} \\ & + \kappa_3 \int_{\Omega} [(w_n + w_{neq})^+]^{p_3+1} + \frac{1}{2} \int_{\Omega} [(u_n + u_{neq})^+]^{p_1} (-u_{neq}) \\ & + \frac{1}{2} \int_{\Omega} [(v_n + v_{neq})^+]^{p_2} (-v_{neq}) + \frac{1}{2} \int_{\Omega} [(w_n + w_{neq})^+]^{p_3} (-w_{neq}) \\ & \leq T + \frac{1}{2} \epsilon_n \|\mathbf{u}_n\|_E; \end{aligned}$$

(where $\kappa_i = \frac{1}{2} - \frac{1}{p_i+1}$) by observing that each term in the expression above is nonnegative, we conclude that the estimate from above holds for each of them, and then

$$(13) \quad \frac{1}{\|\mathbf{u}_n\|_E} \int_{\Omega} [(u_n + u_{neq})^+]^{p_1+1} \rightarrow 0, \quad \frac{1}{\|\mathbf{u}_n\|_E} \int_{\Omega} [(v_n + v_{neq})^+]^{p_2+1} \rightarrow 0,$$

and

$$(14) \quad \frac{1}{\|\mathbf{u}_n\|_E} \int_{\Omega} [(w_n + w_{neq})^+]^{p_3+1} \rightarrow 0.$$

For any $(\phi, \varphi, \psi) \in E$ we get

$$\begin{aligned} & \frac{1}{\|\mathbf{u}_n\|_E} F'(\mathbf{u}_n)[\phi, \varphi, \psi] \\ & = B((U_n, V_n, W_n), (\phi, \varphi, \psi)) - \int_{\Omega} \frac{[(u_n + u_{neq})^+]^{p_1}}{\|\mathbf{u}_n\|_E} \phi \\ & \quad - \int_{\Omega} \frac{[(v_n + v_{neq})^+]^{p_2}}{\|\mathbf{u}_n\|_E} \varphi - \int_{\Omega} \frac{[(w_n + w_{neq})^+]^{p_3}}{\|\mathbf{u}_n\|_E} \psi \rightarrow 0 \end{aligned}$$

which, by using the weak convergence of (U_n, V_n, W_n) and (13), (14), implies that

$$B((U, V, W), (\phi, \varphi, \psi)) = 0$$

this means that (U, V, W) is a solution of $-\Delta(U, V, W)^T = B(U, V, W)^T$, where $B = \begin{bmatrix} \frac{-a+2b}{2} & \frac{a}{2} & \frac{a}{2} \\ \frac{a}{2} & \frac{-a+2b}{2} & \frac{a}{2} \\ \frac{a}{2} & \frac{a}{2} & \frac{-a+2b}{2} \end{bmatrix}$. Since for the matrix B eigenvalue-eigenvector pairs are

$$-a + b, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}; \quad -a + b, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}; \quad \frac{a + 2b}{2}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

(U, V, W) is the unique solution and then it is zero if real eigenvalues of B are not in $\sigma(-\Delta)$.

This gives rise to a contradiction since by definition we have

$$\|(U, V, W)\|_E = 1.$$

We conclude that $\|\mathbf{u}_n\|_E$ is bounded.

It is now simple to see that \mathbf{u}_n admits a convergent subsequence. In fact, up to a subsequence, $(u_n, v_n, w_n) \rightarrow (u, v, w)$ weakly in E , then we calculate the inner product of $(u_n, v_n, w_n) - (u, v, w)$ and $\Psi_{i,j}$ to obtain that the convergence is in fact strong. \square

3.4. The second solution through the minimax theorem.

Now, we may prove.

PROPOSITION 3.1. *There exists a critical point $\mathbf{u} \in E$ for the functional F with $F(\mathbf{u}) \geq \xi > 0$ (and then $\mathbf{u} \neq (0, 0, 0)$, so that it is a second solution).*

Proof. It is a consequence of the minimax theorem, by using the estimates in Lemma 3.3 and 3.5 and the PS condition in Lemma 3.6.

Actually, we set

$$\begin{aligned} S &= \{\mathbf{u} : \mathbf{u} \in E^+, \|\mathbf{u}\|_E = \rho\}, \\ Q &= \{\mathbf{u} = \mathbf{w} + \tau \mathbf{g} : \mathbf{w} \in E^-, \|\mathbf{w}\|_E \leq R, 0 \leq \tau \leq \theta R\}, \end{aligned}$$

and then we just need to check that also the hypotheses on the form of the functional that are required in [3] are satisfied. This is the case if we set $L(u, v, w) = \frac{1}{2}(v + w, u + w, u + v)$: this is linear, bounded, self-adjoint and $P^- \exp(\mu L) : E^- \rightarrow E^-$ is an invertible linear operator

for any $\mu > 0$: actually, it is simple to check that L is invertible with respect to the base we are considering, in fact,

$$\begin{aligned} L\Psi_{i,1} &= \frac{1}{2} \frac{(-\phi_i, 0, \phi_i)}{\sqrt{2\lambda_i}} = -\frac{1}{2}\Psi_{i,1} \quad (i \in \mathbb{N}), \\ L\Psi_{i,2} &= \frac{1}{2} \frac{(\frac{1}{2}\phi_i, 0, \frac{1}{2}\phi_i)}{\sqrt{\frac{3}{2}\lambda_i}} = -\Psi_{i,2} + \sqrt{2}\Psi_{i,3} \quad (i \in \mathbb{N}), \\ L\Psi_{i,3} &= \frac{1}{2} \frac{(2\phi_i, 2\phi_i, 2\phi_i)}{\sqrt{3\lambda_i}} = \Psi_{i,3} \quad (i \in \mathbb{N}); \end{aligned}$$

then $\exp(\mu L)$ is invertible too and takes the form

$$\begin{aligned} \exp(\mu L)\Psi_{i,1} &= \exp\left(-\frac{1}{2}\mu\right)\Psi_{i,1} \quad (i \in \mathbb{N}), \\ \exp(\mu L)\Psi_{i,2} &= \exp(-\mu)\Psi_{i,2} + \exp\left(\sqrt{2}\mu\right)\Psi_{i,3} \quad (i \in \mathbb{N}), \\ \exp(\mu L)\Psi_{i,3} &= \exp(\mu)\Psi_{i,3} \quad (i \in \mathbb{N}), \end{aligned}$$

which shows that it maps E^- onto itself and is invertible on it. \square

Finally, we may conclude the proof of Theorem 1.2.

Proof of Theorem 1.2. Proposition 3.1 implies Theorem 1.2 by the minimax lemma.

References

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