Korean J. Math. **16** (2008), No. 4, pp. 527–535

THE NUMBER OF THE CRITICAL POINTS OF THE STRONGLY INDEFINITE FUNCTIONAL WITH ONE PAIR OF THE TORUS-SPHERE VARIATIONAL LINKING SUBLEVELS

Tacksun Jung and Q-Heung Choi*

Abstract. Let $I \in C^{1,1}$ be a strongly indefinite functional defined on a Hilbert space H. We investigate the number of the critical points of I when I satisfies one pair of Torus-Sphere variational linking inequality. We show that I has at least two critical points when I satisfies one pair of Torus-Sphere variational linking inequality with $(P.S.)^*_c$ condition. We prove this result by use of the limit relative category and critical point theory on the manifold with boundary.

1. Introduction and statement of the main result

Let $I \in C^{1,1}$ be a strongly indefinite functional defined on a Hilbert Space H. In this paper, we investigate the number of the critical points of I when I satisfies one pair of Torus-Sphere variational linking inequalities and $(P.S.)^*_c$ condition. We show that I has at least two critical points when I has the sublevel set satisfying one pair of Torus-Sphere variational linking inequalities and satisfying the $(P.S.)^*_c$ condition. We prove this result by use of the limit relative category and critical point theory on the manifold with boundary. In the case that I is not strongly indefinite functional, Marino, A., Micheletti, A.M., Pistoia, Schechter, M., Tintarev. K., and Rabinowitz, P., proved in Theorem (3.4) of [4], [7] and [8] a theorem of existence of two solutions when I satisfies one pair of Sphere-Torus variational linking inequality by the mountain pass theorem and degree theory. Marino, A., Micheletti, A. M. and Pistoia, A.

Received November 7, 2008. Revised November 29, 2008.
2000 Mathematics Subject Classification: 35A15.
Key words and phrases: strongly indefinite functional, torus-sphere variational linking inequality, $(P.S.)^*_c$ condition, critical point theory, limit relative category.

*Corresponding author.

proved in Theorem (8.4) of [5] a theorem of existence of three solutions when I satisfies two pairs of Sphere-Torus variational linking inequalities and $(P.S.)_c$ condition by the mountain pass theorem and degree theory. In this paper we obtain the following results for the strongly indefinite functional case:

THEOREM 1.1. *(One pair of Torus-Sphere variational link) Let H be a Hilbert space with a norm $\|\cdot\|$, which is topological direct sum of the three subspaces X_0, X_1 and X_2. Let $I \in C^{1,1}(H,R)$ be a strongly indefinite functional. Assume that*

(1) $\dim X_1 < +\infty$;

(2) *There exist a small number $\rho > 0$, $r > 0$ and $R > 0$ such that $r < R$ and*

$$\sup_{\Sigma_R(S_1(\rho),X_0)} I < \inf_{S_r(X_1 \oplus X_2)} I,$$

where

$$S_1(\rho) = \{u \in X_1 |\ \|u\| = \rho\},$$
$$S_r(X_1 \oplus X_2) = \{u \in X_1 \oplus X_2 |\ \|u\| = r\},$$
$$B_r(X_1 \oplus X_2) = \{u \in X_1 \oplus X_2 |\ \|u\| \le r\},$$

$$\begin{aligned}\Sigma_R(S_1(\rho),X_0) &= \{u = u_1 + u_2 |\ u_1 \in S_1(\rho), u_2 \in X_0, \|u_1\| = \rho, \\ &\quad 1 \le \|u_1 + u_2\| = R\} \cup \{u = u_1 + u_2 |\ u_1 \in S_1(\rho), \\ &\quad \|u_1\| = \rho, 1 \le \|u_2\| \le R\},\end{aligned}$$

$$\begin{aligned}\Delta_R(S_1(\rho),X_0) & \\ &= \{u = u_1 + u_2 |\ u_1 \in S_1(\rho), u_2 \in X_0, \|u_1\| = \rho, \\ &\quad 1 \le \|u_1 + u_2\| \le R\};\end{aligned}$$

(3) $\beta = \sup_{\Delta_R(S_1(\rho),X_0)} I < +\infty$;

(4) *$(P.S.)^*_c$ condition holds for any $c \in [\alpha, \beta]$ where*

$$\alpha = \inf_{S_r(X_1 \oplus X_2)} I;$$

(5) *There exists one critical point e in $X_0 \oplus X_2$ with $I(e) < \alpha$.*
Then there exist at least two distinct critical points except e, u_i, $i = 1, 2$, in X_1, of I with

$$\inf_{S_r(X_1 \oplus X_2)} I \le I(u_i) \le \sup_{\Delta_R(S_1(\rho),X_0)} I.$$

For the proof of the main result we use the critical point theory on the manifold with boundary. Since the functional I is strongly indefinite functional, it is convenient to use the notion of the limit relative category instead of the relative category and the $(P.S.)^*_c$ condition which is a version of the Palais-Smale condition. We restrict the functional I to the manifold C with boundary, where C is introduced in section 3. We study the geometry and topology of the sub-levels of I and $\tilde{I}$ and investigate the limit relative category of the sub-level sets of $\tilde{I}$ and $(P.S.)^*_c$ condition in C. By the facts that the number of the limit relative category $\mathrm{cat}^*_{(C,\tilde{\Sigma_R})}(\tilde{\Delta_R})$ is equal to 2 and the critical point theory on the manifold with boundary, we obtain at least two distinct critical points of $\tilde{I}$, so we obtain at least two distinct critical points of I.

2. Critical Point Theory on the manifold with boundary

Now, we consider the critical point theory on the manifold with boundary. Let H be a Hilbert space and M be the closure of an open subset of H such that M can be endowed with the structure of C^2 manifold with boundary. Let $f : W \to R$ be a $C^{1,1}$ functional, where W is an open set containing M. For applying the usual topological methods of critical points theory we need a suitable notion of critical point for f on M. Since the functional $I(u)$ is strongly indefinite, the notion of the $(P.S.)^*_c$ condition and the limit relative category (see [2]) is a useful tool for the proof of the main theorem.

Definition 2.1. *If $u \in M$, the lower gradient of f on M at u is defined by*

$$grad^-_M f(u) = \begin{cases} \nabla f(u) & \text{if } u \in int(M), \\ \nabla f(u) + [< \nabla f(u), \nu(u) >]^- \nu(u) & \text{if } u \in \partial M, \end{cases} \tag{2.1}$$

where we denote by $\nu(u)$ the unit normal vector to ∂M at the point u, pointing outwards. We say that u is a lower critical for f on M, if $grad^-_M f(u) = 0$.

Let $(H_n)_n$ be a sequence of closed finite dimensional subspace of H with $\dim H_n < +\infty$, $H_n \subset H_{n+1}$, $\cup_{n\in N} H_n$ is dense in H.

Let $M_n = M \cap H_n$, for any n, be the closure of an open subset of H_n and has the structure of a C^2 manifold with boundary in H_n. We assume that for any n there exists a retraction $r_n : M \to M_n$. For given $B \subset H$, we will write $B_n = B \cap H_n$.

Definition 2.2. *Let $c \in R$. We say that f satisfies the $(P.S.)^*_c$ condition with respect to $(M_n)_n$, on the manifold with boundary M, if for any sequence $(k_n)_n$ in N and any sequence $(u_n)_n$ in M such that $k_n \to \infty$, $u_n \in M_{k_n}$, $\forall n$, $f(u_n) \to c$, $grad^-_{M_{k_n}} f(u_n) \to 0$, there exists a subsequence of $(u_n)_n$ which converges to a point $u \in M$ such that $grad^-_M f(u) = 0$.*

Let Y be a closed subspace of M.

Definition 2.3. *Let B be a closed subset of M with $Y \subset B$. We define the relative category $cat_{M,Y}(B)$ of B in (M,Y), as the least integer h such that there exist $h+1$ closed subsets U_0, $U_1, \ldots$, U_h with the following properties:*
$B \subset U_0 \cup U_1 \cup \ldots \cup U_h$;
$U_1, \ldots, U_h$ are contractible in M;
$Y \subset U_0$ and there exists a continuous map $F : U_0 \times [0,1] \to M$ such that

$$\begin{aligned} F(x,0) &= x \qquad \forall x \in U_0, \\ F(x,t) &\in Y \qquad \forall x \in Y,\ \forall t \in [0,1], \\ F(x,1) &\in Y \qquad \forall x \in U_0. \end{aligned}$$

If such an h does not exist, we say that $cat_{M,Y}(B) = +\infty$.

Definition 2.4. *Let (X,Y) be a topological pair and $(X_n)_n$ be a sequence of subsets of X. For any subset B of X we define the limit relative category of B in (X,Y), with respect to $(X_n)_n$, by*

$$cat^*_{(X,Y)}(B) = \limsup_{n\to\infty} cat_{(X_n,Y_n)}(B_n).$$

Let Y be a fixed subset of M. We set

$$\mathcal{B}_{\mathrm{i}} = \{\mathrm{B} \subset \mathrm{M} |\ \mathrm{cat}^*_{(\mathrm{M,Y})}(\mathrm{B}) \geq \mathrm{i}\},$$

$$c_i = \inf_{B \in \mathcal{B}_{\mathrm{i}}} \sup_{x \in B} f(x).$$

We have the following multiplicity theorem, which was proved in [6].

Theorem 2.1. *Let $i \in N$ and assume that*
(1) $c_i < +\infty$,
(2) $\sup_{x \in Y} f(x) < c_i$,

(3) *the* $(P.S.)^*_{c_i}$ *condition with respect to* $(M_n)_n$ *holds.*
Then there exists a lower critical point x *such that* $f(x) = c_i$. *If*

$$c_i = c_{i+1} = \ldots = c_{i+k-1} = c,$$

then

$$cat_M(\{x \in M | f(x) = c,\ grad^-_M f(x) = 0\}) \geq k.$$

3. Proof of Theorem 1.1

Let H be a Hilbert space with a norm $\|\cdot\|$ and $H = X_0 \oplus X_1 \oplus X_2$ with $\dim X_1 < \infty$. Let $I \in C^{1,1}(H, R)$ be a strongly indefinite functional. Let $(H_n)_n$ be a sequence of closed subspaces of H with finite dimension and such that for all n,

$$X_1 \subset H_n, \qquad P_{X_i} \cdot P_{H_n} = P_{H_n} \cdot P_{X_i} (= P_{X_i \cap H_n}), \qquad i = 0, 1, 2,$$

where, for all given subspace X of H, P_X is the orthogonal projection from H onto X. Set

$$C = \{u \in H |\ \|P_{X_1} u\| \geq 1\}.$$

Then C is the smooth manifold with boundary. Let $C_n = C \cap H_n$. Let us define a functional $\psi : H \backslash (X_0 \oplus X_2) \longrightarrow H$ by

$$\psi(u) = u - \frac{P_{X_1} u}{\|P_{X_1} u\|} = P_{X_0 \oplus X_2} u + \left(1 - \frac{1}{\|P_{X_1} u\|}\right) P_{X_1} u. \tag{3.1}$$

We have

$$\psi'(u)(v) = v - \frac{1}{\|P_{X_1} u\|} \left(P_{X_1} v - \left\langle \frac{P_{X_1} u}{\|P_{X_1} u\|}, v \right\rangle \frac{P_{X_1} u}{\|P_{X_1} u\|} \right). \tag{3.2}$$

Let us introduce the constrained functional $\tilde{I} : C \to H$ by

$$\tilde{I} = I \cdot \psi.$$

Then $\tilde{I} \in C^{1,1}_{loc}$. We note that if $\tilde{u}$ is the critical point of $\tilde{I}$ and lies in the interior of C, then $u = \psi(\tilde{u})$ is the critical point of I. We also note that

$$\|\text{grad}^-_C \tilde{I}(\tilde{u})\| \geq \|P_{X_0 \oplus X_2} \nabla I(\psi(\tilde{u}))\|, \qquad \forall \tilde{u} \in \partial C. \tag{3.3}$$

Let us set

$$\begin{aligned}
\tilde{S_r} &= \psi^{-1}(S_r(X_1 \oplus X_2)), \\
\tilde{\Sigma_R} &= \psi^{-1}(\Sigma_R(S_1(\rho), X_0), \\
\tilde{\Delta_R} &= \psi^{-1}(\Delta_R(S_1(\rho), X_0).
\end{aligned}$$

Then $\tilde{S_r}$, $\tilde{\Sigma_R}$ and $\tilde{\Delta_R}$ has the same topological structure as that of S_r, Σ_R and Δ_R.

From condition (2), we have

$$\sup_{\tilde{\Sigma_R}} \tilde{I} = \sup_{\Sigma_R(S_1(\rho),X_0)} I < \inf_{S_r(X_1 \oplus X_2)} I = \inf_{\tilde{S_r}} \tilde{I}. \tag{3.4}$$

From the condition (4), $\tilde{I}$ satisfies the $(P.S.)^*_c$ condition on C with respect to (C_n) $(C_n = C \cap H_n)$ for any c such that

$$\inf_{\tilde{S_r}} \tilde{I} = \inf_{S_r(X_1 \oplus X_2)} I \leq c \leq \sup_{\Delta_R(S_1(\rho),X_0)} I = \sup_{\tilde{\Delta_R}} \tilde{I}. \tag{3.5}$$

Next, we claim that $\text{cat}^*_{(C,\tilde{\Sigma_R})}(\tilde{\Delta_R}) = 2$. Let us set

$$\Sigma_n = \Sigma_R(S_1(\rho), X_0) \cap H_n, \qquad \Delta_n = \Delta_R(S_1(\rho), X_0) \cap H_n,$$

$$\tilde{\Sigma_n} = \tilde{\Sigma_R} \cap H_n \qquad \text{and} \qquad \tilde{\Delta_n} = \tilde{\Delta_R} \cap H_n.$$

We consider a continuous deformation $r : \tilde{S_r} \backslash X_2 \times [0,1] \to \tilde{S_r} \backslash X_2$ such that

$\cdot\ r(x,0) = x, \qquad \forall x \in \tilde{S_r} \backslash X_2,$

$\cdot\ r(x,t) = x, \qquad \forall x \in \tilde{S_r} \cap X_1 \quad \forall t \in [0,1],$

$\cdot\ r(x,1) \in \tilde{S_r} \cap X_1 \qquad \forall x \in \tilde{S_r} \backslash X_2.$

Now we can define, if $x = x_0 + x_1 + x_2$, $x_i \in X_i$, $i = 0,1,2$, $t \in [0,1]$,

$$r_1(x,t) = x_0 + \|x_1 + x_2\| r\left(\frac{x_1 + x_2}{\|x_1 + x_2\|}, t\right).$$

Using r_1 we construct, for all n, a continuous deformation $\eta_n : C_n \times [0,1] \to C_n$ such that

$\cdot\ \eta_n(x,0) = x \qquad \forall x \in C_n,$

$\cdot\ \eta_n(x,t) = x \qquad \forall x \in \tilde{\Delta}_n, \qquad \forall t \in [0,1],$

$\cdot\ \eta_n(x,1) \in \tilde{\Delta}_n \qquad \forall x \in C_n.$

$\cdot\ \eta_n(x,t) \in C_n \backslash \tilde{S_r}, \qquad \forall x \in C_n \backslash \tilde{S_r}, \qquad \forall t \in [0,1].$

The existence of η_n implies that $\text{cat}_{(C_n,\tilde{\Sigma_n})}(\tilde{\Delta_n}) = \text{cat}_{(\tilde{\Delta_n},\tilde{\Sigma_n})}(\tilde{\Delta_n})$; moreover the pair $(\tilde{\Delta_n}, \tilde{\Sigma_n})$ is homeomorphic to the pair (Δ_n, Σ_n) and (Δ_n, Σ_n) is homeomorphic to the pair $(\mathcal{B}^{p+1} \times \mathcal{S}^{q-1}, \mathcal{S}^p \times \mathcal{S}^{q-1})$, so $(\tilde{\Delta_n}, \tilde{\Sigma_n})$ is homeomorphic to the pair $(\mathcal{B}^{p+1} \times \mathcal{S}^{q-1}, \mathcal{S}^p \times \mathcal{S}^{q-1})$, where $p = \dim(X_0) \cap$

H_n, $q = \dim(X_1) \cap H_n$ and we are denoting by $\mathcal{B}^r$ and $\mathcal{S}^r$ the r-dimensional ball and the r-dimensional sphere, respectively. This implies that $\text{cat}_{(C_n,\tilde{\Sigma_n})}(\tilde{\Delta_n}) = 2$ (in the case $q = 1$ a connection argument can be used, otherwise this is a consequence of the fact that cuplength $(\mathcal{B}^{p+1} \times \mathcal{S}^{q-1}, \mathcal{S}^p \times \mathcal{S}^{q-1}) = 1$ and (b) of (3.7) in [7]). Thus $\text{cat}_{(C_n,\tilde{\Sigma_n})}(\tilde{\Delta_n}) = 2$, so we have $\text{cat}^*_{(C,\tilde{\Sigma_R})}(\tilde{\Delta_R}) = 2$. Thus we prove the claim. Let us set

$$\mathcal{A}_1 = \{A \subset C|\ \text{cat}^*_{(C,\tilde{\Sigma_R})}(A) \geq 1\}, \qquad \mathcal{A}_2 = \{A \subset C|\ \text{cat}^*_{(C,\tilde{\Sigma_R})}(A) \geq 2\}.$$

Since $\text{cat}^*_{(C,\tilde{\Sigma_R})}(\tilde{\Delta_R}) = 2$, $\tilde{\Delta_R} \in \mathcal{A}_1$ and $\tilde{\Delta_R} \in \mathcal{A}_2$. Let us set

$$\tilde{c_1} = \inf_{A\in\mathcal{A}_1} \sup_{\tilde{u}\in A} \tilde{I}(\tilde{u}) \qquad \text{and} \qquad \tilde{c_2} = \inf_{A\in\mathcal{A}_2} \sup_{\tilde{u}\in A} \tilde{I}(\tilde{u}).$$

From the condition (3) and $\tilde{\Delta_R} \in \mathcal{A}_i$, $i = 1, 2$, it follows that

$$\tilde{c_i} = \inf_{A\in\mathcal{A}_i} \sup_{\tilde{u}\in A} \tilde{I}(\tilde{u}) \leq \sup_{\tilde{u}\in\tilde{\Delta_R}} \tilde{I}(\tilde{u}) = \sup_{u\in\Delta_R(S_1(\rho),X_0)} I(u) < \infty, i = 1, 2.$$

It is easily checked that for $\tilde{\Sigma_R} \subset A \in \mathcal{A}_i$, $i = 1, 2$,

$$\sup_{\tilde{u}\in\tilde{\Sigma}_R} \tilde{I}(\tilde{u}) \leq \sup_{\tilde{u}\in A} \tilde{I}(\tilde{u}),$$

and hence $\sup_{\tilde{u}\in\tilde{\Sigma}_R} \tilde{I}(\tilde{u}) \leq \inf_{A\in\mathcal{A}_i} \sup_{\tilde{u}\in A} \tilde{I}(\tilde{u}) = \tilde{c_i}$. From the condition (4), $\tilde{I}$ satisfies the $(P.S.)^*_c$ condition on C with respect to (C_n) for any c with (3.5). Thus, by Theorem 2.1, there exist two critical points $\tilde{u_1}$, $\tilde{u_2}$ such that

$$\tilde{I}(\tilde{u_1}) = \tilde{c_1} \qquad \text{and} \qquad \tilde{I}(\tilde{u_2}) = \tilde{c_2}.$$

We claim that

$$\inf_{\tilde{u}\in\tilde{S_r}} \tilde{I}(\tilde{u}) \leq \tilde{c_1} \leq \tilde{c_2} \leq \sup_{\tilde{u}\in\tilde{\Delta_R}} \tilde{I}(\tilde{u}).$$

In fact, since $\text{cat}^*_{(C,\tilde{\Sigma_R})}(\tilde{\Delta_R}) = 2$, $\tilde{\Delta_R} \in \mathcal{A}_2$ and hence

$$\tilde{c_2} = \inf_{A\in\mathcal{A}_2} \sup_{\tilde{u}\in A} \tilde{I}(\tilde{u}) \leq \sup_{\tilde{u}\in\tilde{\Delta_R}} \tilde{I}(\tilde{u}), \forall A \in \mathcal{A}_2.$$

For the proof of $\tilde{c_1} \geq \inf_{\tilde{u}\in\tilde{S_r}} \tilde{I}(\tilde{u})$, we construct a deformation $\eta'_n : C_n\backslash\tilde{S_r} \times [0, 1] \to C_n\backslash\tilde{S_r}$ such that

$\cdot\ \eta'_n(x, 0) = x\ \forall x \in C_n\backslash\tilde{S_r}$,

$\cdot\ \eta'_n(x, t) = x\ \forall x \in \tilde{\Sigma_n}\ \forall t \in [0, 1]$,

$\cdot\ \eta'_n(x,1) \in \tilde{\Sigma}_n\ \forall x \in C_n$.
Actually η'_n can be defined taking the restriction of η_n on $C_n \backslash \tilde{S}_r$ followed by a retraction of $\tilde{\Delta}_n \backslash \tilde{S}_r$ to $\tilde{\Sigma}_n$. The existence of η'_n, for all n, implies that any $A \in \mathcal{A}_1$ must intersect $\tilde{S}_r$, so $\sup \tilde{I}(A) \geq \inf_{\tilde{u}\in\tilde{S}_r} \tilde{I}(\tilde{u})$, $\forall A \in \mathcal{A}_1$. So we have that $\tilde{c}_1 = \inf_{A\in\mathcal{A}_1} \sup_{\tilde{u}\in A} \tilde{I}(\tilde{u}) \geq \inf_{\tilde{u}\in\tilde{S}_r} \tilde{I}(\tilde{u})$. Thus we prove the claim. Setting $u_i = \psi(\tilde{u}_i)$, $i = 1, 2$, we have

$$\begin{aligned} \inf_{u\in S_r(X_1\oplus X_2)} I(u) &= \inf_{\tilde{u}\in\tilde{S}_r} \tilde{I}(\tilde{u}) \leq \tilde{I}(\tilde{u_1}) = I(u_1) \leq \tilde{I}(\tilde{u_2}) = I(u_2) \\ &\leq \sup_{\tilde{u}\in\tilde{\Delta}_R} \tilde{I}(\tilde{u}) = \sup_{u\in\Delta_R(S_1(\rho),X_0)} I(u). \end{aligned}$$

Next, we claim that $\tilde{u_i} \notin \partial C$, that is $u_i \notin X_0 \oplus X_2$, which implies that u_i are critical points of I in X_1. For this we assume by contradiction that $u_i \in X_0 \oplus X_2$. From (3.3), $\|\mathrm{grad}_C^- \tilde{I}(\tilde{u})\| \geq \|P_{X_0\oplus X_2} \nabla I(\psi(\tilde{u}))\|$, $\forall \tilde{u} \in \partial C$ and $P_{X_0\oplus X_2}\nabla I(u_i) = 0$, namely u_i, $i = 1, 2$, are critical points for $I|_{X_0\oplus X_2}$. Just notice that, for fixed $w_0 \in X_0$ the functional $w_2 \mapsto I(w_0 + w_2)$ is weakly convex in X_2, while, for fixed $w_2 \in X_2$ the functional $w_0 \mapsto I(w_0 + w_2)$ is strictly concave in X_0. Moreover e is the critical point in $X_0 \oplus X_2$ with $I(e) < \alpha = \inf_{S_r(X_1\oplus X_2)} I$. If $u_1 = w_0 + w_2$ is another critical point, we have

$$I(e) \leq I(w_2) \leq I(w_0 + w_2) = I(u_1) \leq I(w_0) \leq I(e),$$

so we have $I(u_1) = I(e)$. Similary we have $I(u_2) = I(e)$, so we have $I(u_1) = I(u_2) = I(e) < \alpha$, which is absurd for the fact that $\alpha = \inf_{u\in S_r(X_1\oplus X_2)} I(u) \leq I(u_1) \leq I(u_2) \leq \sup_{u\in\Delta_R(S_1(\rho),X_0)} I(u) = \beta$. Thus $u_i \notin X_0 \oplus X_2$, $i = 1, 2$. Moreover it is easily checked that there is no critical point $u \in X_0 \oplus X_2$ such that $I(u) \in [\alpha, \beta]$. Hence u_i, $i = 1, 2$, are critical points of I, in X_1. Thus we prove the theorem.

References

[1] Q. H. Choi and jtsT. Jung, *An application of a variational reduction method to a nonlinear wave equation,* J. Differential Equations, **117**, 390-410 (1995).
[2] Fournier, g., Lupo, D. Ramos, M., and willem, M., *Limit relative category and ctritical point theory*, Dynam. Report, **3**(1993), 1-23.
[3] D. Lupo and A.MA. M. Micheletti, *Two applications of a three critical points theorem,* J. Differential Equations **132**, 222-238 (1996).

[4] Marino. A., Micheletti, A. M. and Pistoia, A. *Same variational results on semilinear problems with asymptoticallly nonsymmetric behaviour*, Nonlinear Analysis "A Tribute in honour of G. Pordi", S.U.S. Pisa (1991).
[5] Marino, A., Micheletti, A. M. and Pistoia, A., *A nonsymmetric asymptotically linear elliptic problem*, Topol. Meth. Nonlin. Anal., (1994), **4**, 289-339.
[6] A. Marino and SacC. Saccon, *Nabla theorems and multiple solutions for some noncooperative elliptic systems,* Sezione Di Annalisi Mathematica E Probabilita, Dipartimento di Mathematica, universita di Pisa, 2000.
[7] Rabinowitz, P., *Minimax methods in critical point theory with applications to differential equations*, C.B.M.S. Reg. Conf. Ser. in Math. 6, American Mathematical Society, Providence, R1,(1986).
[8] Schechter, M. and TinTintarev, K., *Pairs of critical points produved by linking subsets with application to semilinear elliptic problems*, Bull. Soc. Math. Belg., (1992), **44(3)**, Ser. B, 249-261.

Department of Mathematics
Kunsan National University
Kunsan 573-701, Korea
E-mail: tsjung@kunsan.ac.kr

Department of Mathematics Education
Inha University
Incheon 402-751, Korea
E-mail: qheung@inha.ac.kr