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SLLN FOR WEIGHTED SUMS OF LEVEL-WISE INDEPENDENT FUZZY RANDOM VARIABLES

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Abstract. Guan and Li [2] obtained SLLN for weighted sums of level-wise independent fuzzy random variables. In this paper, a generalization of Guan and Li [2] is obtained by using the Skorokhod metric on $F(R)$.

1. Introduction

In recent years, strong laws of large numbers(SLLN) for sums of independent fuzzy random variables have received much attention by several researchers and have been studied by Klement et al. [9], Colubi et al. [1], Molchanov [10], Joo et al. [6], Joo and Kim [8], and so on. Recently, Joo [5] obtained a SLLN for independent and convexly tight fuzzy random variables with respect to the Skorokhod metric d_s which was introduced by Joo and Kim [7].

It is one of significant problems how we can generalize strong laws of large numbers for sums of fuzzy random variables to the case of weighted sums. Related to this problem, some results are obtained by Guan and Li [2], Hyun et al.[4] under restrictive conditions.

The purpose of this paper is to obtain a generalization of Guan and Li [2] by using the Skorokhod metric d_s which was introduced by Joo and Kim [7].

2. Preliminaries

Let R denote the real line. A fuzzy number is a fuzzy set $\tilde{u} : R \to [0, 1]$ with the following properties ;

(1) $\tilde{u}$ is normal, i.e., there exists $x \in R$ such that $\tilde{u}(x) = 1$.

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(2) $\tilde{u}$ is upper semicontinuous.
(3) supp $\tilde{u} = cl\{x \in R : \tilde{u}(x) > 0\}$ is compact.
(4) $\tilde{u}$ is convex, i.e. $\tilde{u}(\lambda x + (1-\lambda)y) \geq \min(\tilde{u}(x), \tilde{u}(y))$ for $x, y \in R$ and $\lambda \in [0,1]$.

Let $F(R)$ be the family of all fuzzy numbers. For a fuzzy set $\tilde{u}$, if we define

$$L_\alpha \tilde{u} = \begin{cases} \{x : \tilde{u}(x) \geq \alpha\}, 0 < \alpha \leq 1, \\ supp\ \tilde{u}, \alpha = 0, \end{cases}$$

then, it follows that $\tilde{u}$ is a fuzzy number if and only if $L_1\tilde{u} \neq \phi$ and $L_\alpha \tilde{u}$ is a closed bounded interval for each $\alpha \in [0,1]$. From this characterization of fuzzy numbers, a fuzzy number $\tilde{u}$ is completely determined by the closed intervals $L_\alpha \tilde{u} = [u^l_\alpha, u^r_\alpha]$. By the theorem of Goetschel and Voxman [3], we can identify a fuzzy number $\tilde{u}$ with the family of closed intervals $\{[u^l_\alpha, u^r_\alpha] : 0 \leq \alpha \leq 1\}$.

The linear structure on $F(R)$ is defined as usual;

$$(\tilde{u} \oplus \tilde{v})(z) = \sup_{x+y=z} \min(\tilde{u}(x), \tilde{v}(y)),$$

$$(\lambda \tilde{u})(z) = \begin{cases} \tilde{u}(z/\lambda), & \lambda \neq 0 \\ \tilde{0}(z), & \lambda = 0, \end{cases}$$

where $\tilde{0} = I_{\{0\}}$ denotes the indicator function of $\{0\}$.

We can define L^1-metric d_1 and uniform metric d_∞ on $F(R)$ as follows;

$$d_1(\tilde{u}, \tilde{v}) = \int_0^1 \max(|u^l_\alpha - v^l_\alpha|, |u^r_\alpha - v^r_\alpha|)\ d\alpha$$

$$\begin{aligned} d_\infty(\tilde{u}, \tilde{v}) &= \sup_{0 \leq \alpha \leq 1} \max(|u^l_\alpha - v^l_\alpha|, |u^r_\alpha - v^r_\alpha|) \\ &= \max(\sup_{0 \leq \alpha \leq 1} |u^l_\alpha - v^l_\alpha|, \sup_{0 \leq \alpha \leq 1} |u^r_\alpha - v^r_\alpha|). \end{aligned}$$

The norm of $\tilde{u} \in F(R)$ is defined by

$$\|\tilde{u}\| = d_\infty(\tilde{u}, \tilde{0}) = \max(|u^l_0|, |u^r_0|).$$

It is well known that $(F(R), d_1)$ is separable but is not complete, and that $(F(R), d_\infty)$ is complete but is not separable (For details, see Klement et al. [9]). Joo and Kim [7] introduced the Skorokod metric d_s on $F(R)$ which makes it a separable and topologically complete metric space as follows:

Definition 2.2. *Let T denote the class of strictly increasing, continuous mapping of $[0,1]$ onto itself. For $\tilde{u}, \tilde{v} \in F(R)$, we define*

$$d_s(\tilde{u}, \tilde{v}) = \inf\{\epsilon > 0 : \text{there exists a } t \in T \text{ such that } \sup_{0 \le \alpha \le 1} |t(\alpha) - \alpha| \le \epsilon \text{ and } d_\infty(\tilde{u}, t(\tilde{v})) \le \epsilon\},$$

where $t(\tilde{v})$ denotes the composition of $\tilde{v}$ and t.

It follows immediately that d_∞-convergence implies d_s-convergence and d_s-convergence implies d_1-convergence. But the converses are not true. (For details, see Joo and Kim [7])

3. Main result

Let $(\Omega, \mathcal{A}, P)$ be a probability space. A fuzzy number valued function

$$\tilde{X} : \Omega \to F(R), \tilde{X} = \{[X^l_\alpha, X^r_\alpha] : 0 \le \alpha \le 1\}$$

is called a fuzzy random variable if for each $\alpha \in [0,1]$, X^l_α and X^r_α are random variable in the usual sense. Now we assume that the space $F(R)$ is considered as the metric space endowed with the metric d_s, unless otherwise stated.

A fuzzy random variable $\tilde{X}$ is called integrable if $E\|\tilde{X}\| < \infty$. The expectation of integrable fuzzy random variable $\tilde{X}$ is a fuzzy number defined by

$$E(\tilde{X}) = \{[EX^l_\alpha, EX^r_\alpha] \ : \ 0 \le \alpha \le 1\}.$$

Let $\{\tilde{X}_n\}$ be a sequence of integrable fuzzy random variables and $\{\lambda_{ni}\}$ be a Toeplitz sequence, i.e., $\{\lambda_{ni}\}$ is a double array of real numbers satisfying

(1) For each i, $\lim_{n\to\infty} \lambda_{ni} = 0$;

(2) There exists $C > 0$ such that $\sum_{i=1}^{\infty} |\lambda_{ni}| \le C$ for each n.

Now, we write $\tilde{X}_n = \{[X^l_{n,\alpha}, X^r_{n,\alpha}] : 0 \le \alpha \le 1\}$ and assume the following condition:

(3.1): For each $\epsilon > 0$, there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = 1$ of $[0,1]$ such that for all n,

$$\max(\max_{1\le k\le m} E|X^l_{n,\alpha^+_{k-1}} - X^l_{n,\alpha_k}|, \max_{1\le k\le m} E|X^r_{n,\alpha^+_{k-1}} - X^r_{n,\alpha_k}|) < \epsilon.$$

The next theorem implies that if $\{\tilde{X}_n\}$ is identically distributed, then it satisfies the condition (3.1).

Theorem 3.1. (a). *Let* $E\|\tilde{X}\| < \infty$. *Then for each* $\epsilon > 0$, *there exists a partition* $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = 1$ *of* $[0,1]$ *such that*

$$\max(\max_{1\le k\le m} E|X^l_{\alpha^+_{k-1}} - X^l_{\alpha_k}|, \max_{1\le k\le m} E|X^r_{\alpha^+_{k-1}} - X^r_{\alpha_k}|) < \epsilon.$$

(b). *If* $\{\tilde{X}_n\}$ *is identically distributed and* $E\|\tilde{X}_1\| < \infty$, *then it satisfies the condition* (3.1).

Proof. See Hyun et al.[4].

For the purpose of this paper, the notion of level-wise independence for fuzzy random variables is defined as follows: □

Definition 3.2. *A sequence of fuzzy random variables* $\{\tilde{X}_n\}$ *is called level-wise independent if for each* $\alpha \in [0,1]$, *a sequence of random vectors* $\{(X^l_{n,\alpha}, X^r_{n,\alpha})\}$ *is independent.*

Guan and Li [2] obtained SLLN for weighted sums of level-wise independent fuzzy random variables by assuming that $\{\frac{1}{n} \oplus_{i=1}^n \lambda_{ni} E\tilde{X}_i\}$ is convergent with respect to d_∞ instead of (3.1). The following theorem is a generalization of Guan and Li [2] because d_∞-convergence implies d_s-convergence.

Theorem 3.3. *Let* $\{\tilde{X}_n\}$ *be a sequence of level-wise independent fuzzy random variables such that for some* $\tilde{u} \in F(R)$,

$$\lim_{n\to 0} d_s(\oplus_{i=1}^n \lambda_{ni} E\tilde{X}_i, \tilde{u}) = 0. \tag{3.2}$$

Suppose that there exists a nonnegative random variable ξ *with* $E\xi^{1+\frac{1}{\gamma}} < \infty$ *for some* $\gamma > 0$ *such that for each* n,

$$P(\|\tilde{X}_n\| \geq \lambda) \leq P(\xi \geq \lambda) \quad \textit{for all } \lambda > 0.$$

Then for a Toeplitz sequence $\{\lambda_{ni}\}$ *satisfying* $\max_{1\leq i\leq n} |\lambda_{ni}| = O(n^{-\gamma})$,

$$\lim_{n\to\infty} d_\infty(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i, \oplus_{i=1}^n \lambda_{ni}E\tilde{X}_i) = 0 \quad a.s.$$

Proof. Let $\epsilon > 0$. By (3.2), there exists a $t \in T$ such that for large n,

$$d_\infty(\oplus_{i=1}^n \lambda_{ni}E\tilde{X}_i, t(\tilde{u})) < \epsilon/3. \tag{3.3}$$

By applying Lemma 3.3 of Joo and Kim [8] to $\tilde{v} = t(\tilde{u})$, we can find a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = 1$ of $[0,1]$ such that

$$max(\max_{1\leq k\leq m} |v^l_{\alpha^+_{k-1}} - v^l_{\alpha_k}|, \max_{1\leq k\leq m} |v^r_{\alpha^+_{k-1}} - v^r_{\alpha_k}|) < \epsilon/3. \tag{3.4}$$

Since

$$\begin{aligned} &|\sum_{i=1}^n \lambda_{ni}(EX^l_{i,\alpha_k} - EX^l_{i,\alpha^+_{k-1}})| \\ \leq &|\sum_{i=1}^n \lambda_{ni}EX^l_{i,\alpha_k} - v^l_{\alpha_k}| + |\sum_{i=1}^n EX^l_{i,\alpha^+_{k-1}} - v^l_{\alpha^+_{k-1}}| \; + |v^l_{\alpha^+_{k-1}} - v^l_{\alpha_k}|, \end{aligned}$$

(3.3) and (3.4) imply that for large n,

$$|\sum_{i=1}^n \lambda_{ni}(EX^l_{i,\alpha_k} - EX^l_{i,\alpha^+_{k-1}})| < \epsilon. \tag{3.5}$$

We note that for $\alpha_{k-1} < \alpha \leq \alpha_k$,

$$\begin{aligned} &|\sum_{i=1}^n \lambda_{ni}(X^l_{i,\alpha} - EX^l_{i,\alpha})| \\ &\leq |\sum_{i=1}^n \lambda_{ni}(X^l_{i,\alpha_k} - EX^l_{i,\alpha^+_{k-1}})| + |\sum_{i=1}^n \lambda_{ni}(X^l_{i,\alpha^+_{k-1}} - EX^l_{i,\alpha_k})| \\ &\leq |\sum_{i=1}^n \lambda_{ni}(X^l_{i,\alpha_k} - EX^l_{i,\alpha_k})| + |\sum_{i=1}^n \lambda_{ni}(X^l_{i,\alpha^+_{k-1}} - EX^l_{i,\alpha^+_{k-1}})| \\ &\qquad\qquad + 2|\sum_{i=1}^n \lambda_{ni}(EX^l_{i,\alpha_k} - EX^l_{i,\alpha^+_{k-1}})|. \end{aligned}$$

Thus by (3.5) we have that for large n,

$$\sup_{\alpha_{k-1}<\alpha\le\alpha_k} |\sum_{i=1}^{n} \lambda_{ni}(X_{i,\alpha}^l - EX_{i,\alpha}^l)|$$
$$\le |\sum_{i=1}^{n} \lambda_{ni}(X_{i,\alpha_k}^l - EX_{i,\alpha_k}^l)| + \sum_{i=1}^{n} |\lambda_{ni}(X_{i,\alpha_{k-1}^+}^l - EX_{i,\alpha_{k-1}^+}^l)| + 2\epsilon.$$

Consequently, we obtain that for large n,

$$\sup_{0\le\alpha\le1} |\sum_{i=1}^{n} \lambda_{ni}(X_{i,\alpha}^l - EX_{i,\alpha}^l)| \le \max_{1\le k\le m} |\sum_{i=1}^{n} \lambda_{ni}(X_{i,\alpha_k}^l - EX_{i,\alpha_k}^l)|$$
$$+ \max_{1\le k\le m} |\sum_{i=1}^{n} \lambda_{ni}(X_{i,\alpha_{k-1}^+}^l - EX_{i,\alpha_{k-1}^+}^l)| + 2\epsilon.$$

Since the first two terms on the right hand converge to 0 almost surely by Rohatgi's results [11],

$$\sup_{0\le\alpha\le1} |\sum_{i=1}^{n} \lambda_{ni}(X_{i,\alpha}^l - EX_{i,\alpha}^l)| \le 2\epsilon \ a.s. \text{ for large } n.$$

Similarly it can be proved that

$$\sup_{0\le\alpha\le1} |\sum_{i=1}^{n} \lambda_{ni}(X_{i,\alpha}^r - EX_{i,\alpha}^r)| \le 2\epsilon \ a.s. \text{ for large } n.$$

Therefore, for large n,

$$d_\infty(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i, \oplus_{i=1}^n \lambda_{ni}E\tilde{X}_i) \le 2\epsilon \ a.s.$$

which completes the proof. □

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