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A BERBERIAN TYPE EXTENSION OF FUGLEDE-PUTNAM THEOREM FOR QUASI-CLASS A OPERATORS

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Abstract. Let $\mathscr{L}(\mathscr{H})$ denote the algebra of bounded linear operators on a separable infinite dimensional complex Hilbert space $\mathscr{H}$. We say that $T \in \mathscr{L}(\mathscr{H})$ is a quasi-class A operator if

$$T^*|T^2|T \geq T^*|T|^2T.$$

In this paper we prove that if A and B are quasi-class A operators, and B^* is invertible, then for a Hilbert-Schmidt operator X

$$AX = XB \text{ implies } A^*X = XB^*.$$

1. Introduction

Recall ([1], [5]) that $T \in \mathscr{L}(\mathscr{H})$ is called *p-hyponormal* if for $p \in (0,1]$

$$(T^*T)^p \geq (TT^*)^p,$$

and T is called *paranormal* if for all unit vector $x \in \mathscr{H}$

$$||T^2x|| \geq ||Tx||^2.$$

Following [5] and [6] we say that $T \in \mathscr{L}(\mathscr{H})$ belongs to *class* A if

$$|T^2| \geq |T|^2.$$

Recall ([9]) that T is called *p-quasihyponormal* if for $p \in (0,1]$

$$T^*(T^*T)^pT \geq T^*(TT^*)^pT.$$

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For brevity, we shall denote classes of p-hyponormal operators, p-quasihyponormal operators, paranormal operators, and class A operators by $\mathcal{H}(p)$, $\mathcal{QH}(p)$, $\mathcal{PN}$, and $\mathcal{A}$, respectively. It is well known that

$$(1.1) \qquad \mathcal{H}(p) \subset \mathcal{A} \subset \mathcal{PN} \text{ and } \mathcal{H}(p) \subset \mathcal{QH}(p) \subset \mathcal{PN}.$$

Recently, Jeon and Kim ([8]) considered an extension of class A operators and p-quasihyponormal operators.

DEFINITION 1.1. We say that $T \in \mathscr{L}(\mathscr{H})$ is *quasi-class* A if

$$T^*|T^2|T \geq T^*|T|^2T.$$

For brevity, we shall denote the set of quasi-class A operators by $\mathcal{QA}$. As shown in [8], the class of quasi-class A operators properly contains classes of class A operators and p-quasihyponormal operators, i.e., the following inclusion holds;

$$(1.2) \qquad \mathcal{H}(p) \subset \mathcal{QH}(p) \subset \mathcal{QA} \text{ and } \mathcal{H}(p) \subset \mathcal{A} \subset \mathcal{QA}$$

In view of inclusions (1.1) and (1.2), it seems reasonable to expect that operators in class $\mathcal{QA}$ are paranormal. But there exists an example which is not paranormal but quasi-class A([8]).

A familiar Fuglede-Putnam theorem is as follows.

PROPOSITION 1.2. *Let A, B, and X be in $\mathscr{L}(\mathscr{H})$. If A and B are normal, then*

$$AX = XB \text{ implies } A^*X = XB^*.$$

In [2] S. K. Berberian relaxes the hypotheses on A and B in the above theorem at the cost of requiring X to be of Hilbert-Schmidt class(denoted $X \in \mathscr{C}_2$, for definitions and details see [10]) as follows.

PROPOSITION 1.3. *Let $A,\ B \in \mathscr{L}(\mathscr{H})$ and $X \in \mathscr{C}_2$. Then*

$$AX = XB \text{ implies } A^*X = XB^*$$

under either of the following hypotheses:

(i) *A and B^* are hyponormal,*
(ii) *B is invertible and $||A|| \cdot ||B^{-1}|| \leq 1$*

In [4, Theorem 2] T. Furuta relaxed the hyponormality on A and B^* to k-quasihyponormality(to be defined below).

Recall [3] that an operator $T \in \mathscr{L}(\mathscr{H})$ ia said to be k-quasihyponormal if $T^{*k}(T^*T - TT^*)T^k \geq 0$ for some non-negative integer k. It is well known that, for $k \geq 2$, the class of k-quasihyponormal operators has no inclusion relations with classes of the former mentioned operators.

In this paper, we prove an analogue result of T. Furuta as follows.

THEOREM 1.4. *Let $A \in \mathcal{QA}$ and $B^* \in \mathcal{QA}$ be invertible. Then for $X \in \mathscr{C}_2$*

$$AX = XB \text{ implies } A^*X = XB^*.$$

The following result immediately follows.

COROLLARY 1.5. *Let $A \in \mathcal{A}$(resp. $A \in \mathcal{QH}(p)$) and $B^* \in \mathcal{A}$(resp. $B^* \in \mathcal{QH}(p)$) be invertible. Then for $X \in \mathscr{C}_2$*

$$AX = XB \text{ implies } A^*X = XB^*.$$

2. Proofs

In this section we give a proof of Theorem1.4, modifying T. Furuta's arguments in the proof of [4, Theorem 2]. We need some lemmas. Recall from [8] that

LEMMA 2.1. *Let $T \in \mathcal{QA}$ and T not have a dense range. Then T has the following matrix representation:*

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \text{ on } \overline{ran(T)} \oplus ker(T^*),$$

where $A \in \mathcal{A}$. Furthermore, $\sigma(T) = \sigma(A) \cup \{0\}$.

From the above lemma we immediately have

COROLLARY 2.2. *If $T \in \mathcal{QA}$ is invertible, then $T \in \mathcal{A}$.*

In [2] an operator $\mathscr{T}$ on $\mathscr{C}_2$ is defined by, for $A, B \in \mathscr{L}(\mathscr{H})$,

$$\mathscr{T}X = AXB.$$

Then, as in [2], simple calculations show that $\mathscr{T}^*X = A^*XB^*$ and also

$$A, B \geq 0 \text{ implies } \mathscr{T} \geq 0. \tag{2.1}$$

LEMMA 2.3. *If $A, B^* \in \mathcal{QA}$, then the operator $\mathscr{T}$ belongs to $\mathcal{QA}$.*

Proof. From the hypotheses of A and B^*, and (2.1) we have

$$\begin{aligned} & (\mathscr{T}^*|\mathscr{T}^2|\mathscr{T} - \mathscr{T}^*|\mathscr{T}|^2\mathscr{T})X \\ = \; & (A^*|A^2|A - A^*|A|^2A)XB|B^{*2}|B^* + A^*|A|^2AX(B|B^{*2}|B^* - B|B^*|^2B^*) \\ \geq \; & 0, \end{aligned}$$

which shows that $\mathscr{T}$ is a quasi-class A operator on $\mathscr{C}_2$. □

Proof of Theorem 1.4. If $S \in \mathscr{L}(\mathscr{H})$ is invertible, let $\mathscr{T}$ on $\mathscr{C}_2$ be defined by

$$\mathscr{T}Y = AYS^{-1}.$$

Since B^* is invertible quasi-class A, B^* is just invertible class A by Corollary 2.2, and $(B^*)^{-1} = (B^{-1})^*$ is also class A by [11]. So it follows that from Lemma 2.3 that $\mathscr{T} \in \mathcal{QA}$. The hypotheses $AX = XB$ implies $\mathscr{T}X = X$ and from the fact $\mathscr{T} \in \mathcal{QA}$ it follows (use the Hölder-McCarthy inequality[5])

$$\begin{aligned} ||\mathscr{T}^*X||^2 &= \langle \mathscr{T}^*X, \mathscr{T}^*X \rangle \\ &= \langle \mathscr{T}^*\mathscr{T}\mathscr{T}X, \mathscr{T}^*\mathscr{T}\mathscr{T}X \rangle \\ &\leq \langle \mathscr{T}^*|\mathscr{T}^2|\mathscr{T}X, X \rangle \\ &= \langle (\mathscr{T}^{*2}\mathscr{T}^2)^{\frac{1}{2}}X, X \rangle \\ &\leq \langle (\mathscr{T}^{*2}\mathscr{T}^2)X, X \rangle^{\frac{1}{2}} \cdot ||X|| \\ &= ||X||^2, \end{aligned}$$

which implies

$$||\mathscr{T}^*X - X||^2 \leq ||\mathscr{T}^*X||^2 - 2||X||^2 + ||X||^2 \leq 0.$$

Hence we have $\mathscr{T}^*X = X$, i.e., $A^*X = XB^*$. □

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